Research Statement

David Jekel

1 Overview

My research is in *free probability* which focuses on the interplay between *von Neumann algebras* and *random matrices*; in particular, I study these objects through the lenses of *entropy*, *optimal transport*, *stochastic control*, and *continuous model theory*. A von Neumann algebra with a tracial state can be understood as a non-commutative version of the algebra $L^\infty(\Omega,\mu)$ associated probability space (Ω,μ), but the classification and structure of von Neumann algebras is much more complicated than that of classical probability spaces. Certain von Neumann algebras are the appropriate objects to describe the behavior of certain random $n \times n$ matrices in the limit as $n \to \infty$. The benefits of this connection go two ways: The infinite-dimensional objects (von Neumann algebras) shed light in the limiting behavior of random $n \times n$ matrices for large finite n, while the matrix approximations also yield some structural results about the von Neumann algebras that would otherwise be intractable.

Topics: I have made contributions in the following areas:

(1) *Non-commutative independence and convolution:* The non-commutative setting allows several versions of independence (such as free, boolean, and monotone), which can be studied using combinatorics of moments as well as associated analytic functions. I worked on relating different types of independence in a general framework and proving associated limit theorems for additive convolutions (see §6).

(2) *Free entropy theory:* There are several analogs of entropy for non-commutative random variables. I worked on relating these notions together and strengthening their connections with the classical entropy of associated random matrix models (see §5).

(3) *Non-commutative optimal transport:* Optimal transport studies the most efficient way to transform one probability distribution into another. I have worked on understanding the large-n behavior of optimal transport for $n \times n$ matrix models, and the associated optimal transport theory for the free limit. This resulted in connections with optimal transport in *quantum information theory* which I am currently exploring (see §7).

(4) *Model theory of von Neumann algebras:* Model theory is a branch of mathematical logic which studies how objects behave with respect to first-order sentences, and which has recently been applied to *metric* structures such as tracial von Neumann algebras [26]. I studied several model-theoretic concepts, such as definability, model completeness, and elementary equivalence, for von Neumann algebras. I am working on integrating the model-theoretic framework with free probability and random matrices (see §8).

Applications: My research programme and techniques have applications in the following areas:

(1) *Structure of von Neumann algebras:* My work on non-commutative change of coordinates or transport of measure exhibited hidden free product decompositions of tracial von Neumann algebras (see §3). Other work on 1-bounded entropy used matrix approximations to prove structural properties of free products and other similar von Neumann algebras (see §4).

(2) *Large deviations for random matrices:* Several past results and future projects on free entropy are aimed at advancing the large deviations theory for Gaussian random matrices studied by [6]. One of the ingredients is the study of stochastic control problems, which also connects random matrices with mean field games (see §5).

(3) *Spin glass theory:* Juspreet Sandhu, Jonathan Shi, and I used the analytic function theory of free independence to design a spectral algorithm for finding the ground state of the Sherrington-Kirkpatrick model [57].

Collaboration and mentorship: I place a high priority on interdisciplinary collaboration and supporting early-career researchers. I regularly organize online reading groups for people to become familiar with a new topic, which can also become a springboard for joint projects. Moreover, many of my papers are joint work with graduate students and sometimes even undergraduates [39, 24, 56, 27, 28]. In §9, I give a sample of potential undergraduate projects related to non-commutative probability.

2 Motivation and Background

While the theory of von Neumann algebras has undergone several revolutions over the decades, and it continues to generate new discoveries and questions, in part because of its deep connections with geometric group theory, ergodic theory, and quantum information theory. For the present discussion, the most important motivation come from *groups* and from *probability spaces*.

Von Neumann algebras from groups: A countable group G has a left regular representation π on the Hilbert space $\ell^2(G)$, which maps a group element to the unitary operator u_g of left translation given by $u_g \delta_h = \delta_{gh}$ where $(\delta_h)_{h \in G}$ is the standard orthonormal basis for $\ell^2(G)$. The *group von Neumann algebra* $L(G)$ is the von Neumann algebra generated by the operators u_g. Thus, $L(G)$ is a completion of the group algebra $\mathbb{C}G$ in a certain topology. Although taking the completion erases a lot of the finer algebraic information about the group, various geometric properties for the group, such as amenability, property (T), free products, tensor products translate to the von Neumann algebraic setting. Remarkably, there are many recent examples of groups that can be uniquely recovered from the group von Neumann algebras. In the case of free groups, much less is known; for instance, the following question is still open.

Question 2.1. *Let F_d be the free group on d generators. Is $L(F_n)$ isomorphic to $L(F_m)$ for $n \neq m$?*

Random matrices and free products: A remarkable set of tools for studying the free group von Neumann algebras was provided by their connection with random $N \times N$ matrices in the large-N limit and free products of von Neumann algebras. Wigner in the 1950's had described the large-N behavior of the spectrum of a Gaussian random $N \times N$ self-adjoint matrix [93], but Voiculescu's work described systematically the large-N behavior of several *independent* random $N \times N$ matrices. An illustrative example is the following theorem.

Theorem 2.2 (Voiculescu). *Let $U_1^{(N)}$, ..., $U_d^{(N)}$ be $N \times N$ matrices chosen uniformly at random from the unitary group (according to Haar measure). Write $U_{-j}^{(N)} = (U_j^{(N)})^* = (U_j^{(N)})^{-1}$ for $j = 1, \dots, d$. Then for any $i_1, \dots, i_k \in \pm\{1, \dots, d\}$, we have almost surely*

$$\lim_{N \to \infty} \operatorname{tr}_N(U_{i_1}^{(N)} \dots U_{i_k}^{(N)}) = \begin{cases} 1, & \text{if } g_{i_1} \dots g_{i_k} = e \text{ in } F_d \\ 0, & \text{otherwise} \end{cases}$$

where F_d is the free group on generators $g_1, \dots, g_d$, and $g_{-j} = g_j^{-1}$. In other words, $N \times N$ independent random unitary matrices behave, in the limit as $N \to \infty$, exactly like the generators of the free group viewed as elements of the group von Neumann algebra $L(F_d)$.

There is a *free product* operation for tracial von Neumann algebras that generalizes the free product operation for groups. Motivated by the connection with probability theory, we say that subalgebras A and B in a tracial von Neumann algebra M are *freely independent* if they are in free product position, i.e. the subalgebra they generate is $A * B$. We also say that A is *freely complemented* in M if $M = A * B$ for some subalgebra B.

Von Neumann algebras as non-commutative probability spaces: The connection with random matrices gave new significance to the long-standing paradigm that von Neumann algebras are a non-commutative analog of probability spaces. A classical probability space is a measure space $(\Omega, \mathcal{F}, P)$ where $P(\Omega) = 1$. The elements of Ω represent "random outcomes" of an experiment, and a *random variable* is a function $X : \Omega \to \mathbb{R}$. The *expectation* of a random variable is $\int X(\omega)\, dP(\omega)$ (when defined), or intuitively the average value of X over all possible outcomes. The space of bounded random variables is $L^\infty(\Omega, \mathcal{F}, P)$ is a von Neumann algebra and the expectation is a linear map $E : L^\infty(\Omega, \mathcal{F}, P)$ is an example of a tracial state. In fact, all commutative von Neumann algebras have this form. Therefore, a von Neumann algebra

M can be regarded as an "algebra of non-commutative random variables"; the "expectation" is replaced by a linear map $\tau : M \to \mathbb{C}$ satisfying analogous properties. Specifically, we want $\tau(1) = 1$ and $\tau(x^*x) \geq 0$ with equality if and only if $x = 0$, and often we want the traciality property $\tau(xy) = \tau(yx)$. A central example is the matrix algebra $M = M_n(\mathbb{C})$ with the normalized trace $\tau = \operatorname{tr}_n = (1/n) \operatorname{Tr}_n$.

Maximal amenable subalgebras: Just like one can probe the structure of a group by studying subgroups, mathematicians have explored the structure of $L(F_n)$ (and free products more generally) by trying to classify the amenable subalgebras. *Amenability* is a property of a tracial von Neumann algebra that generalizes amenability for groups; it is equivalent to being an inductive limit of finite-dimensional tracial $*$-algebras by a seminal theorem of Connes [19]. If (M, τ) is a tracial von Neumann algebra, then a *maximal amenable subalgebra* is an amenable $\mathcal{A} \subseteq \mathcal{M}$ that is not contained in any larger amenable subalgebra of $\mathcal{M}$. Popa showed the following.

Theorem 2.3 (Popa [73])**.** *If $M = A * B$ with A amenable and B nontrivial, then A is maximal amenable in M. In particular, any freely complemented amenable subalgebra of $L(F_d)$ is maximal amenable.*

The question of a converse to the last sentence has gained some attention recently.

Question 2.4. *Is every maximal amenable subalgebra of $L(F_d)$ freely complemented?*

A positive answer to this question would place strong restrictions on the structure of amenable subalgebras in $L(F_n)$, which seem too good to be true. However, one positive indication was my result from [46] showing free complementation in a situation where it was highly nonobvious; see §3.

A related question is whether every amenable subalgebra inside $L(F_d)$ is contained in a *unique* maximal amenable subalgebra; this was conjectured by Peterson and Thom, motivated by their study of subgroups inside groups with similar analytic properties to free groups [72]. A positive answer to this question was given recently through the work of Hayes on free entropy theory [38] combined with a deep result in random matrix theory which has several proofs due Belinschi-Capitaine [3] and Bordenave-Collins [11].

Theorem 2.5 (Peterson-Thom conjecture)**.** *Every amenable subalgebra of $L(F_n)$ is contained in a unique maximal amenable subalgebra.*

I have worked on several joint projects that give applications and extensions of this result (see §4).

3 Free complementation through change of coordinates

During my Ph.D. studies, I showed a positive answer to Question 2.4 in a large family of subalgebras of $L(F_d)$. This result is expressed in terms of self-adjoint rather than unitary generators for $L(F_d)$. Let $S_1, \dots, S_d$ be freely independent self-adjoint non-commutative random variables where the probability distribution of each S_j is Wigner's semicircular measure. These operators $(S_1, \dots, S_d)$ describe the large-n behavior of independent random matrices $S_1^{(n)}, \dots, S_d^{(n)}$ chosen from the Gaussian unitary ensemble (that is, Gaussian random variables in the space of self-adjoint matrices with appropriate normalizations), in an analogous way to Theorem 2.2. Moreover, the von Neumann algebra $\mathrm{W}^*(S_1, \dots, S_d)$ that they generate is isomorphic to $L(F_d)$. In general, I will denote by $\mathrm{W}^*(X_1, \dots, X_d)$ the von Neumann algebra generated by certain operators $X_1, \dots, X_d$.

Theorem 3.1 (J. 2020)**.** *Let $(S_1, \dots, S_d)$ be a free semicircular family. Let g be a non-commutative self-adjoint polynomial. There exists $\epsilon_g > 0$ such that if $|t| < \epsilon_g$, then $\mathrm{W}^*(S_1 + tg(S_1, \dots, S_d))$ is freely complemented in $\mathrm{W}^*(S_1, \dots, S_d)$.*

The proof is based on probabilistic techniques, specifically applying transport of measure to the conditional distributions of the random variables.

Non-commutative probability distributions: A natural way to choose "coordinates" for a tracial von Neumann algebra (M, τ) is to fix a d-tuple $\mathbf{x} = (x_1, \dots, x_d)$ that generates M; the analog in classical probability would be a d-tuple of random variables $\mathbf{X} = (X_1, \dots, X_d)$ that generates the underlying σ-algebra. Two generating tuples are said to have the *same distribution* if $\tau(p(\mathbf{x})) = \tau(p(\mathbf{y}))$ for all non-commutative $*$-polynomials p in d variables; here non-commutative polynomials are objects like

$$p(x_1, x_2, x_3) = 3x_1 - x_2^2 x_3 + x_3 x_2^* x_1 + 5x_3^* x_1 x_2.$$

The analogous concept in classical probability is that bounded random variables $\mathbf{X} = (X_1, \dots, X_d)$ and $\mathbf{Y} = (Y_1, \dots, Y_d)$ agree in distribution if $E[p(\mathbf{X})] = E[p(\mathbf{Y})]$ for all polynomials in z_1, ..., z_d and $\overline{z}_1$, ..., $\overline{z}_d$. Agreement in distribution means that the algebras generated by $\mathbf{x}$ and $\mathbf{y}$ are isomorphic by mapping x_j to y_j for each j.

Change of coordinates and transport of measure: Suppose we have two tuples $\mathbf{X}$ and $\mathbf{Y}$ coordinatizing von Neumann algebras, and we want to show $\mathrm{W}^*(\mathbf{X}) \cong \mathrm{W}^*(\mathbf{Y})$. An isomorphism $\mathrm{W}^*(\mathbf{X}) \to \mathbf{W}^*(\mathbf{Y})$ would not necessarily map $\mathbf{X}$ to $\mathbf{Y}$. Rather, we seek to construct an isomorphism through "change of coordinates", that is, by sending $f(\mathbf{X})$ to $\mathbf{Y}$ for some invertible function f. This is a direct analog of the problem in probability theory of *transport of measure*, that is, finding a function that will transform one given probability distribution to another. However, in stark contrast to classical atomless probability spaces, there are continuum many tracial von Neumann algebras that are not isomorphic, and do not even embed into each other [70], hence any result establishing non-commutative transport of measure already gives a nontrivial statement about isomorphism.

Free Gibbs laws: The question of non-commutative transport of measure is more tractable when $\mathbf{X}$ and $\mathbf{Y}$ have *free Gibbs distributions*, meaning there is potential function $V(x_1, \dots, x_d)$ that serves as something like the log of the probability density for X_1, ..., X_d, and a similar function for $\mathbf{y}$. Such non-commutative random variables arise in the large N limit from random matrix tuples $(X_1^{(N)}, \dots, X_d^{(N)})$ with probability density given by a constant times $e^{-N^2 V}$ on $M_N(\mathbb{C})^d$. See [8, 33, 34, **?**]. When V is *equal* to the quadratic function, then X_1, ..., X_d specializes to the free semicircular family mentioned above.

When $\mathbf{S} = (S_1, \dots, S_d)$ is a free semicircular family, and $\mathbf{X} = (X_1, \dots, X_d)$ is given by a convex potential V, Guionnet and Shlyakhtenko [35] showed that $\mathrm{W}^*(\mathbf{S}) \cong \mathrm{W}^*(\mathbf{X})$ by constructing some function f such that $f(\mathbf{S})$ has the same distribution as $\mathbf{X}$. I upgraded this result to construct a transport function f that was *triangular* in the sense that the jth component $f_j(S_1, \dots, S_d)$ only depended on X_1, ..., X_j. (For triangular transport in classical probability theory, see [10].)

Theorem 3.2. *Let $\mathbf{S} = (S_1, \dots, S_d)$ be a free semicircular family (again the analog of Gaussian), and let $\mathbf{X} = (X_1, \dots, X_d)$ be distributed according to a free Gibbs law with log-density V which is sufficiently smooth and sufficiently close to a quadratic function. Then there is an isomorphism $\phi : \mathrm{W}^*(\mathbf{S}) \to \mathrm{W}^*(\mathbf{X})$ that maps $\mathrm{W}^*(S_1, \dots, S_j)$ onto $\mathrm{W}^*(X_1, \dots, X_j)$ for $j = 1$, ..., d. In particular, $\mathrm{W}^*(X_1, \dots, X_j)$ is freely complemented in $\mathrm{W}^*(X_1, \dots, X_d)$.*

This theorem implies Theorem 3.1 since we can compute the log-density of a perturbation $S_1 + tg_1(\mathbf{S})$, ..., $S_d + tg_d(\mathbf{S})$ using a change of variables formula.

Techniques in the proof: The arguments imitate classical constructions for transport of measure, using a partial differential equation to relate the potential V with the transport function f. This requires having suitable spaces of non-commutative functions; naturally, one would want to take some completion of the space of non-commutative polynomials, similar to the way that classical smooth functions $\mathbb{R}^d \to \mathbb{R}^d$ can be approximated by classical (commutative) polynomials. The sticking point is to make sure that the functions spaces are closed under the operations needed to solve the differential equations. In [45, 46], I used non-commutative Lipschitz functions, and my analysis gave improved control over how the corresponding functions $f^{(N)}$ for the $N \times N$ matrix models converged in the large-N limit.

The free Wasserstein manifold: In [53], Li and Shlyakhtenko and I gave versions of these results for non-commutative smooth functions. Moreover, we described a systematic approach to non-commutative transport of measure results [53, §8], by formalizing the relationship between perturbations of a non-commutative smooth log-density V and transport of measure by a perturbation of the identity function. We developed the non-commutative analog of Wasserstein geometry, a formalism that treats the set of probability distributions on $\mathbb{R}^d$ as an infinite-dimensional manifold with a Riemannian metric that is closely related to transport of measure (see for instance [64, 69, 68]).

Ongoing work with Félix Parraud and Evangelos Nikitopoulos aims to refine our understanding of the transport maps $F^{(n)}$ constructed for the $n \times n$ matrix models by giving an asymptotic expansion in powers of $1/n^2$. As preparation for this project, I gave in [51] a new combinatorial treatment of Parraud's formula [71] for the asymptotic expansion of $\mathbb{E} \operatorname{tr}_n(p(\mathbf{S}^{(n)}))$ for GUE random matrices.

Conjecture 3.3 (in progress with Parraud and Nikitopoulos)**.** *Let V be a non-commutative smooth log-density that is sufficiently close to the quadratic potential, let $(X_1^{(n)}, \dots, X_d^{(n)})$ be the associated random* *matrix models, and let $(S_1^{(n)}, \dots, S_d^{(n)})$ be a GUE random matrix tuple. There exist functions $F^{(n)}$ on d-tuples of self-adjoint matrices such that $F^{(n)}(\mathbf{S}^{(n)}) \sim \mathbf{X}^{(n)}$ in probability distribution and there exist non-commutative smooth functions F_0, F_1, …, such that $F^{(n)} = \sum_{j=0}^k n^{-2j} F_j + O(n^{-2k+2})$ for any $k \in \mathbb{N}$.*

However, all of these results assuming strong convexity hypotheses for V, and the general case remains a challenging open problem which continues to motivate my research programme.

Question 3.4. *Consider random matrix tuples $(X_1^{(N)}, \dots, X_d^{(N)})$ with probability density given by a constant times $e^{-N^2 V}$ on $M_N(\mathbb{C})^d$ for some non-commutative smooth function V. Under what conditions will $(X_1^{(N)}, \dots, X_d^{(N)})$ converge in non-commutative law as $N \to \infty$ to some family $(X_1, \dots, X_d)$? Under what conditions will $\mathrm{W}^*(X_1)$ be freely complemented in $\mathrm{W}^*(X_1, \dots, X_d)$?*

4 Free entropy and structural properties

Free entropy: Motivated in part by Question 2.1, Voiculescu defined several analogs of entropy in free probability theory. The free entropy $\chi(\mathbf{x})$ and the associated *free entropy dimension* $\delta(\mathbf{x})$ are based on measuring the amount of matrix approximations for the distribution of $\mathbf{x}$ (more precisely, it is an exponential growth rate of the volume of spaces of $N \times N$ matrices with approximately the same distribution as $\mathbf{x}$) [87, 88]. Though Question 2.1 remains open, free entropy theory was used to prove impressive indecomposability results about $L(F_n)$, for instance (a) it has no approximate center [88], (b) it cannot arise as the von Neumann algebra of classical dynamical system [88], (c) it cannot be expressed as a nontrivial tensor product [29].

Hayes [37] gave a massive generalization of these indecomposability results by showing that $L(F_d)$ cannot be generated from an amenable subalgebra A by iteratively adding in unitaries that normalize A through transfinite induction. He used a version of free entropy coming out of the work of Jung [59], which we will denote h. Unlike free entropy dimension, the Jung-Hayes entropy $h(\mathbf{x})$ is known to be an invariant the von Neumann algebra $\mathrm{W}^*(\mathbf{x})$, that is, it is independent of the choice of coordinates. The Jung-Hayes entropy $h(A : M)$ is thus well-defined for any inclusion of tracial von Neumann algebras $A \subseteq M$, and it also relates well to various von Neumann algebraic operations and properties. For instance, Hayes, Kunnawalkam Elayavalli, and I showed that that if a II_1 factor M has property (T), an important rigidity property for von Neumann algebras, then $h(M : M) < \infty$ [39].

Free entropy approach to maximal amenability: In joint work with Hayes, Nelson, and Sinclair, we gave a new proof and generalization of Theorem 2.3, where the condition of amenability was replaced with vanishing of h. The proof relied on techniques of exponential concentration of measure as well as random matrix approximations of non-commutative conditional probability distributions (somewhat related to the triangular transport results in the previous section).

Theorem 4.1 (Hayes-J.-Nelson-Sinclair 2021 [42])**.** *Suppose $M = A * B$ where A and B are diffuse (that is, atomless) and Connes-embeddable (*some *matrix approximations for them exist). If $C \subseteq M$ satisfies that $h(C : M) = 0$ and $A \cap C$ is diffuse, then $C \subseteq A$. In particular, if $h(A : M) = 0$, then there is no larger subalgebra C containing A with $h(C : M) = 0$.*

Based on the analogy with dynamical entropy, this paper defined the *Pinsker algebra* of A inside M as the maximal von Neumann subalgebra with $h(A : M) = 0$. Thus, the previous theorem says that if $h(A) = 0$, then A is a Pinsker algebra in $A * B$. Recent joint work with Kunnawalkam Elayavalli studied Pinsker algebras inside the *ultrapower* $(A * B)^{\mathcal{U}}$ of $A * B$. The ultrapower $M^{\mathcal{U}}$ is a much larger von Neumann algebra containing M; roughly speaking, the elements of $M^{\mathcal{U}}$ represent all possible limiting behaviors of sequences from M.

Theorem 4.2 (J.-Kunnawalkam Elayavalli 2024)**.** *Let $M = A * B$ be a free product of tracial von Neumann algebras that are Connes-embeddable. Suppose that $h(A) = 0$ and $h(B) = 0$. Then the Pinsker algebras of A and B in $M^{\mathcal{U}}$ are freely independent of each other. In particular, if A and B are amenable, then any maximal amenable subalgebras of $M^{\mathcal{U}}$ containing A and B respectively must be freely independent.*

This result is much stronger than Theorem 4.1 since the Pinsker algebra of A inside $M^{\mathcal{U}}$ can be much larger than the Pinsker algebra of A inside M. As an example the Pinsker algebra of A contains the relative commutant $A' \cap M^{\mathcal{U}} = \{x \in M^{\mathcal{U}} : xa = ax \text{ for } a \in A\}$. Thus, our theorem recovers in many cases the results of Houdayer and Ioana on free indepedence of approximate commutants of A and B in $M^{\mathcal{U}}$ [43], which first inspired our work. In particular, we have the following consequence for random matrix theory:

Corollary 4.3 (J.-Kunnawalkam Elayavalli 2024)**.** *Let $U_1^{(n)}$ and $U_2^{(n)}$ be independent Haar unitary random matrices. Let $B_1^{(n)}$ and $B_2^{(n)}$ be any random matrices that are bounded in operator norm. If $\|U_j^{(n)} B_j^{(n)} - B_j^{(n)} U_j^{(n)}\|_2 \to 0$ almost surely, then $B_1^{(n)}$ and $B_2^{(n)}$ are almost surely asymptotically freely independent.*

Consequences of the solution to the Peterson-Thom conjecture: The solution to the Peterson-Thom conjecture by [38, 3, 11] has further structural consequences for free group von Neumann algebras. Indeed, Hayes [38] showed that [3, 11] implies that a subalgebra A of $L(F_n)$ has $h(A) = 0$ if and only if A is amenable. In general, there are many non-amenable algebras with $h(A) = 0$; for instance, if h is a tensor product. Thus, this result places tight restrictions on the subalgebras that can occur inside $L(F_n)$. In joint work [41], we detailed some of the consequences of this result, including the resolution of the coarseness conjecture of [37, Conjecture 1.12] and [75, Conjecture 5.2].

Theorem 4.4 (Hayes-J.-Kunnawalkam Elayavalli 2023)**.** *Let A be a maximal amenable subalgebra of $L(F_d)$. Then A is* coarse*, meaning that the orthogonal complement of A in M embeds into $(A \otimes A)^{\oplus \infty}$ as an A-A-bimodule.*

This relates to the question of free complementation because any freely complemented subalgebra A is automatically coarse. We also showed that two maximal amenable subfactors are either unitarily conjugate or they are completely disjoint in the sense of Popa's intertwining [74, §6].

Next, together with Brent Nelson, we adapted these results to type III_1 von Neumann algebras. In the type III_1 setting, the von Neumann algebra M is equipped with a state $\varphi : M \to \mathbb{C}$ that does not satisfy the traciality condition $\varphi(xy) = \varphi(yx)$. In particular, this makes it unclear how to define appropriate matrix approximations for generators $(X_1, \dots, X_d)$ for M. However, Tomita-Takesaki theory shows that any type III_1 von Neumann algebra M can be generated by a group action of $\mathbb{R}$ on $B(H) \overline{\otimes} N$ where N *does* admit a tracial state; $B(H) \overline{\otimes} N$ is called the *continuous core* of M. We were able to study the notion of Pinsker algebra in terms of the continuous core, and hence obtain the following result.

Theorem 4.5 (Hayes–J.–Kunnawalkam Elayvalli–Nelson 2024)**.** *Suppose that M is a type III_1 von Neumann algebra with continuous core $B(H) \otimes L(F_\infty)$. (This includes certain free Araki-Woods factors.) Then all the Pinsker algebras in M must be amenable, and in particular M satisfies the analog of the coarseness conjecture (Theorem 4.4) and the Peterson-Thom conjecture (Theorem 2.5).*

5 Free entropy, large deviations, and stochastic analysis

Two notions of free entropy: In classical probability theory, the entropy of a probability density ρ on $\mathbb{R}^d$ is given by $h(\rho) = -\int \rho \log \rho \, dx$ (up to sign conventions). It is difficult to extend this to the multivariate non-commutative setting because there is not a direct analog of probability density. For non-commutative probability distributions of a d-tuple $\mathbf{x} = (x_1, \dots, x_d)$, Voiculescu defined two different analogs of the classical differential entropy.

(1) The "microstates approach" [87, 88] is based on measuring the amount of matrix approximations for the distribution as mentioned in the last section.

(2) The "infinitesimal approach" is based on the idea that the rate of change of the entropy of a distribution evolving with time according to the heat equation should be the Fisher information [89].

Biane, Capitaine, and Guionnet showed in general that $\chi(\mathbf{X}) \leq \chi^*(\mathbf{X})$ through their study of large deviations theory for matrix Brownian motion. The question of equality for $d > 1$ is still largely open.

Question 5.1. *Let $\mathbf{X}$ be a d-tuple from a tracial von Neumann algebra. Under what conditions does $\chi(\mathbf{X}) = \chi^*(\mathbf{X})$? In particular, does this hold if $\mathbf{X}$ arises from a free Gibbs law associated to some potential V as in §3?*

Equality of χ and χ^* for certain free Gibbs laws: Dabrowski [22] gave the first proof that $\chi(\mathbf{X}) = \chi^*(\mathbf{X})$ for a large class of non-commutative d-tuples, specifically those given as free Gibbs laws for a convex potential V as in §3. In [45], I gave a more elementary proof based on precisely controlling the large-N behavior of functions associated to the $N \times N$ random matrix models. This is closely related to my work on transport and the free Wasserstein manifold [46, 53].

Simplified proof and generalization of $\chi \leq \chi^*$: The Biane-Capitaine-Guionnet result that $\chi \leq \chi^*$ came out of a much more general framework on large deviations and used a lot of stochastic analysis. However, a simpler proof of the isolated result $\chi \leq \chi^*$ was desirable because of its widespread use in applications to von Neumann algebras. Myself and Jennifer Pi, a graduate student at UC Irvine at the time, gave such a simplified proof of $\chi \leq \chi^*$, while at the same time generalizing it to conditional entropy. The argument makes precise connections between the functions for finite N and their large N limit, giving a rigorous form to Voiculescu's original heuristics. A key ingredient is an observation in [81, Appendix] that interprets $\chi(\mathbf{X})$ as the maximum possible limit of the classical entropy of $X^{(n)}$ over sequences $(\mathbf{X}^{(n)})_{n\in\mathbb{N}}$ of random matrix models approximating $\mathbf{X}$.

Stochastic optimization approach: It is clear from [6, 22, 45] that in order to study Question 5.1 for free Gibbs laws associated to non-convex log-densities V, we need a better understanding of what happens when a density of the form $e^{-n^2V^{(n)}}$ on d-tuples of self-adjoint matrices evolve according to the heat equation. Writing the density at time t as $e^{-n^2V_t^{(n)}}$, we have the equation

$$\partial_t V_t^{(n)} = \Delta^{(n)} V_t^{(n)} - \frac{1}{2}\|\nabla V_t^{(n)}\|_2^2,$$

where $\Delta^{(n)}$ is the Laplacian with appropriate dimensional normalization. Both [6] and [22] express $V_t^{(n)}$ as

$$V_t^{(n)}(x_0) = \inf\left\{\mathbb{E}[V^{(n)}(X_t) + \frac{1}{2}\int_0^t \|\alpha_s\|_2^2\, ds] : X_t = x_0 + \int_0^t \alpha_s\, dS_s^{(n)}\right\}, \tag{5.1}$$

where $S_s^{(n)}$ is a self-adjoint matrix Brownian motion and α_s a matrix-valued process adapted to the associated filtration. Hence, in order to answer Question 5.1, we need to understand what happens to (5.1) in the large-n limit. While the large-n limit of matrix Brownian motion is understood to be a semicircular Brownian motion, it is unknown how the Brownian motion interacts with the infimum in the large-n limit. Three recent and ongoing projects are at least partly motivated by this question:

(1) With my postdoc supervisor Todd Kemp and his student Evangelos Nikitopoulos, we revisited the foundations of free stochastic differential equations and studied how they relate with the non-commutative smooth functions in [53].

(2) Joint work in progress with Wilfrid Gangbo, Kyeongsik Nam, and Aaron Palmer studies stochastic control problems such as (5.1) in a more general framework showing the connection between random matrices and mean field games. In this paper, we allow the non-commutative probability space and filtration for the stochastic optimization problem to vary as we take the infimum.

(3) Joint work with David Gao on direct integrals of tracial von Neumann algebras gives a better understanding of the interplay between classical randomness and the realization of large-n limits in terms of ultraproducts [28, §6]. Our framework should allow the construction of a non-commutative probability space and filtration that would accurately describe the large-n limit of (5.1).

6 Non-commutative independence and its applications

In classical probability theory, independence of random variables $X_1, \dots, X_d$ gives a canonical way of determining the joint (mixed) moments. In the non-commutative setting of operator algebras, there are several different types of independence, including free [85], boolean [80], and monotone independence [66]. There is a host of others as well if one's definition of independence is sufficiently broad.

Unifying framework for non-commutative independence: There have been several important results unifying different types of non-commutative independence, for instance [78, 67]. My contributions to this area are as follows:

(1) In [44] and [47], I described the parallel between stochastic processes for free, boolean, and monotone independence, encompassing analytic aspects, combinatorial aspects, and the operator models.

(2) Weihua Liu and I described a general operad framework for independences defined by trees [54]. This was a systematic way to address combinations such as taking X_1 and X_2 boolean independent, and then taking X_3 freely independent of X_1 and X_2.

(3) In [24], I worked on the complex-analytic aspects of these general types of independence with Ethan Davis (an undergrad at the time) and Zhichao Wang (a graduate student of Todd Kemp).

(4) Janusz Wysoczański and Lahcen Oussi asked about [54] during several conferences, which led to a new project giving general limit theorems for non-commutative independence [55].

Graph products: A joint project with Charlesworth, de Santiago, Kunnawalkam Elayavalli, Hayes, and Nelson [15] studies the W*-algebraic properties of *graph products* of operator algebras [32, 13], a construction that mixes free product and tensor product operations (thus, mixing classical independence and free independence). We constructed matrix models for graph independence using random $n \times n$ permutation matrices (as [14] did for unitary matrices) using the combinatorial techniques of [65] and [2]. These matrix models provided a new proof that soficity of groups is preserved by graph products. We then used them to study the Jung-Hayes entropy for graph product von Neumann algebras [16]. We also studied fundamental von Neumann algebraic properties for graph products, such as relative amenability [17].

Application to spin glass theory: Juspreet Sandhu, Jonathan Shi, and I gave an algorithm for an optimization problem associated to the Sherrington-Kirkpatrick model [57]. Briefly, the problem requires us to maximize $\langle x, Ax \rangle$ for $x \in [-1,1]^n$, where A is a Gaussian random matrix. We modified the objective function in the interior of the domain by adding a potential arising from spin glass theory, and the increments in the algorithm were chosen based on the Hessian of the objective. This Hessian matrix is a diagonal matrix plus the original Gaussian matrix, which we analyzed using the complex-analytic toolkit for free probability, as well as the techniques for strong convergence from [18].

7 Non-commutative optimal transport

Wasserstein distance and Monge-Kantorovich duality: In §3, I described work on transport of measure in the free setting. Transport of measure gives rise to *Wasserstein distance* given two (classical or non-commutative) probability distributions μ and ν, the Wasserstein distance is the minimum distance $\|\mathbf{X} - \mathbf{Y}\|_{L^2}$ over pairs $(\mathbf{X}, \mathbf{Y})$ of random variables such that $\mathbf{X}$ has distribution μ and $\mathbf{Y}$ has distribution ν. A pair that realizes this minimal distance is called an *optimal coupling.* The free version of the Wasserstein distance was studied in [9]. A crucial question for the development of Wasserstein geometry in random matrix theory is the following.

Question 7.1. *Let $\mathbf{X}^{(n)}$ and $\mathbf{Y}^{(n)}$ be d-tuples of random matrices; for instance, consider those given by a smooth log-density V as in §3. Suppose that $\mathbf{X}^{(n)}$ and $\mathbf{Y}^{(n)}$ have a large-n limit described by non-commutative random variables $\mathbf{X}$ and $\mathbf{Y}$. Does the classical Wasserstein distance of $\mathbf{X}^{(n)}$ and $\mathbf{Y}^{(n)}$ converge to the free Wasserstein distance of $\mathbf{X}$ and $\mathbf{Y}$?*

This question was open even for a nice convex V for which free monotone transport was studied in [35]; despite having the transport maps in hand, it was unclear whether there might not be some exotic coupling that produces a shorter distance. In [53], we showed that the couplings studied by [35] are indeed optimal, arguing using Legendre transforms of convex functions of non-commuting variables. Then Wilfrid Gangbo, Kyeongsik Nam, Dimitri Shlyakhtenko, and I developed Legendre transforms and Monge-Kantorovich duality systematically for the non-commutative setting.

Connections with quantum information: We discovered that the free Wasserstein distance can be expressed in terms of factorizable quantum channels. Leveraging this connection, we showed that the behavior of free optimal couplings can be quite wild: Even if the two variables $\mathbf{X}$ and $\mathbf{Y}$ we are trying to couple each individually come from finite-dimensional algebras, they may require an infinite-dimensional algebra to optimally couple. Moreover, a negative solution to Connes embedding problem [58] implies a negative answer to Question 7.1 in general.

Todd Kemp and I supervised an undergraduate project of Junchen Zhao at UCSD which gave more explicit examples of this high-dimensionality phenomenon for optimal couplings. Moreover, I am currently working on finding a wider class of explicit examples with Magdalena Musat and Mikael Rørdam at the University of Copenhagen.

Free and quantum Wasserstein distance: Motivated by the connections between free optimal transport and quantum channels, I have organized an online reading group for discussion on non-commutative Wasserstein distance. Recently, we have learned and connected various notions of Wasserstein distances that have been used in quantum information, which will likely lead to results that unify free and quantum optimal transport.

8 Model theory of von Neumann algebras

Model theory of von Neumann algebras: In 2013-2014, researchers began to apply model-theoretic concepts and tools to von Neumann algebras [26]. In particular, there was interest in classifying von Neumann algebras up to *elementary equivalence* (which means two objects have the same first-order theory) and determining which properties were preserved by elementary equivalence. For instance, Goldbring, Kunnawalkam Elayavalli, Pi, and I described a certain uniform super McDuff property that is preserved by elementary equivalence [30]. As a postdoc with Ilijas Farah in 2023-2024, I worked on quantifier elimination and model completeness with Farah and Pi [27]. This also motivated a recent work with Gao on direct integrals and elementary equivalence [28].

Model theory and free probability: I became interested in model theory of von Neumann algebra for its potential applications to free probability theory. Indeed, continuous model theory concerns predicates that take real values rather than true/false values, and uses sup and inf rather than $\forall$ and $\exists$ as quantifiers; thus, the first-order formulas for tracial von Neumann algebras take the form

$$\varphi^M(x_1, \dots, x_n) = \sup_{z_1} \inf_{z_2} \dots \sup_{z_{2m-1}} \inf_{z_{2m}} F(\tau(p_1(\mathbf{x}, \mathbf{z})), \dots, \tau(p_k(\mathbf{x}, \mathbf{z}))), \tag{8.1}$$

where $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{z} = (z_1, \dots, z_{2m})$ and the sup's and inf's in z_1, …, z_{2m} are taken over the unit ball in M. In §5, we saw that the large deviations rate function should be given by a formula with an infimum. Optimal transport naturally results in variational problems as well. In contrast to atomless classical probability spaces, tracial von Neumann algebras do not admit quantifier elimination, which means that there is no simpler way to express the values of these formulas without using a supremum of infimum. This suggests that the non-commutative probability distribution of $\mathbf{x}$ as usually defined, which only looks at the formulas without any supremum or infimum, provide insufficient information about the tuple $\mathbf{x}$ to adequately address these natural variational problems.

For this reason, I considered the *type*, which expresses the value of all formulas, as the correct analog of probability distribution in the non-commutative setting, and that the concepts studied up to this point should be adapted for types.

(1) I developed the analog of free entropy for types in [49], as well as sketching many open question about how free probability and independence relate with model theory.

(2) I developed the analog of Jung–Hayes entropy for types in [48], and as an application showed that an ultraproduct of matrix algebras has infinite Jung–Hayes entropy.

(3) I studied the non-commutative Wasserstein distance for types in [50]. This allowed for a Monge-Kantorovich duality with improved continuity conditions compared to [31]. It also provided insight into the definable closure for von Neumann algebras.

Building on this work, I plan to relate entropy and Wasserstein distance for types. In particular, I want to show that the free entropy χ is concave along Wasserstein geodesics, meaning that if $(\mathbf{X}, \mathbf{Y})$ is an optimal coupling of two types, then $\chi((1-t)\mathbf{X} + t\mathbf{Y})$ is concave for $t \in [0, 1]$.

9 Proposal for student projects

Here I give a self-contained description of non-commutative independences and potential undergraduate projects in this area, related to the work in §6.

Non-commutative probability spaces: Given a probability space (Ω, μ) the set of *bounded random variables* is $L^\infty(\Omega, \mu)$. These bounded random variables form a $*$-algebra, that is, they are closed under vector-space operations, multiplication, and complex conjugation. The expectation defines a linear map $E : L^\infty(\Omega, \mu) \to \mathbb{C}$. In non-commutative probability theory, we replace $L^\infty(\Omega, \mu)$ with a non-commutative algebra $\mathcal{A}$. The algebra $\mathcal{A}$ still has vector-space operations and multiplication, but we don't necessarily have $ab = ba$. The analog of complex conjugation is a $*$-operation such that $a \mapsto a^*$ is conjugate-linear and $(ab)^* = b^* a^*$. A central example is $\mathcal{A} = M_n(\mathbb{C})$ the algebra of $n \times n$ complex matrices; here the $*$-operation corresponds to taking the adjoint. In the non-commutative setting, the expectation is replaced by a state $\varphi : \mathcal{A} \to \mathbb{C}$, that is, a linear map satisfying $\varphi(1) = 1$ and $\varphi(a^* a) \geq 0$. In the case $\mathcal{A} = M_n(\mathbb{C})$, we could take φ to be the normalized trace $\varphi(A) = \operatorname{tr}_n(A) = (1/n) \operatorname{Tr}_n(A)$; we could also consider $\varphi(A) = \langle v, Av \rangle$ for some unit vector $v \in \mathbb{C}^n$.

Non-commutative independence: In classical probability theory, random variables $X_1, \ldots, X_d$ are independent if for any functions $f_1, \ldots, f_d$, we have

$$\mathbb{E}[f_1(X_1) \ldots f_d(X_d)] = \mathbb{E}[f_1(X_1)] \ldots \mathbb{E}[f_d(X_d)].$$

We can think of independence as giving a rule for determining the joint moments of $X_1, \ldots, X_d$ from the moments of the individual random variables. There are several different types of independence in the non-commutative setting, including free [85], boolean [80], and monotone independence [66]. There is a host of other types of independence if one's definition is sufficiently broad. Several frameworks have been introduced for unifying different types of non-commutative independence, for instance [78, 67]. Weihua Liu and I described a general operad framework for independences defined by trees [54] which encompassed many different types of independence, and described systematically what happens when these independences are combined in various ways, such as taking X_1 and X_2 boolean independent, and then taking X_3 freely independent of X_1 and X_2.

This reduced the study of various convolution identities to combinatorial manipulations of graphs and trees, something which is ripe for undergraduate projects. The key operation is *composition* of graphs: If G is a directed graph (digraph) on vertex set $\{1, \ldots, n\}$ and $G_1, \ldots, G_n$ are digraphs, then the composition $G(G_1, \ldots, G_n)$ is defined by taking disjoint copies of $G_1, \ldots, G_n$ and then making an edge from every vertex in G_i to every vertex in G_j when there is an edge from i to j in G.

Project 9.1. *How many of the graphs on n vertices can be expressed as a nontrivial composition? Are most graphs indecomposable? For some particular family of graphs (that we would choose for the project), how many ways are there are to decompose it?*

Non-commutative convolutions: In particular, if X_1 and X_2 are independent, then the distribution $\mu_{X_1+X_2}$ of $X_1 + X_2$ is uniquely determined, and in fact it is the convolution $\mu_{X_1+X_2} = \mu_{X_1} * \mu_{X_2}$. Each type of independence has a corresponding notion of convolution; whereas in the classical setting, Fourier transforms are used to compute the convolution, these non-commutative convolutions are computed using various complex-analytic functions associated to the distribution, all of which are derived from the Cauchy transform

$$G_\mu(z) = \int_{\mathbb{R}} (z - t)^{-1} \, d\mu(t).$$

In [44] and [47], I described the parallel between stochastic processes and differential equations for the Cauchy transforms for free, boolean, and monotone independence.

The Cauchy transform allows explicit computations of the free, boolean, and monotone convolution of various probability distributions (often atomic distributions, those with an analytic density, or some mixture), and thus provides many opportunities for undergraduates with background in probability and complex analysis. I studied the complex-analytic aspects of digraph convolutions together with Ethan Davis (an undergraduate at the time) and Zhichao Wang (a graduate student at the time). Later, Janusz Wysoczański and Lahcen Oussi and I worked on general limit theorems for digraph convolutions. A sufficient condition to apply our results to a family of digraphs G_n is for normalized counts of digraph homomorphisms to converge. This motivates the following project.

Project 9.2. *Pick a sequence of digraphs G_n (e.g. grids, iterated compositions, trees). For digraphs, G, G', let $\mathrm{Hom}(G', G)$ be the set of digraph homomorphisms. For your chosen family of graphs G_n, does $\lim_{n\to\infty} \mathrm{Hom}(G', G_n)$ exist for all directed trees G', and can you compute it?*

We obtained for families of multi-regular digraphs an explicit description of central limit distributions. Here we assume the graph has m different sets of vertices, and t_j is the portion of vertices in the jth set, and each vertex in the ith set has edges going to an $a_{i,j}$ portion of the vertices in the jth set.

Project 9.3. *For a probability measure μ, let $K_\mu(z) = z - 1/G_\mu(z)$. For $i, j = 1, \ldots, n$, let $a_{i,j}$ and t_j be weights in $[0, 1]$ such that $a_{i,j} \leq t_j$. There are probability measures ν_j such that*

$$zK_{\nu_i}(z) - \sum_{j=1}^{n} a_{i,j} K_{\nu_i}(z) K_{\nu_j}(z) = 1,$$

and the central limit distribution associated to these parameters satisfies

$$K_\mu(z) = \sum_{j=1}^{n} t_j K_{\nu_j}(z).$$

Compute the central limit distributions either algebraically or numerically for various parameters t_j and $a_{i,j}$. Determine when μ has atoms and how large the atoms are.

Another similar project would be to study the central limit distributions associated to Project 9.2.

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