

Research Statement

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1. INTRODUCTION

Free probability is a theory of non-commuting random variables that describes the large- n behavior of many families of $n \times n$ random matrices. On the applied side, free probability has been used for many problems where random matrices arise in data analysis, neural networks, communications, and finance, while on the side of pure mathematics, it has deep connections with von Neumann algebras, which serve as a non-commutative analog of probability spaces. The strength of free probability lies in its ability to connect the finite-dimensional random matrix models with the infinite-dimensional von Neumann algebras. The benefits of this connection go two ways: The infinite-dimensional objects (von Neumann algebras) shed light on the limiting behavior of random $n \times n$ matrices for large finite n , while the matrix approximations also yield some structural results about the von Neumann algebras that would otherwise be intractable.

The broad aim of my research in free probability is to develop non-commutative analogs of classical objects in probability theory and information theory (such as probability measures, independence, entropy, optimal transportation distance), which coherently describe the large- n behavior of the corresponding quantities for the random matrix models. Furthermore, I use these probabilistic tools to analyze the resulting von Neumann algebras.

I prioritize interdisciplinary collaboration and supporting early-career researchers. I regularly organize online reading groups for people to become familiar with a new topic, which can also become a springboard for joint projects. Moreover, many of my papers are joint work with graduate students and sometimes even undergraduates.

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2. RANDOM MATRIX THEORY

2.1. Motivation and background from a random matrix viewpoint. Random matrix theory has clear relevance to any problems that deal with large matrices of data, and in particular for understanding how random noise or measurement errors affect the spectrum or singular values of a large matrix. For instance, suppose that the true values are represented by a matrix X and the empirically observed values are $X + Z$ where Z is a Gaussian random matrix independent of X . Focusing on square matrices for simplicity, a prototypical question for this setting is the following *denoising problem*:

Question 2.1. *Given $X + Z$ and given a polynomial f , find a function g so that $g(X + Z)$ best approximates $f(X)$ in L^2 distance.*

My results on conditional expectations entail a solution to this problem for a certain class of random matrix models; see Theorem 2.3 below. Indeed, the optimizer should be the orthogonal projection or conditional expectation of $f(X)$ onto $\{g(X + Z) : g \text{ polynomial}\}$. Here the inner product is given by $\langle A, B \rangle_{\text{tr}_n} = \mathbb{E} \text{tr}_n(A^*B)$ where tr_n is the normalized trace on $n \times n$ matrices and \mathbb{E} is the probabilistic expectation. So understanding the orthogonal projection requires us to know the behavior of $\text{tr}_n(f(X)g(X + Z))$ for all polynomials f and g . This can be determined from the *joint distribution* of the random matrices X and Z ; but X and Z do not commute, it is not sufficient to know $\text{tr}_n(X^j Z^k)$ for all j, k , but rather we need to know the traces of all possible non-commutative products such as $XZ^2X^3ZX^2Z$.

Free probability theory, developed in large part by Voiculescu, allows (for instance) an explicit description of large- n asymptotics of $\text{tr}_n(XZ^2X^3ZX^2Z)$ in terms of traces of polynomials in X alone, when Z is a GUE

matrix independent of X . GUE stands for ‘‘Gaussian unitary ensemble,’’ and it means that Z has Gaussian entries with mean zero and variance $1/n$ that are independent up to Hermitian symmetry. The spectral distribution of a GUE matrix converges in the large- n limit to the semicircular measure $(1/2\pi)\sqrt{4-x^2} dx$ on the interval $[-2, 2]$, as observed by Wigner. However, the joint distribution of two matrices in the limit cannot be described by a classical measure due to the non-commutativity, so a new framework is needed.

The fundamental objects of study in free probability are *non-commutative probability spaces*. In classical probability, bounded random variables form the L^∞ space of the underlying measure space, while in the non-commutative setting, we replace L^∞ by a non-commutative algebra of operators on a Hilbert space.

- Specifically, we consider a *von Neumann algebras* $M \subseteq B(H)$, which is closed under algebraic operations, adjoints, and limits in weak operator topology.
- The *expectation* in classical probability is replaced in the non-commutative setting by a linear functional τ from the given von Neumann algebra M into the complex numbers, usually a *state* or even a *tracial state*.
- Elements (X_1, \dots, X_m) of A are viewed as *non-commutative random variables* and their *joint distribution* is an object encoding the values of moment expressions such as $\tau(X_{i_1} \dots X_{i_k})$ for any string i_1, \dots, i_k .
- For $X_1, \dots, X_m \in M$, we denote by $W^*(X_1, \dots, X_m)$ the von Neumann subalgebra that they generate.

Voiculescu’s key insight in [62] is that the large- n behavior of independent random matrices is described by *free independence*, an analog of independence for non-commuting variables that resembles the behavior of free products of groups. Given a von Neumann algebra M with expectation τ , subalgebras M_1, \dots, M_N are freely independent if whenever $i_1 \neq i_2 \neq \dots \neq i_k$ and $X_1 \in M_{i_1}, \dots, X_k \in M_{i_k}$ with $\tau(X_j) = 0$, then $\tau(X_1 \dots X_k) = 0$.

Random matrices $(X_1^{(n)}, \dots, X_m^{(n)})$ are said to *converge in non-commutative law to x almost surely* if

$$\lim_{n \rightarrow \infty} \text{tr}_n [p(X_1^{(n)}, \dots, X_m^{(n)})] = \tau[p(X_1, \dots, X_m)]$$

for every non-commutative polynomial p . Voiculescu’s work implies that the following: Suppose $X_1^{(n)}, \dots, X_m^{(n)}$ converge almost surely in non-commutative law to (X_1, \dots, X_m) . Let $Z_1^{(n)}, \dots, Z_m^{(n)}$ be independent GUE matrices. Let Z_1, \dots, Z_m be a *free semicircular family*, that is, elements in a von Neumann algebra (M, τ) with Wigner semicircular spectral distribution, with $W^*(X_1, \dots, X_m)$ and $W^*(Z_1), \dots, W^*(Z_m)$ freely independent. Then $(X_1^{(n)}, \dots, X_m^{(n)}, Z_1^{(n)}, \dots, Z_m^{(n)})$ converges almost surely in non-commutative law to $(X_1, \dots, X_m, Z_1, \dots, Z_m)$.

2.2. Results on conditional expectations. One of the core results from my work in [35] and [45] is that for certain random matrix models, the classical conditional expectation of $f(X_1^{(n)}, \dots, X_m^{(n)})$ given $X_1^{(n)}, \dots, X_k^{(n)}$ (where $k < m$) agrees with the conditional expectation in the free setting, or roughly speaking the conditional expectation given $X_1^{(n)}, \dots, X_k^{(n)}$ can be exchanged with the large- n limit. To set the stage, I recall that for a von Neumann algebra M with a tracial state τ , and a von Neumann subalgebras $N \subseteq M$, there is a non-commutative conditional expectation $E_N : M \rightarrow N$ which is given by the orthogonal projection with respect to the inner product $\langle X, Y \rangle_\tau = \tau(X^*Y)$. Finally, $\|X\|_{2,\tau} = \tau(X^*X)^{1/2}$ is the non-commutative L^2 -norm.

Question 2.2. Let $(X_1^{(n)}, \dots, X_m^{(n)})$ be random matrices that converge almost surely in non-commutative law to (X_1, \dots, X_m) . Then does

$$(2.1) \quad \lim_{n \rightarrow \infty} \|\mathbb{E}[p(X_1^{(n)}, \dots, X_m^{(n)}) \mid X_1^{(n)}, \dots, X_k^{(n)}]\|_{2,\text{tr}_n} = \|E_{W^*(X_1, \dots, X_k)}[p(xX_1, \dots, X_m)]\|_{2,\tau}?$$

The exchange of limits and conditional expectations is quite subtle even in classical probability theory, and even harder in the non-commutative setting. For instance, given $X = (X_1, \dots, X_m)$ and a freely independent semicircular family $Z = (Z_1, \dots, Z_m)$, it is an open question whether $t \mapsto \|E_{W^*(X+tZ)}[Z]\|_{2,\tau}$ is continuous, and this turns out to be extremely important for the problem of unifying several different definitions of free entropy, called χ and χ^* [63]. Indeed, if one could prove that $\|\mathbb{E}[Z^{(n)} \mid X^{(n)} + tZ^{(n)}]\|_{2,\text{tr}_n}$ converges to $\|E_{W^*(X+tZ)}[Z]\|_{2,\tau}$ for a reasonably general class of random matrix models, this would prove that $\chi(X) = \chi^*(X)$.

I proved that in fact all these results are true in the case of “free log-concave measures.”

Theorem 2.3. *Let $V(x)$ be a function of the form $V(x) = f(\tau(p_1(x)), \dots, \tau(p_k(x)))$ where $f : \mathbb{C}^k \rightarrow \mathbb{R}$ is continuous, and define $V^{(n)}$ on the matrix space $(\mathbb{M}_n)_{\text{sa}}^m$ by $V^{(n)}(x) = f(\text{tr}_n(p_1(x)), \dots, \text{tr}_n(p_k(x)))$. Consider a random matrix tuples $X^{(n)}$ with probability density on $(\mathbb{M}_n)_{\text{sa}}^m$ given by*

$$(2.2) \quad d\mu^{(n)}(X) = \frac{1}{C^{(n)}} e^{-n^2 V^{(n)}(X)} dX,$$

where $C^{(n)}$ is a normalizing constant. Assume that $V^{(n)}(x) - (c/2)\|x\|_{2, \text{tr}_n}^2$ is convex for some $c > 0$.

- $X^{(n)}$ converges in non-commutative law almost surely to some X
- The convergence (2.1) of conditional expectations holds.
- The two free entropies $\chi(X)$ and $\chi^*(X)$ agree.
- The denoising problem Question 2.1 for $X^{(n)}$ can be solved fairly explicitly in the large- n limit.

The problem of when $\chi(X) = \chi^*(X)$ has been open for over 20 years. The most significant progress was Biane–Capitaine–Guionnet’s work on large deviations theory for random matrices which implied as a corollary that $\chi \leq \chi^*$ in general. My work in [34], I gave another proof of the unpublished result of Dabrowski [13] that showed that $\chi(\mu) = \chi^*(\mu)$ for free Gibbs laws associated to convex potentials. These are the only known general results showing $\chi = \chi^*$ outside the single-variable setting.

2.3. Other matrix models. Besides the matrix models mentioned so far, I have studied many other matrix models with special structure. For instance, in [43], we use free probability to describe the limiting behavior of *general* Gaussian matrices under mild hypotheses. In [8], we use random permutation matrices to construct matrix models for graph products of operator algebras which allow mixtures of free and classical independence (a generalization of graph products of groups, including right-angled Artin and Coxeter groups). Related work on graph product von Neumann algebras is in [9, 10, 11], and I also worked on other types of independence that can arise in the non-commutative setting in [45, 14, 46].

3. FREE GROUP VON NEUMANN ALGEBRAS

3.1. Motivation from groups and operator algebras. Much of my work has focused on the structural properties of the free group von Neumann algebras $L(F_m)$, which is generated by the left shift operators associated to the regular representation $F_m \rightarrow B(\ell^2 F_m)$. Despite being one of the first examples, these algebras are very challenging to understand (for instance, we still do not know if they are isomorphic for different values of m). In general the passage from the group to the group von Neumann algebra erases a lot of the finer algebraic information about the group, although various geometric properties for the group, such as amenability, property (T), free products, tensor products translate to the von Neumann algebraic setting.

The free group von Neumann algebras have driven a lot of research in von Neumann algebras over the decades; one of the landmark results was Popa’s proof that the subalgebra generated by the first generator g_1 is *maximal amenable*, that is, it cannot be embedded into any larger amenable subalgebra. We still do not know whether every maximal amenable subalgebra A in $L(F_m)$ arises from a free product decomposition $L(F_m) = A * B$ (in this case it is said to be *freely complemented*).

3.2. Triangular transport. My Ph.D. research [36, 35] showed a large class of maximal abelian subalgebras in $L(F_m)$ are freely complemented, probably the first nonlinear example that did not arise from a decomposition of the group. The technique was instead based on smooth change of coordinates for the free setting. For this it was convenient to work in terms of a free semicircular family Z_1, \dots, Z_m rather than free Haar unitaries (they generate the same von Neumann algebra). I showed the following:

Theorem 3.1. *Let f_1, \dots, f_m be self-adjoint non-commutative polynomials in m variables, let $Z = (Z_1, \dots, Z_m)$ and let $X = (X_1, \dots, X_m) = (Z_1 + \varepsilon f_1(Z), \dots, Z_m + \varepsilon f_m(Z))$. Then for ε sufficiently small, there is an isomorphism $\phi : W^*(Z) \rightarrow W^*(X)$ that maps $W^*(Z_1, \dots, Z_k)$ onto $W^*(X_1, \dots, X_k)$ for $k = 1, \dots, m$. In particular, $W^*(X_1, \dots, X_j)$ is freely complemented in $W^*(Z_1, \dots, Z_m)$ for each $k = 1, \dots, m-1$.*

The isomorphism is given by finding invertible functions g_j such that $g_1(X_1), g_2(X_1, X_2), \dots, g_m(X_1, \dots, X_m)$ have the same joint distribution as the semicircular family (Z_1, \dots, Z_m) , or generate an isomorphic von Neumann algebra. Thus, the construction is analogous to change of coordinates or transport for classical

probability measures, which refers to finding a function f that will push forward a given measure μ to another given measure ν . In fact, the non-commutative functions g in my result arise from the large- n limit of classical transport functions for the random matrix models. Earlier work of Guionnet and Shlyakhtenko [24] introduced change of variables for the “log-density” V in the free setting which mirrored the classical change of variables, while my work showed that this change of variables could be done in a triangular way, while also showing a stronger form of asymptotic approximation of the matrix transport functions by the free version. Joint work with Li and Shlyakhtenko also studied smooth change of coordinates more systematically in the non-commutative setting [44].

3.3. Absorption for subalgebras via matrix approximations. In [32], we used free entropy theory to study a certain absorption phenomenon for free products. A von Neumann algebra is *amenable* if and only if A is an inductive limit of finite-dimensional algebras [12]. Popa’s result on maximal amenability was generalized to the amenable absorption property that if $M = A * B$ and P is amenable and $P \cap A$ is diffuse (nontrivial in the sense of having no minimal projections), then $P \subseteq A$. We gave a free entropy proof of this fact (in the case where A and B admit *some* matrix approximations), which applies not only to amenable P but to the much more general class of P with vanishing 1-bounded entropy in the sense of Hayes [26].

The joint work [42] showed an even stronger result that applied to subalgebras in the ultrapower $M^{\mathcal{U}}$. An ultrapower $M^{\mathcal{U}}$ is the von Neumann algebra whose elements are bounded sequences (x_n) in M , up to asymptotic equivalence along an ultrafilter \mathcal{U} , i.e. $[x_n] = [y_n]$ in $M^{\mathcal{U}}$ if and only if $\lim_{n \rightarrow \mathcal{U}} \|x_n - y_n\|_{L^2} = 0$. Intuitively, the ultrapower $M^{\mathcal{U}}$ is a larger algebra containing M , such that behaviors that can happen *approximately* in M can happen *exactly* in $M^{\mathcal{U}}$.

Theorem 3.2. *If $M = A * B$ and A and B admit matrix approximations as above, and if P and Q are subalgebras in the ultraproduct $M^{\mathcal{U}}$ with $A \cap P$ diffuse and $B \cap Q$ diffuse, then $A \wedge P$ and $B \wedge Q$ are freely independent.*

This recovered both the result of [32] and (in certain cases) the result of Houdayer and Ioana that the relative commutants $A' \cap M^{\mathcal{U}}$ and $B' \cap M^{\mathcal{U}}$ are freely independent [33]. I have also studied the applications of 1-bounded entropy to the structural properties of von Neumann algebras in many other works [28, 29, 30, 31, 37].

3.4. Future Plans. In the long-term, I want to determine whether every maximal amenable subalgebra of $L(F_m)$ is freely complemented, without any assumptions on smoothness or special structure. One recent breakthrough was Hayes’ approach in [27] to showing the Peterson–Thom conjecture from [55] that every diffuse amenable subalgebra has a *unique* maximal amenable extension, which implied that the maximal amenable algebras have various analytic properties that resemble the freely complemented case. Nonetheless, I am not convinced that every maximal amenable subalgebra is freely complemented. One tool that could help in this general setting is to analyze geometric and topological properties of the matrix microstate spaces (rather than merely analyzing their volume or covering numbers), analogously to Austin’s study of model spaces in the ergodic theory setting [2]. The properties of the microstate spaces would provide some *necessary* conditions for free complementation.

4. FREE INFORMATION GEOMETRY

4.1. Motivation from classical information geometry. *Information geometry* refers to the study of optimal transportation (Wasserstein) distances and information measures (such as entropy) on a space of probability measures. This geometric viewpoint, which treats the space of probability measures as an infinite-dimensional Riemannian manifold is a powerful heuristic that has shaped current research in partial differential equations, optimization, and data analysis; see e.g. [50, 49, 54]. Many results in information geometry have already inspired corresponding results in free probability, but we do not yet have a systematic geometry; for instance, we do not yet have an analog of McCann’s result that entropy is concave along geodesics [51].

4.2. Free optimal transport theory. One of the key ingredients in information geometry is the *Wasserstein distance*. The L^2 -Wasserstein distance $d_W(\mu, \nu)$ of two distributions μ and ν on \mathbb{R}^m is the minimum value of $\|X - Y\|_{L^2}$ where X and Y are m -tuples of random variables in some probability space with joint distributions μ and ν respectively. The definition in the non-commutative setting is analogous [6], and a pair

(X, Y) is called an *optimal coupling* of two non-commutative probability distributions (μ, ν) if $\|X - Y\|_{L^2}$ achieves the minimal distance.

Before my work, a key result of optimal transport called Monge-Kantorovich duality had not yet been studied in the free setting, which meant that it was hard to verify that a transport function was optimal in even after having the function in hand as in [24]. In its basic form for the L^2 Wasserstein distance, MK duality asserts that for every pair of distributions (μ, ν) , there exist convex functions (φ, ψ) on \mathbb{R}^m such that $\varphi(X) + \psi(Y) \geq \langle X, Y \rangle$ with equality in the case that X and Y are an optimal coupling of (μ, ν) . In [19], we gave an analog in the free setting, which required defining a new, less smooth class of convex functions.

In the same work [19], we proved various “negative” results that illustrated the difficulty of optimal transport in the non-commutative setting, using connections with other topics in operator algebras as well as factorizable quantum channels. The connection with quantum channels also led to recent and ongoing collaborations on the relationship between free Wasserstein distances and quantum Wasserstein distances as in [15], including development of Wasserstein distances for quantum permutation groups [1].

4.3. Free entropy and model theory. One of the other main quantities of concern for information geometry is the *entropy* of a probability distribution, which very roughly speaking measures how much information or randomness it contains. As mentioned in §2.2, there are several candidates for entropy in the free setting, and part of the developing free information geometry is finding the “right” definition. Ideally, one would like to have random matrix models such that their classical entropy and Wasserstein distances converge to their free counterparts. Unfortunately, I showed in [40] that this is actually impossible in general. The core issue, in my view, is the fact that optimal transport inherently deals with variational problems with suprema and infima. However, in the non-commutative setting, the functions typically used as an analog of continuous scalar-valued functions are not closed under partial suprema and infima, reflecting the model-theoretic property that von Neumann algebras do not have quantifier elimination [17, 16, 18]; in contrast, $L^\infty[0, 1]$ does have quantifier elimination, making the classical probability setting much tamer!

This motivated my development of model-theoretic versions of free entropy in [38]. The main idea is to enlarge the space of test functions; while random matrix models usually use test functions of the form $F(\tau(p_1(X)), \dots, \tau(p_k(X)))$ where p_1, \dots, p_k are non-commutative polynomials and $F : \mathbb{C}^k \rightarrow \mathbb{R}$ is continuous, we now consider test functions of the form

$$(4.1) \quad \varphi^M(X_1, \dots, X_n) = \sup_{Z_1} \inf_{Z_2} \dots \sup_{Z_{2m-1}} \inf_{Z_{2m}} F(\tau(p_1(X, Z)), \dots, \tau(p_k(X, Z))),$$

where $X = (X_1, \dots, X_n)$ and $Z = (Z_1, \dots, Z_{2m})$ and the sup’s and inf’s in Z_1, \dots, Z_{2m} are taken over the unit ball in M . The formula thus encodes information about how X interacts with the ambient von Neumann algebra in terms of variational problems. Then X and X' are said to have the same *type* if $\varphi^M(X) = \varphi^M(X')$ for all X, X' .

Despite its complicated setup, the type has a natural motivation in terms of symmetry, which is parallel to what happens for classical probability measures: Two tuples X and X' in the ultrapower M^U have the same type if and only if they are conjugate by an automorphism. Furthermore, there is a natural Wasserstein distance for types which corresponds to the minimal distance between automorphism orbits. In [39], I proved a well-behaved Monge–Kantorovich on the type space. Then in [40] I showed that this version of entropy *does* have reasonable behavior along Wasserstein geodesics.

4.4. Future plans. In order to study entropy and the associated PDE theory for the non-commutative setting, it is necessary not only to consider deterministic optimization problems, but also stochastic optimization problems. Indeed, considering the measure $\mu^{(n)}$ and $X^{(n)}$ as in Theorem 2.3 given by a potential V as in (2.1). Let $Z^{(n)}$ be an independent family of GUE matrices. It is well known that the probability density of $X^{(n)} + t^{1/2}Z^{(n)}$ satisfies the heat equation. In order to study this in the large- n limit, we write the density as $(1/C^{(n)})e^{-n^2V_t^{(n)}}$ and observe that

$$(4.2) \quad \partial_t V_t^{(n)} = \frac{1}{2} \Delta V_t^{(n)} - \frac{1}{2} \|\nabla V_t^{(n)}\|_{2, \text{tr}_n}^2,$$

which is the prototypical Hamilton–Jacobi–Bellman equation on the space of matrices.

Question 4.1. *Given a non-commutative test function V as in Theorem 2.3 but without assuming V is convex, describe the large- n limiting behavior of $V_t^{(n)}$ in terms of dimension-independent free probabilistic limit.*

My earlier work showed that the case of strongly convex V is well-behaved, but the problem with the non-convex setting is that we do not know if the free version of the equation will have adequate regularity to apply the same techniques. The solution of the equation in general is given by a stochastic optimization problem, which raises again the issue that suprema and infima have a more complicated behavior in the non-commutative setting. In joint work with Gangbo, Nam, and Palmer [20], I work on developing stochastic optimal control theory in the free setting systematically. The next stage (in progress with different coauthors) will be to combine the stochastic setup with the model-theoretic setup. From there, we will obtain McCann's concavity for the free entropy setting and work on settling the unification problem for free entropy in general. In the long-term I will obtain a clear answer to the remaining questions around large deviations for random matrices from [5].

5. APPLICATIONS OF FREE PROBABILITY

5.1. Contractibility of the unitary group. Another nonobvious application of free probability techniques was recent work that showed that the unitary group of an arbitrary II_1 factor is contractible in the strong operator topology [41]. Contractibility was proved in the 1960's for the unitary group of $B(H)$ and the 1970's for von Neumann algebras of type II_∞ and III . For the type II_1 case, Popa and Takesaki [58] showed in 1993 that the unitary group is contractible when M is stable under tensorization by the amenable factor \mathcal{R} (hence this was not known for free group von Neumann algebras for instance). I gave a short proof this year for the general case using a completely different approach: I deform the unitaries by multiplying by approximately free elements (obtained using Popa's theorem [57]) so that their spectral distribution converges to the Haar measure, and then to apply another transformation by functional calculus to bring them close to 1.

In ongoing and future work, I will give applications of these approximate freeness techniques to bundles of von Neumann algebras over compact spaces.

5.2. Algorithms in spin glass theory. In a collaboration between mathematics and computer science, Juspreet Sandhu, Jonathan Shi, and I applied free probability and random matrix theory to study spin glass models [47]. These are statistical mechanical models involving the interaction of a large number of sites with assigned spins and a Hamiltonian describing their interaction. Our work focused on the Sherrington–Kirkpatrick model where any two sites can interact (the geometry is described by a complete graph). We gave a new algorithm for finding the ground state of the Sherrington–Kirkpatrick model. In more pedestrian terms, the problem requires us to maximize $\langle x, Ax \rangle$ for $x \in [-1, 1]^n$, where A is a Gaussian random matrix with real entries (essentially GOE). The value of the optimum was predicted in celebrated work of Parisi, which expressed the optimum value via a variational problem involving a PDE and a probability measure on $[0, 1]$. It took decades to rigorously establish this result mathematically, and Talagrand's achievements in this area [61] won him the Abel prize, but there are still many open questions about the regularity of the optimizer.

The goal in our work was to find a vector in $[-1, 1]^n$ that nearly achieves the optimum value, with runtime that is polynomial in n (though not necessarily polynomial in the error tolerance). Here note that the brute force approach of testing every corner takes exponential time in n , so we heavily rely on the structure of the problem and the Parisi formula. We follow an approach used by Subag [60] for the much simpler geometry of the sphere. We modify the objective function $\langle x, Ax \rangle$ in the interior of the domain by adding a potential obtained from the Parisi solution. We used free probability theory to analyze the behavior of the GOE matrix A together with the Hessian of the potential. This allowed us to find a near-maximizer by an iteration starting at the origin, where the direction for the next step is determined from the Hessian.

Several sequel papers are currently in progress, which will extend the choices of Hamiltonians in our model as well as connect it with other algorithms.

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