Non-commutative transport of measure for operator algebras

David Jekel [1]

University of California, San Diego

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Welcome to the non-commutative world

classical	non-commutative
$C(\Omega)$ for Ω compact Hausdorff	unital C^*-algebra $\mathcal{A}$
$L^\infty(\Omega, \mathcal{F})$ for $(\Omega, \mathcal{F})$ measurable	W^*-algebra $\mathcal{M}$

A C^**-algebra* is $*$-subalgebra of $B(H)$, closed with respect to the operator norm. Every commutative unital C^*-algebra is isomorphic to $C(\Omega)$ for some compact Hausdorff space Ω.

A W^**-algebra* (or *von Neumann algebra*) is a $*$-subalgebra of $B(H)$, closed with respect to the weak operator topology. Every commutative W^*-algebra is isomorphic to $L^\infty(\Omega, \mathcal{F})$ for some measurable space $(\Omega, \mathcal{F})$.

Welcome to the non-commutative world

classical	non-commutative
$C(\Omega)$ for Ω compact Hausdorff	unital C^*-algebra $\mathcal{A}$
probability measure μ on Ω	state ϕ on $\mathcal{A}$
$L^\infty(\Omega, \mathcal{F})$ with prob. measure μ	W^*-algebra $\mathcal{M}$ with normal state ϕ

A *state* on a unital C^*-algebra $\mathcal{A}$ is a linear functional satisfying $\phi(1) = 1$ and $\phi(a^* a) \geq 0$.

By the Riesz representation theorem, every state on $C(\Omega)$ has the form $\phi(f) = \int f \, d\mu$ for a Borel probability measure μ.

Given a state on a C^*-algebra $\mathcal{A}$, one can create a separation-completion of $\mathcal{A}$ into a von Neumann algebra $\mathcal{M}$ that is faithfully represented on the GNS space $L^2(\mathcal{A}, \phi)$. This is the non-commutative version of passing from $C(\Omega)$ to $L^\infty(\Omega, \mu)$.

Traces are nicer

classical	non-commutative
$C(\Omega)$ for Ω compact Hausdorff	unital C^*-algebra $\mathcal{A}$
probability measure μ on Ω	trace τ on $\mathcal{A}$
$L^\infty(\Omega, \mathcal{F})$ with prob. measure μ	W^*-algebra $\mathcal{M}$ with faithful normal trace

A *trace* on $\mathcal{A}$ is a state that satisfies $\tau(xy) = \tau(yx)$. In the commutative case, every state is a trace. Traces are nice because they "behave more like finite measures than general states."

The non-commutative world is much more complicated ...

We are no longer in Kansas

Classical: Two complete probability spaces (Ω_1, μ_1) and (Ω_2, μ_2) are isomorphic if and only if the atoms of μ_1 and μ_2 have the same sizes. In particular, if μ_1 and μ_2 are diffuse (no atoms), then the two measure spaces are isomorphic (in other words, μ_1 can be transported to μ_2 by some essentially bijective function).

Every probability space occurs as a quotient of $[0, 1]$ with Lebesgue measure.

Non-commutative: (McDuff) There are continuum many non-isomorphic WOT-separable II_1-factors (a II_1-*factor* is a tracial W^*-algebra with trivial center which is "diffuse").

(Ozawa) There is no WOT-separable II_1 factor that contains an isomorphic copy of all other WOT-separable II_1 factors.

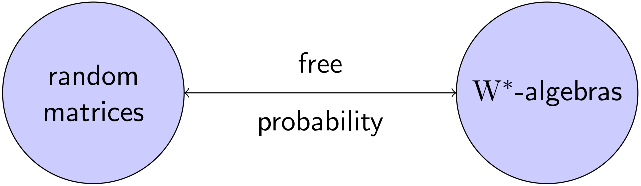
We are no longer in Kansas

Classical: Consider the space of Borel probability measures on $[-R, R]^d$. Finite linear combinations of δ masses are dense in the weak-$*$ topology.

Non-commutative: Let $\Sigma_{d,R}$ be the space of traces on the universal free product $C([-R, R])^{*d}$. Such a trace represents the *non-commutative law* of a d-tuple of self-adjoint operators $(X_1, \dots, X_d)$ from a tracial W^*-algebra $(\mathcal{M}, \tau)$ with $\|X_i\| \leq R$. Indeed, a trace on $C([-R, R])^d$ is obtained by composing τ with the $*$-homomorphism $C([-R, R])^{*d} \to \mathcal{M}$ that sends the canonical generators to X_1, ..., X_d.

One non-commutative analogue of finitely supported measures would be the non-commutative laws of d-tuples from finite-dimensional tracial W^*-algebras. For such "finitary" laws to be dense in $\Sigma_{d,R}$ is equivalent to the long-standing *Connes embedding problem* which has recently been shown to have a negative solution through an equivalent formulation in quantum information theory (Ji, Natarajan, Vidick, Wright, and Yuen).

Building a bridge



At this point, it should be clear that non-commutative transport of measure does not always exist. However, certain tuples $(X_1, \dots, X_d)$ from $(\mathcal{M}, \tau)$ can be very well approximated by d-tuples of random self-adjoint matrices, in light of several fundamental results in free probability theory (due to Voiculescu, Guionnet, and many others). This gives us some hope that classical results about transport of measure might be carried over to the non-commutative setting in special cases.

Free Gibbs laws

Free Gibbs laws are non-commutative laws which arise as the large-N limit of certain random matrix models.

Consider a probability measure $\mu^{(N)}$ on $M_N(\mathbb{C})_{sa}^d$ (d-tuples of self-adjoint matrices) given by

$$d\mu^{(N)}(x) = \text{constant}\, e^{-N^2 V^{(N)}(x)}\, dx,$$

where dx is Lebesgue measure and $V^{(N)} : M_N(\mathbb{C})_{sa}^d \to \mathbb{R}$ is some function "with a formula independent of N," something like

$$V^{(N)}(x_1, \dots, x_d) = \frac{1}{N} \operatorname{Tr}(p(x_1, \dots, x_d)),$$

where p is a non-commutative polynomial. Let $X^{(N)} = (X_1^{(N)}, \dots, X_d^{(N)})$ be a random d-tuple of matrices chosen according to the measure $\mu^{(N)}$.

Free Gibbs laws

Let $\|x\|_2 = (\sum_{j=1}^d (1/N)\operatorname{Tr}(x_j^* x_j))^{1/2}$.

Theorem (≈ Guionnet & Maurel-Segala 2006, Guionnet & Shlyakhtenko 2009, Guionnet & Shlyakhtenko & Dabrowski 2016)

Suppose that $V^{(N)}(x) = (1/N)\operatorname{Tr}(f(x))$ where f is a non-commutative polynomial or power series and that $V^{(N)}(x) - (c/2)\|x\|_2^2$ is convex for some $c > 0$. Then there exists some d-tuple X from a tracial W^*-algebra $(\mathcal{M}, \tau)$ such that

$$\lim_{N\to\infty} \frac{1}{N}\operatorname{Tr}(p(X^{(N)})) = \tau(p(X))$$

almost surely for every non-commutative polynomial p.

Remark

The limit is deterministic, which makes sense given the phenomenon of concentration of measure in high dimensions (cf. Bakry-Émery criterion).

The free Gaussian

In the case where $V^{(N)}(x) = (1/2)\|x\|_2^2$, the random matrix $S^{(N)}$ is called the *Gaussian unitary ensemble*.

(Wigner) The spectral distribution of each $S_j^{(N)}$ converges to the *semicircle law* $(1/2\pi)\sqrt{4-t^2}\chi_{[-2,2]}(t)\,dt$.

(Voiculescu) The matrices $S^{(N)}$ become asymptotically *freely independent* as $N \to \infty$, and as a consequence the W^*-algebra generated by S is isomorphic to the free group W^*-algebra $L(\mathbb{F}_d)$.

S is called a *standard free semicircular family*, and we will denote its non-commutative law by σ. This is a canonical non-commutative law that we hope to transport other non-commutative laws to.

Non-commutative tranport of measure

Theorem (Guionnet & Shlyakhtenko 2014, Dabrowski & Guionnet & Shlyakhtenko 2016)

Let $X^{(N)}$ and X be as in the previous theorem. Then the associated von Neumann algebra $W^*(X_1, \dots, X_d)$ is isomorphic to $L(\mathbb{F}_d)$ (the Gaussian case).

One natural proof strategy:

- Let $S^{(N)}$ be a Gaussian unitary ensemble, let $\sigma^{(N)}$ be the associated probability measure, and let S be a free semicircular family describing the large-N limit.
- Using classical techniques (see e.g. Otto-Villani 2000), you can construct some bijective function $F^{(N)}$ such that $(F^{(N)})_* \mu^{(N)} = \sigma^{(N)}$ or equivalently $F^{(N)}(X^{(N)}) \sim S^{(N)}$ in distribution.
- Note that $F^{(N)}$ describes an isomorphism between $(M_N(\mathbb{C})^d_{\text{sa}}, \mu^{(N)})$ and $(M_N(\mathbb{C})^d_{\text{sa}}, \sigma^{(N)})$.

Non-commutative transport of measure

- Arrange and prove that $F^{(N)}$ has good asymptotic behavior as $N \to \infty$, and it approaches some non-commutative function F in the limit. (But what does this mean??)
- Same for inverse function of $F^{(N)}$.
- Then $F(X) \sim S$ in non-commutative law, and we get an isomorphism $\mathrm{W}^*(X) \cong \mathrm{W}^*(S)$.

This strategy, so to speak, imports the classical results about transport of measure into the non-commutative world across the bridge of free probability.

The papers mentioned above did not actually prove it this way; they worked directly at the level of the non-commutative random variables X and S. They deduced *a posteriori* that their non-commutative transport functions approximated the matrix optimal transport functions in L^2.

A new functional calculus

In order to carry out our strategy, we want to define an appropriate space of "functions of d non-commuting self-adjoint variables" which will contain our hypothetical F.

While the non-commutative or fully matricial functions of Taylor are a good model for "non-commutative complex analysis," our aims here are more in the spirit of real analysis and PDE, and we wish to incorporate the trace into our construction of functions.

Classical continuous functions on $\mathbb{R}^d$ can be approximated uniformly on balls by polynomials. In the same way, our functions will be approximated on operator-norm balls by *trace polynomials*, functions such as

$$f(x_1, x_2, x_3) = \operatorname{tr}(x_1)x_2^2 + \operatorname{tr}(x_1 x_3)\operatorname{tr}(x_2)x_3 x_1 x_2 - 2\operatorname{tr}(x_3^2).$$

A new functional calculus

If f is a d-variable trace polynomial and $(\mathcal{M}, \tau)$ is a tracial W^*-algebra and $X \in \mathcal{M}_{\mathrm{sa}}^d$, then f can be evaluated on X by substituting $X_1, \dots, X_d$ and τ for the formal symbols $x_1, \dots, x_d$ and tr.

We define a uniform 2-norm on the R-ball for trace polynomials by

$$\|f\|_{2,R} = \sup\{\|f(X)\|_2 : X \in \mathcal{M}_{\mathrm{sa}}^d, (\mathcal{M}, \tau) \text{ tracial } \mathrm{W}^*\text{-algebra}, \|X_j\| \leq R\},$$

where $\|Y\|_2 = \tau(Y^* Y)$. Let $\overline{\mathrm{TrP}}_d$ be the completion of the space of trace polynomials with respect to the family of seminorms $(\|\cdot\|_{2,R})_{R>0}$.

Remark

Since we are using the 2-norm rather than operator norm, the functions defined in this way are not the direct analogue of $C(\mathbb{R}^d)$ if we think that continuous functions on compact sets correspond to C^*-algebras. However, these functions are continuous with respect to $\|\cdot\|_2$ on each operator-norm ball.

Asymptotic approximation

The functions defined above make sense to evaluate on any d-tuple of self-adjoint operators from a tracial W^*-algebra. In particular, they can be evaluated on d-tuples of self-adjoint matrices (where we use the normalized trace $(1/N)\operatorname{Tr}$ on $M_N(\mathbb{C})$).

Thus, $\overline{\mathrm{TrP}}_d$ is suitable for describing the asymptotic behavior of certain sequences of functions $f^{(N)}$ on $M_N(\mathbb{C})_{\mathrm{sa}}^d$. The notion of asymptotic approximation uses (again) uniform approximation with respect to $\|\cdot\|_2$ on an operator-norm ball.

Definition

Let $f^{(N)} : M_N(\mathbb{C})_{\mathrm{sa}}^d \to M_N(\mathbb{C})$. We say that $f^{(N)} \rightsquigarrow f$ if for every $R > 0$, we have

$$\lim_{N \to \infty} \sup \{ \|f^{(N)}(x) - f(x)\|_2 : x \in M_N(\mathbb{C})_{\mathrm{sa}}^d, \|x_j\| \leq R \} = 0.$$

Asymptotic approximation

This notion of asymptotic approximation $f^{(N)} \rightsquigarrow f$ is preserved under the following operations (with mild hypotheses on the functions in question):

- Linear combinations.
- Composition.
- Limits.
- Application of the heat semigroup $e^{t\Delta/N}$ on functions on $M_N(\mathbb{C})_{sa}^d$. (This is where it is important to use trace polynomials rather than merely non-commutative polynomials.)

Combining and iterating these operations allow us to "build" many sequences of functions associated to the random matrix measures $\mu^{(N)}$, and hence to get control over their asymptotic behavior as $N \to \infty$. In particular, we can build certain functions transporting $\mu^{(N)}$ to $\sigma^{(N)}$ (constructed by the same methods as in Otto and Villani's proof of the Talagrand inequality). This is a bunch of technical work.

Non-commutative transport of measure revisited

Theorem (J. 2018-2019)

Suppose that $V^{(N)} : M_N(\mathbb{C})^d_{\mathrm{sa}} \to \mathbb{R}$ *satisfies that* $V^{(N)}(x) - (c/2)\|x\|_2^2$ *is convex and* $V^{(N)}(x) - (C/2)\|x\|_2^2$ *is concave for some* $0 < c < C$*, and suppose that* $\nabla V^{(N)}$ *is asymptotic to some* $f \in (\overline{\mathrm{TrP}}_d)^d_{\mathrm{sa}}$ *as* $N \to \infty$*. Let* $X^{(N)}$ *be the associated random matrix tuple as above. Then there exists some* d*-tuple* X *from a tracial* W^**-algebra* $(\mathcal{M}, \tau)$ *such that*

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr}(p(X^{(N)})) = \tau(p(X))$$

almost surely for every non-commutative polynomial p*. Also, there exist some* $F, G \in (\overline{\mathrm{TrP}}_d)^d_{\mathrm{sa}}$ *such that* $F \circ G = G \circ F = \mathrm{id}$ *and* $F(X) \sim S$*. Furthermore,* F *is obtained as the large-*N *limit of some* $F^{(N)}$ *with* $F^{(N)}(X^{(N)}) = S^{(N)}$*.*

Triangular transport of measure

Theorem (J. 2019)

We can arrange that $F^{(N)}$ and F are lower-triangular functions in the sense that

$$F(x_1, \dots, x_d) = (F_1(x_1), F_2(x_1, x_2), \dots, F_d(x_1, \dots, x_d)).$$

This implies that there is an isomorphism $\phi : \mathrm{W}^*(X_1, \dots, X_d) \to \mathrm{W}^*(S_1, \dots, S_d)$ such that

$$\phi(\mathrm{W}^*(X_1, \dots, X_k)) = \mathrm{W}^*(S_1, \dots, S_k) \text{ for } k = 1, \dots, d.$$

The idea of the proof is to transport the conditional distribution of X_d given X_1, ..., X_{d-1} to that of a Gaussian independent of X_1, ..., X_{d-1}, then transport the conditional distribution of X_{d-1} given X_1, ..., X_{d-2} to Gaussian, and so forth.

Work in progress and goals

- Define a tracial non-commutative analogue of $C^k(\mathbb{R}^d)$.
- Triangular transport for the C^*-algebras generated by X and S, not just the W^*-algebras.
- Develop non-commutative PDE theory.
- Find easier constructions of transport.
- Study non-commutative optimal transport.
- What happens when the convexity assumption on V is removed?