Applications of free probability to W^*-bundles [1]

David Jekel
University of Copenhagen

Scandinavian Operator Algebras Workshop
Lund 2026-06-08

[1] Funded by the European Union, Marie Skłodowska-Curie Actions

Motivation

Traditionally, von Neumann algebras are viewed as noncommutative measure spaces and C^*-algebras as noncommutative topological spaces.

There are nonetheless many problems in operator algebras that include both measure-theoretic and topological aspects.

For instance, the abstract approach to classification of C^*-algebras uses the relationship between ultraproducts of von Neumann algebras and C^*-algebras and an intermediate class of tracially complete C^*-algebras.

Today's topic

Elements of a von Neumann algebra that vary in a continuous way with respect to SOT, and more generally bundles of von Neumann algebras.

Review of von Neumann algebras

The *strong operator topology (SOT)* on $B(H)$ is the topology generated by the maps $T \mapsto Th$ for $h \in H$.

A *von Neumann algebra* is a $*$-subalgebra M of $B(H)$ which is closed in the SOT (in particular, it is a C^*-algebra).

A von Neumann algebra is called a *factor* if it has trivial center $Z(M) = \mathbb{C}1$.

A state τ on M is *tracial* if $\tau(xy) = \tau(yx)$.
It is *normal* if it is SOT-continuous on the unit ball.
it is *faithful* if $\tau(x^*x) = 0$ implies $x = 0$.

A II_1-*factor* is an infinite-dimensional factor which has a tracial state (which is automatically unique and normal and faithful).

Review of von Neumann algebras

If (M, τ) is a von Neumann algebra with a faithful normal tracial state, we write

$$\|x\|_2 = \tau(x^* x)^{1/2}$$

and consider the GNS space $L^2(M, \tau)$.

We can throw away the original H and consider M acting on $L^2(M, \tau)$ since the representation faithful and preserves the σ-SOT.

In the representation on $L^2(M, \tau)$, the SOT on the unit ball is equivalent to the $\|\cdot\|_2$-topology.

Contractibility of the unitary group

Question

Is the unitary group $U(M)$ of a von Neumann algebra contractible with respect to SOT?

To prove contractibility, we want a homotopy $h : [0,1] \times U(M) \to U(M)$ such that $h(0,u) = u$ and $h(1,u) = 1$.

In a von Neumann algebra every unitary u can be written as e^{ix} for some x self-adjoint by Borel functional calculus, and hence u is connected to the identity by the path e^{itx}.

However, it is unclear how to choose a logarithm x that depends continuously on u, especially if the spectral measure of u has atoms.

Contractibility of the unitary group

The case of $B(H)$ and factors of type II_∞ and III were settled long ago [7, 14, 4, 5]. For II_1 factors, Popa and Takesaki [17] handled the McDuff case, but the general case was open.

In fact, for infinite factors (those without a tracial state), it is contractible with respect to $\|\cdot\|$, but for II_1 factors it is not [2].

Theorem (J. 2025) [13]

The unitary group of a II_1 factor is contractible with respect to strong operator topology.

- **Step 1:** Deform u so that the spectral measure spreads out towards Haar measure. *Uses approximate freeness.*
- **Step 2:** Apply continuous functional calculus to the push the mass of the spectral measure toward 1.

Use of approximate freeness

Free independence was first described by Avitzour and Voiculescu. $*$-subalgebras $(M_i)_{i \in I}$ in (M, τ) are *freely independent* if whenever $i_1 \neq i_2 \neq \ldots \neq i_k$ and $x_j \in M_{i_j}$, we have

$$\tau\left[(x_1 - \tau(x_1)) \ldots (x_k - \tau(x_k))\right] = 0.$$

This allows the construction of free products of von Neumann algebras (cf. reduced free product of C^*-algebras).

Fact

Suppose that u is a unitary in M and $v = e^{ix}$ is a Haar unitary with x freely independent of u. Then ue^{itx} deforms u into uv, and uv is a Haar unitary.

Use of approximate freeness

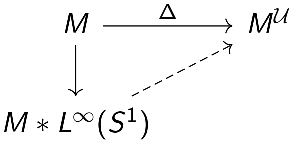
Theorem (Popa 1995) [16]

Let M be a II_1 factor. Then there exists a sequence of unitaries v_n which asymptotically behave as a Haar unitary freely independent from M.

Namely, for all $x_0, \ldots, x_k \in M$, and $r_1, \ldots, r_k \in \mathbb{Z} \setminus \{0\}$,

$$\lim_{n \to \infty} \tau(x_0 v_n^{r_1} (x_1 - \tau(x_1)) \ldots v_n^{r_{k-1}} (x_{k-1} - \tau(x_{k-1})) v_n^{r_k} x_k) = 0.$$

Remark: In terms of ultraproducts, this means that



Aside: Approximate freeness in a C^*-setting

The C^*-algebraic analog of having approximately free Haar unitaries is Robert's notion of selflessness [18].

Definition

A C^*-algebra A with a state τ is *selfless* if there exist unitaries $v_n \in A$ such that for all $a_1, \ldots, a_k$ in A and $*$-polynomials p, we have

$$\lim_{n\to\infty} \|p(a_1, \ldots, a_k, v_n)\|_A = \|p(a_1, \ldots, a_k, v)\|_{A * C(S^1)},$$

where $A * C(S^1)$ is the reduced free product with respect to τ and the Haar state on $C(S^1)$ and v is the canonical generator of $C(S^1)$.

Selflessness implies a lot of desirable C^*-algebraic properties such as strict comparison, simplicity, and stable rank one.

$C_r(\mathbb{F}_m)$ is selfless [1], and obtaining new selfless C^*-algebras is an active area of research.

Contractibility revisited

Given a von Neumann algebra M and a compact Hausdorff space K, let $C_\sigma(K, M)$ be the C^*-algebra of functions $K \to M$ that are bounded in operator norm and σ-SOT continuous.

A unitary in $C_\sigma(K, M)$ is a continuous function from K into $U(M)$. The contractibility of the unitary group of M implies that a unitary in $C_\sigma(K, M)$ is homotopic to the constant 1, so the unitary group of $C_\sigma(K, M)$ is connected.

In this instance, $C_\sigma(K, M)$ behaves like a II_1 factor M. The underlying topology of K does not matter.

A freely independent Haar unitary v comes from "outside" the algebra M, leading to constructions that are agnostic to the particular point in M or in K.

Other problems in $C_\sigma(K, M)$

Along the same lines, there are several other problems about whether well-known properties in von Neumann algebras can be achieved in a continuously varying way.

Comparison of projections

Suppose that p and q are projections in $C_\sigma(K, M)$. Suppose that $\tau(p(t)) \leq \tau(q(t))$ for all $t \in K$. Does there exist a partial isometry $v(t)$ such that $v^*v = p$ and $vv^* \leq q$?

Unitary polar decomposition

Let $x \in C_\sigma(K, M)$. Does there exist a unitary $u \in C_\sigma(K, M)$ such that $x = u|x|$? Hence, x could be approximated in operator norm by invertible elements, meaning that $C_\sigma(K, M)$ would have stable rank one.

Other problems in $C_\sigma(K, M)$

Discrete approximation

Let $x \in C_\sigma(K, M)$ be self-adjoint. When do spectral projections of x exist in $C_\sigma(K, M)$? Does there exist some x' with finite spectrum with $\|x - x'\| < \varepsilon$? This would mean that $C_\sigma(K, M)$ has real rank zero.

In joint work with Maxwell Ryder and Stuart White, we show that the answers to these three questions is essentially yes.

We consider the same problems more generally for bundles of von Neumann algebras over K.

W^*-bundles

Naïve definition: A (tracial) W^*-bundle over K is given by:

- A collection of tracial von Neumann algebras (M_t, τ_t) for $t \in K$.
- A $*$-algebra $\mathcal{M}$ of sections $x : K \to \bigcup_{t \in K} M_t$.

such that for each $x \in \mathcal{M}$,

- $x(t) \in M_t$ for all t,
- $\sup_{t \in K} \|x(t)\| < \infty$,
- $t \mapsto \tau_t(x(t))$ is continuous

and such that

- The unit ball of $\mathcal{M}$ is complete with respect to the uniform 2-norm $\|x\|_{2,K} = \sup_{t \in K} \tau_t(x(t)^* x(t))^{1/2}$,
- $\{x(t) : x \in \mathcal{M}\} = M_t$,
- If $f \in C(K)$, then $f(t)1$ is in $\mathcal{M}$.

W^*-bundles

Ozawa's definition: [15]

- A C^*-algebra $\mathcal{M}$.
- An embedding $C(K) \to Z(\mathcal{M})$.
- A tracial conditional expectation $E : \mathcal{M} \to C(K)$ (meaning $E(xy) = E(yx)$)

such that $\mathcal{M}$ the unit ball of $\mathcal{M}$ is complete with respect to

$$\|x\|_{2,K} = \|E(x^*x)\|_{C(K)}^{1/2}.$$

Fact: These two definitions are equivalent. Here is how you recover the fibers from Ozawa's definition:

For $t \in K$, note that $\tau_t = \delta_t \circ E$ is a trace on $\mathcal{M}$. Let π_t be the GNS representation of $\mathcal{M}$ with respect to τ_t. Then $M_t = \pi_t(\mathcal{M})$ turns out to already be a von Neumann algebra. Sections are given by $x(t) = \pi_t(x)$.

W*-bundles

Fact: These two definitions are equivalent. Here is how you obtain the conditional expectation E from the fiberwise definition.

Given the fibers (M_t, τ_t) define $E(x) \in C(K)$ by

$$E(x)(t) = \tau_t(x(t)),$$

which was assumed to be a continuous function of t!

We also assumed that $f(t)1$ is in $\mathcal{M}$ for all $f \in C(K)$, which gives our embedding $C(K) \to \mathcal{M}$.

The two definitions of the uniform 2-norm agree:

$$\|E(x^*x)\|_{C(K)}^{1/2} = \sup_{t \in K} E(x^*x)(t)^{1/2} = \sup_{t \in K} \tau_t(x(t)^*x(t))^{1/2}.$$

Tracially complete C^*-algebras

A *tracially complete* C^**-algebra* [6] is a pair (A, X) where

- A is a C^*-algebra.
- X is a nonempty closed convex set of tracial states on A.
- The uniform 2-norm $\|a\|_{2,X} = \sup_{\tau \in X} \tau(a^*a)^{1/2}$ defines a norm on A (faithfulness of X).
- The unit ball of A is complete in $\|\cdot\|_{2,X}$.

Theorem (Ozawa) [15]

Suppose that X is a face of $\mathcal{T}(A)$ and that the set of extreme points $K = \partial_e X$ is closed. Then A is isomorphic to a W^*-bundle $\mathcal{M}$ over K where $E(x)(\tau) = \tau(x)$ for $\tau \in K$.

Background: $\mathcal{T}(A)$ is a Choquet simplex, meaning that every point can be uniquely decomposed in terms of extreme points. Extreme points are exactly the traces τ such that $\pi_\tau(A)''$ is a factor.

Selfless W^*-bundles

In order to apply free probability techniques to a W^*-bundle, we want elements that are asymptotically free in a uniform way over the fibers.

Definition (J.–Ryder–White)

Let $\mathcal{M}$ be a W^*-bundle over K. WLOG identify K with a closed subset of $\mathcal{T}(\mathcal{M})$. We say $\mathcal{M}$ is *selfless* if there exist unitaries $v_n \in \mathcal{M}$ such that for all $x_0, \dots, x_k \in \mathcal{M}$, and $r_1, \dots, r_k \in \mathbb{Z} \setminus \{0\}$,

$$\lim_{n\to\infty} \sup_{\tau \in K} |\tau(x_0 v_n^{r_1}(x_1 - \tau(x_1)) \dots v_n^{r_{k-1}}(x_{k-1} - \tau(x_{k-1}))v_n^{r_k} x_k)| = 0.$$

This statement is the same as Popa's asymptotic freeness theorem but we just need the conditions to hold uniformly for all $\tau \in K$, i.e. uniformly in all the fibers $\pi_\tau(\mathcal{M})$.

Selfless W^*-bundles: Examples

1. Every trivial W^*-bundle $C_\sigma(K, M)$ is selfless (by Popa's theorem).
2. Every locally trivial W^*-bundle is selfless.
3. **Patching theorem:** Suppose K is covered by finitely many closed sets K_j and that $\mathcal{M}|_{K_j}$ is selfless for each j. Then $\mathcal{M}$ is selfless.
4. Suppose that $N \subseteq M$ is an irreducible inclusion of II_1 factors ($N' \cap M = \mathbb{C}$). Suppose that $C_\sigma(K, N) \subseteq \mathcal{M} \subseteq C_\sigma(K, M)$. Then $\mathcal{M}$ is selfless (also Popa's theorem).
5. Fix $N \subseteq M$, not necessarily irreducible, and fix $K' \subseteq K$ closed. Then $\mathcal{M} = \{x \in C_\sigma(K, M) : x|_{K'} \in C_\sigma(K', M)\}$ is selfless (from more technical version of patching).
6. There exist selfless W^*-bundles $\mathcal{M}$ such that all the fibers are isomorphic, and $\mathcal{M}$ does not embed into a trivial bundle.

Selfless W^*-bundles: Results

Theorem (J., Ryder, White)

Let $(\mathcal{M}, K)$ be a selfless W^*-bundle. Then $\mathcal{M}$ has the following properties:

1. **Comparison of projections:** If p, q are projections with $\tau(p) \leq \tau(q)$ for all $\tau \in K$, then there exists $v \in \mathcal{M}$ with $v^*v = p$ and $vv^* \leq q$.
2. **Existence of projections:** For every $\varphi \in C(K, [0,1])$, there exists a projection p such that $\varphi(\tau) = \tau(p)$ for all $\tau \in K$.
3. **Real rank zero:** Every self-adjoint $x \in \mathcal{M}$ can be approximated in operator norm by self-adjoints with finite spectrum.
4. **FU property:** Every unitary can be approximated by a unitary with finite spectrum. Hence, $K_1(\mathcal{M}) = 0$.
5. **Stable rank one:** Invertible elements are dense in $\mathcal{M}$ with respect to operator norm.

Recent related results

Farah–Vaccaro [10]: Classification of projections holds for a trivial W^*-bundle, provided that either (1) the base space has covering dimension at most one, or (2) $\mathcal{M}$ is the fiberwise tensor product of a W^*-bundle $\mathcal{N}$ with $L(\mathbb{F}_\infty)$.

Evington [8]: Tracial completions of $\mathcal{Z}$-stable C^*-algebras have strict comparison with respect to the specified set of traces.

Evington–Tikuisis [9]: Stable rank one, and real rank zero hold for factorial tracially complete C^*-algebras with complemented partitions of unity. (For instance, tracial completions of $\mathcal{Z}$-stable C^*-algebras.)

This includes many W^*-bundles, but they must at least have property Gamma. So [9] handles many tracially complete C^*-algebras that we can't, and we handle many W^*-bundles that they can't.

Further consequences

Theorem (J., Ryder, White)

Suppose that $(\mathcal{M}, K)$ is any W^*-bundle satisfying classification of projections (items 1-2 above) and real rank zero (item 3). Then $\mathcal{M}$ has uniform Dixmier averaging:

$\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$, $\forall x \in (\mathcal{M})_1$, $\exists u_1, \ldots, u_n \in U(\mathcal{M})$,

$$\left\| \frac{1}{n} \sum_{j=1}^{n} u_j x u_j^* - E(x) \right\| < \varepsilon.$$

Other consequences of 1-3 (immediate or from known results):

1. $\mathcal{M}$ has strict comparison.
2. The closed convex hull of K is all of $\mathcal{T}(\mathcal{M})$ (no exotic traces).
3. The ideals of $\mathcal{M}$ are classified by closed subsets of K.
4. $K_0(\mathcal{M}) \cong C(K; \mathbb{R})$.

Sample proof: Stable rank one

The proofs of comparison of projections, real rank zero, stable rank one follow the same strategy:

- Formulate the desired property as a G_δ condition (intersection of open conditions indexed by $k \in \mathbb{N}$).
- For each $k \in \mathbb{N}$, use the freely independent element to perturb the given x so it satisfies the kth condition (free probability ingredients).
- Show that the perturbation using an *approximately* free element still satisfies the kth condition.
- Apply Baire category theorem.

I will focus on developing this for the stable rank one case.

Polar decomposition

Goal: The elements of $\mathcal{M}$ which have trivial kernel form a dense G_δ set w.r.t. operator norm (also incidentally w.r.t. the uniform 2-norm).

Lemma

Let $\mathcal{M}$ be a W^*-bundle and $x \in \mathcal{M}$. The following are equivalent:

1. $\pi_\tau(x)$ has trivial kernel for all $\tau \in K$ (or $\mathbb{1}_{\{0\}}(\pi_\tau(x)) = 0$).
2. For every $\varepsilon > 0$, there exists $f \in C_0([0,\infty),[0,1])$ with $f(0) = 1$ and
$$\sup_{\tau \in K} \tau(f(x^* x)) < \varepsilon.$$

Remark: The key point for (1) $\implies$ (2) is to exploit compactness of K. If φ_k is a decreasing sequence of continuous functions on K and $\varphi_k \to 0$, then the convergence must be uniform.

Polar decomposition

Lemma

Suppose that $x \in \mathcal{M}$ satisfies the equivalent conditions of the previous lemma. Then $x = u|x|$ for a unique unitary $u \in \mathcal{M}$.

Remark: We construct u as the limit of $x(x^*x)^{-1/2}(1 - f_k(x^*x))$ where f_k is identically 1 in a neighborhood of 0, and we arrange that the limit as $k \to \infty$ exists in uniform 2-norm.

This comes from the uniformity in condition (2) of the previous lemma.

Lemma

If x is as in the previous lemma, then x can be approximated in operator norm by invertible elements, namely, $u(|x| + \varepsilon)$ for $\varepsilon > 0$.

Formulating a G_δ condition

Goal: The elements of $\mathcal{M}$ which have trivial kernel form a dense G_δ set w.r.t. operator norm.

Intersection of open conditions: Let G_k be the set of x such that there exists $f \in C_0([0,\infty),[0,1])$ satisfying

$$\sup_{\tau \in K} \tau(f(x^* x)) < \frac{1}{k}.$$

Let $G = \bigcap_{k \in \mathbb{N}} G_k$, which is the set of $x \in \mathcal{M}$ with trivial kernel.

Openness of G_k: If some f works for x, then it also works in a neighborhood of x using uniform continuity of the functional calculus.

Free perturbation argument

A *free circular element* z is an operator of the form $x + iy$ where x and y are freely independent semicircular elements. Here semicircular means that

$$\tau(f(x)) = \frac{1}{2\pi} \int_{-2}^{2} f(t)\sqrt{4 - t^2}\, dt$$

Theorem (Belinschi–Yin–Zhong [3])

Suppose x is any element of a tracial von Neumann algebra and z is a circular element freely independent from x. Then $x + \varepsilon z$ has Brown measure which is Lebesgue absolutely continuous. Consequently (by properties of Brown measure due to Haagerup and Larsen [11]), it has trivial kernel.

Free perturbation argument

Consider the (fiberwise) free product of my W^*-bundle $(\mathcal{M}, K)$ with a (N, σ) where N is the von Neumann algebra of a free circular element z.

If $x \in \mathcal{M}$, then $x + \varepsilon z$ has trivial kernel in $(\mathcal{M}, K) * (N, \sigma)$, so there exists f with

$$\sup_{\tau \in K} \tau * \sigma(f(|x + \varepsilon z|^2)) < \frac{1}{k}.$$

Using selflessness, we can choose z_n which simulates the behavior of the free circular element z uniformly over all the traces,

$$\lim_{n \to \infty} \sup_{\tau \in K} |\tau(f(|x + \varepsilon z_n|^2)) - \tau * \sigma(f(|x + \varepsilon z|^2))| = 0.$$

So for large enough n,

$$\sup_{\tau \in K} \tau(f(|x + \varepsilon z_n|^2)) < \frac{1}{k} \qquad \text{hence } x + \varepsilon z_n \in G_k.$$

Stable rank one: Conclusion

Thus, G_k is dense for each k.

By Baire category, $G = \bigcap_{k \in \mathbb{N}} G_k$ is dense G_δ.

Thus, every element in $\mathcal{M}$ can be approximated by an element of trivial kernel, which in turn can be approximated by an invertible element.

Open questions

Question 1

Does there exist a II_1 factor bundle which is not selfless?

Question 2

Is selflessness of a W^*-bundle $(\mathcal{M}, K)$ equivalent to real rank zero plus classification of projections?

Question 3

To what extent do these results generalize to tracially complete C^*-algebras?

Note that existence of approximately free elements implies that the fiber is a factor, and so only makes sense for extreme traces. And if this is to hold *uniformly* on some $E \subseteq \mathcal{T}(\mathcal{M})$, then it holds on $\overline{E}$.

Open questions

Question 4

For a selfless W^*-bundle $(\mathcal{M}, K)$, is every unitary (not merely a dense subset) equal to e^{ix} for some self-adjoint x?

Question 5

Generalize the theory of W^*-bundles beyond the tracial setting. What conditions are needed for the fiberwise modular group to leave $\mathcal{M}$ invariant?

Remark: Houdayer and Marrakchi [12] showed that for a factor M with faithful normal state φ, selflessness in the von Neumann algebraic sense (i.e. Popa's theorem) is equivalent to satisfying the bicentralizer conjecture.

References I

[1] Tattwamasi Amrutam, David Gao, Srivatsav Kunnawalkam Elayavalli, Gregory Patchell. Strict comparison in reduced group C^*-algebras. Invent. Math. 242.3: 639-657 (2025).

[2] H. Araki, M.-S.B. Smith, and L. Smith. On the homotopical significance of the type of von Neumann algebra factors. Commun. Math. Phys., 22:71–88 (1971).

[3] Serban Belinschi, Zhi Yin, and Ping Zhong. The brown measure of a sum of two free random variables, one of which is triangular elliptic. Advances in Mathematics, 441:art. 109562 (2024).

[4] M. Breuer. A generalization of Kuiper's theorem to factors of type II_∞. J. Math. Mech., 16:917–925 (1967).

[5] Willgerodt W. Bruning, J. Eine Verallgemeinerung eines Satzes von N. Kuiper. Math. Ann., 220:47–58 (1976).

References II

[6] José R. Carrión, Jorge Castillejos, Samuel Evington, James Gabe, Christopher Schafhauser, Aaron Tikuisis, and Stuart White. Tracially complete C^*-algebras. Preprint, arXiv:2310.20594 (2023)

[7] J. Dixmier and A. Douady. Champs continus d'espaces hilbertiens et de C^*-algébres. Bulletin de la S.M.F. 91 (1963).

[8] Traces on the uniform tracial completion of $\mathcal{Z}$-stable C^*-algebras. J. London Math. Soc. 111.6, paper no. e70207 (2025).

[9] Samuel Evington and Aaron Tikuisis. The real and stable rank of tracially complete C^*-algebras, arXiv:2604.24206 (2026).

[10] Ilijas Farah and Andrea Vaccaro. Continuous selection of unitaries in II_1 factors. Proceedings of the American Mathematical Society 154: 1609-1622 (2026).

References III

[11] Uffe Haagerup and Flemming Larsen. Brown's spectral distribution measure for R-diagonal elements in finite von Neumann algebras. Journal of Functional Analysis, 176:331–367 (2000).

[12] Cyril Houdayer and Amine Marrakchi. Selfless W^*-probability spaces and Connes' bicentralizer problem. arXiv:2511.11409

[13] David Jekel. The unitary group of a II_1-factor is SOT-contractible. Mathematische Annalen. 393:3109-3117 (2025)

[14] N.H. Kuiper. The homotopy type of the unitary group of Hilbert space. Topology 3:19–30 (1965).

[15] Narutaka Ozawa. Dixmier approximation and symmetric amenability for C^*-algebras. J. Math. Sci. Univ. Tokyo, 20.3:349–374 (2013).

[16] Sorin Popa. Free-independent sequences in type II_1 factors and related problems. Ast´erisque, 232:187–202 (1995).

References IV

[17] Sorin Popa and Masamichi Takesaki. The topological structure of the unitary and automorphism groups of a factor. Comm. Math. Phys., 155:93–101 (1993).

[18] Leonel Robert. Selfless C^*-algebras. Adv. Math., 478: Paper No. 110409, 28 pages (2025).