An operatorial viewpoint on Loewner chains in the complex upper half-plane

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Meta-background

Instead of a formal thesis defense, I am giving a series of talks explaining results in my dissertation, intended to be accessible for general mathematicians.

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Dissertation title: Evolution Equations in Non-commutative Probability

This talks includes some results from the paper "Operator-valued Loewner chains and non-commutative probability", Journal of Functional Analysis 278.10:108452, which will also be in my dissertation.

Meta-background

Goal: This talk will motivate the connections between complex analysis and non-commutative probability (the theory of operators on Hilbert space as "random variables"). We start from the viewpoint of complex analysis. Non-commutative probability in its own right will be explained in the next talk.

Prerequisites: Complex analysis, measure theory, Hilbert spaces and self-adjoint operators.

Outline

1. Cauchy transform representation of functions
2. Operator models for Cauchy transforms
3. Operator model for composition
4. Operator models for Loewner chains

Outline

The Cauchy transform

We begin with a famous result of Nevanlinna that relates certain analytic functions on the upper half-plane with Borel measures on $\mathbb{R}$.

The *upper and lower half-planes* are the regions

$$\mathbb{H}_+ = \{z \in \mathbb{C} : \operatorname{im} z > 0\}$$
$$\mathbb{H}_- = \{z \in \mathbb{C} : \operatorname{im} z > 0\}.$$

If μ is a finite Borel measure on $\mathbb{R}$, the *Cauchy transform* of μ is the function

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z - x}\, d\mu(x).$$

If $\operatorname{im} z > 0$, then $\operatorname{im}(z - x)^{-1} < 0$, and hence G_μ maps $\mathbb{H}_+$ to $\mathbb{H}_-$.

Series expansion at ∞

If μ is a compactly supported, then $G_\mu(z)$ is analytic in a neighborhood of ∞, and we have a power series expansion:

$$\begin{aligned} G_\mu(z) &= \frac{1}{z} \int_{\mathbb{R}} \frac{1}{1 - t/z} \, d\mu(t) \\ &= \frac{1}{z} \int_{\mathbb{R}} \sum_{n=0}^{\infty} \left(\frac{x}{z} \right)^n \, d\mu(x) \\ &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \int_{\mathbb{R}} x^n \, d\mu(x). \end{aligned}$$

The quantity $\mu_n := \int_{\mathbb{R}} x^n \, d\mu(x)$ is called the *nth moment of* μ, and thus $G_\mu(z)$ is a kind of *moment generating function for* μ.

Nevanlinna's theorem

Nevanlinna showed that *any* analytic function $G : \mathbb{H}_+ \to \mathbb{H}_-$ with $\limsup_{y \to +\infty} y|\operatorname{im} G(iy)| < +\infty$ can be represented as G_μ for some finite Borel measure μ. We will focus on the compactly supported case.

Proposition

An analytic function $G : \mathbb{H}_+ \to \mathbb{H}_-$ is the Cauchy transform of a compactly supported measure μ if and only if $G(z)$ has an analytic extension to a neighborhood of ∞ satisfying $G(\overline{z}) = \overline{G(z)}$. Moreover, in this case, we have $G(z) = \|\mu\|/z + O(1/z^2)$, where $\|\mu\|$ is the total mass.

Remark

This is a powerful result because with only a few assumptions, we can represent the function G in the very special form G_μ, which in particular gives us good control over the derivatives of G since $G_\mu^{(k)}(z) = (-1)^k(k-1)! \int_{\mathbb{R}} (z-x)^{-k-1}\, d\mu(x)$.

Self-maps of the upper half-plane

Definition

The *F-transform* of a measure μ is the function $F_\mu : \mathbb{H}_+ \to \mathbb{H}_+$ given by $F_\mu(z) = 1/G_\mu(z)$.

Corollary

A function $F : \mathbb{H}_+ \to \mathbb{H}_+$ is the F-transform of a compactly supported probability measure if and only if $\underline{K(z)} = z - F(z)$ has an extension to a neighborhood of ∞ with $K(\overline{z}) = \overline{K(z)}$.

Sketch of proof.

Let us just explain $\Longleftarrow$. Let $G(z) = 1/F(z) = 1/(z - K(z))$. Since $K(z)$ is analytic at ∞, we can use the geometric series expansion $1/(z - K(z)) = (1/z)\sum_{n=0}^\infty (K(z)/z)^n$ to get the properties of G described in the previous proposition. $G(z) = 1/z + O(1/z^2)$, so the total mass of μ is 1. □

Translation between complex analysis and probability

A *hull* is a compact set $S \subseteq \overline{\mathbb{H}}_+$ such that $\Omega := \mathbb{H}_+ \setminus S$ is simply connected.

The Riemann mapping theorem tells us that there is a conformal map $F : \mathbb{H}_+ \to \Omega$, which is analytic at ∞. This F is unique if we impose the normalization $F(z) = z + O(1/z)$.

Then if we write

$$F(z) = z - \frac{t}{z} + O\left(\frac{1}{z^2}\right),$$

the number $t > 0$ is called the *half-plane capacity* of the hull S, which is some measure of how "large" S is.

Translation between complex analysis and probability

By the previous corollary, F can be represented as $F_\mu = 1/G_\mu$ for a compactly supported probability measure μ.

A power series computation reveals that

$$F_\mu(z) = z - \mu_1 - \frac{\mu_2 - \mu_1^2}{z} + O\left(\frac{1}{z^2}\right).$$

Thus, our normalization of F on the previous slide is $\mu_1 = 0$, or the *mean* of μ is zero.

The half-plane capacity t is equal to $\mu_2 - \mu_1^2$, which is the *variance* of μ, a standard probabilistic measurement of how "spread out" μ is.

Outline

F-transforms and Cauchy transforms

Proposition

Let μ be a compactly supported probability measure on $\mathbb{R}$. Then there exists a real number b and a compactly supported measure on $\mathbb{R}$ such that

$$F_\mu(z) = z - b - G_\sigma(z).$$

Conversely, every such pair (b, σ) has a corresponding μ.

There are several ways to prove this. For instance, it can be deduced from Nevanlinna's theorem above after showing that $\operatorname{im} F_\mu(z) \geq \operatorname{im} z$.

I'll give a proof using operators on Hilbert space, which contains the kernel of two important ideas in non-commutative probability theory:

1. representing measures as spectral distributions of operators,
2. combinatorial manipulation of moment generating functions.

Operator representation of measures

Let μ be a compactly supported probability measure on $\mathbb{R}$. Consider the Hilbert space $L^2(\mu)$.

Let X be the operator $L^2(\mu) \to L^2(\mu)$ of multiplication by x, that is,

$$(Xf)(x) = xf(x).$$

Observe that X is a bounded self-adjoint operator with operator norm $\|X\| = \sup\{|x| : x \in \operatorname{supp}(\mu)\}$.

Let $\xi \in L^2(\mu)$ be the function which is identically 1. Then for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\langle \xi, (z - X)^{-1} \xi \rangle = \int_{\mathbb{R}} 1 \cdot \frac{1}{z - x} 1 \, d\mu(x) = G_\mu(z).$$

Here $(z - X)^{-1}$ is short for $(zI - X)^{-1}$ in $B(H)$.

Operator representation of measures

In fact, for any $f \in C_0(\mathbb{R})$, we have

$$\langle \xi, f(X)\xi \rangle = \int_{\mathbb{R}} f(x)\, d\mu(x),$$

where $f(X)$ means the application of the function f to the operator X through functional calculus. This last equation means precisely that μ is the *spectral measure* associated to the operator X and the vector ξ.

Recall that in general, for any self-adjoint operator X and vector ξ in a Hilbert space $\mathcal{H}$, such a spectral measure is guaranteed to exist by the spectral theorem, and the total mass of the measure is $\|\mu\| = \|\xi\|^2$.

Proof of Proposition 1

Let's write $\mathcal{H} = L^2(\mu)$. Let $P : \mathcal{H} \to \mathcal{H}$ be the projection onto $\mathbb{C}\xi$, that is, $Pf = \xi\langle \xi, f\rangle$, or in bracket notation $P = |\xi\rangle\langle\xi|$. Let $Q = 1 - P$.

In the proposition, we will use $b = \langle \xi, X\xi\rangle$ and σ will be the spectral measure of QXQ with respect to the vector $QX\xi$, which means in particular that

$$G_\sigma(z) = \langle QX\xi, (z - QXQ)^{-1} QX\xi\rangle.$$

So the relation $F_\mu(z) = z - b - G_\sigma(z)$ that we want to prove can be rephrased as

$$\frac{1}{\langle \xi, (z - X)^{-1}\xi\rangle} = z - \langle \xi, X\xi\rangle - \langle QX\xi, (z - QXQ)^{-1} QX\xi\rangle$$

Proof of Proposition 1

There is an efficient but perhaps unenlightening proof for this relation using resolvent identities. Instead, I'll show an argument with combinatorial insight.

Denote

$$\begin{aligned} K(z) &= \langle \xi, X\xi \rangle - \langle QX\xi, (z - QXQ)^{-1} QX\xi \rangle \\ &= \langle \xi, X\xi \rangle - \langle \xi, XQ(z - QXQ)^{-1} QX\xi \rangle. \end{aligned}$$

We need to show that $(z - K(z))^{-1} = \langle \xi, (z - X)^{-1} \xi \rangle = G_\mu(z)$.

For z in a neighborhood of ∞, we have

$$(z - K(z))^{-1} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{K(z)}{z} \right)^n.$$

Proof of Proposition 1

Let's find an elegant way to express $K(z)/z$. Note that

$$XQ(z - QXQ)^{-1}QX = XQ\left(\frac{1}{z}\sum_{m=0}^{\infty}\frac{1}{z^m}(QXQ)^m\right)QX$$
$$= \sum_{k=1}^{\infty}\frac{1}{z^k}X(QX)^k.$$

Hence,

$$X + XQ(z - QXQ)^{-1}QX = \sum_{k=0}^{\infty}\frac{1}{z^k}X(QX)^k.$$

Now take the inner product with ξ and divide by z again:

$$\frac{K(z)}{z} = \sum_{k=0}^{\infty}\frac{1}{z^{k+1}}\langle\xi, X(QX)^k\xi\rangle = \sum_{k=0}^{\infty}\langle\xi, z^{-1}X(Qz^{-1}X)^k\xi\rangle.$$

Proof of Proposition 1

In short, $K(z)/z$ is given by taking the inner product with ξ on both sides to all the "strings" produced by alternating the "letters" $z^{-1}X$ and Q, with $z^{-1}X$ occurring at both the start and the end.

Then we plug this into the power series expansion of $(z - K(z))^{-1}$:

$$(z - K(z))^{-1} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{K(z)}{z} \right)^n = z^{-1} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \langle \xi, z^{-1} X (Q z^{-1} X)^k \xi \rangle \right)^n.$$

By distributing multiplication over addition, we get

$$z^{-1} \sum_{n \geq 0} \sum_{k_1, \dots, k_n \geq 0} \langle \xi, z^{-1} X (Q z^{-1} X)^{k_1} \xi \rangle \dots \langle \xi, z^{-1} X (Q z^{-1} X)^{k_n} \xi \rangle$$

Proof of Proposition 1

Because $P = |\xi\rangle\langle\xi|$, we can replace each occurrence of "$\xi\rangle\langle\xi$," by P, so it is

$$z^{-1} \sum_{n \geq 0} \sum_{k_1, \dots, k_n \geq 0} \langle \xi, z^{-1} X (Q z^{-1} X)^{k_1} P \dots P z^{-1} X (Q z^{-1} X)^{k_n} \xi \rangle.$$

The recipe to generate these terms is as follows: pick $n \geq 0$; then choose n different strings of $z^{-1}X$'s and Q's where the number of Q's is k_1, ..., k_n; then concatenate these n strings together with a P in between consecutive terms.

In this way, we will produce all possible strings like $z^{-1}X$, then P or Q, then $z^{-1}X$, then P or Q, ..., ending with $z^{-1}X$.

Proof of Proposition 1

If we fix $\ell \geq 0$ and consider all such strings that have ℓ occurrences of $z^{-1}X$, with all possible choices of P or Q in each position, then like in the binomial theorem, we would get

$$z^{-1}X(P+Q)z^{-1}X\dots(P+Q)z^{-1}X$$

with ℓ occurrences of $z^{-1}X$. But $P+Q=1$, so this is just $(z^{-1}X)^\ell$. So therefore,

$$(z-K(z))^{-1} = z^{-1}\sum_{\ell\geq 0}\langle \xi, (z^{-1}X)^\ell \xi\rangle,$$

which by the geometric series expansion again gives us

$$(z-K(z))^{-1} = \langle \xi, (z-X)^{-1}\xi\rangle = G_\mu(z),$$

which completes the direction $\mu \rightsquigarrow (b,\sigma)$ of the Proposition.

Proof of Proposition 1

In the other direction, we start by building a Hilbert space $\mathcal{K}$, a self-adjoint operator Y, and a vector ζ with

$$G_\sigma(z) = \langle \zeta, (z - Y)^{-1} \zeta \rangle$$

Then consider the Hilbert space $\mathcal{H} = \mathbb{C}\xi \oplus \mathcal{K}$, where ξ is assumed to be a unit vector, and the operator $X : \mathcal{H} \to \mathcal{H}$ given in bracket notation by

$$X = b|\xi\rangle\langle\xi| + |\xi\rangle\langle\zeta| + |\zeta\rangle\langle\xi| + Y$$

Let μ be the spectral measure associated to X and ξ. Then retracing our previous argument, we have $P = |\xi\rangle\langle\xi|$ and $Q = 1 - P = \mathrm{Proj}_{\mathcal{K}}$. Also, $\langle \xi, X\xi \rangle = b$ and $X\xi = \zeta$, and $QXQ = Y$, so we will get back the original b and σ.

Remarks on Proposition 1

We have just showed that there is a bijection between compactly supported probability measures μ and pairs (b, σ) of a real number and a finite compactly supported Borel measure given by $F_\mu(z) = z - b - G_\sigma(z)$ (bijection since a measure is uniquely determined by the Cauchy transform).

This bijection can be realized using operator models: To get (b, σ) from μ, we cut the operator into four pieces $PXP = bP$, PXQ, QXP, and QXQ. For the other direction, we assembled the operator X out of the four pieces $b|\xi\rangle\langle\xi|$, $|\xi\rangle\langle\zeta|$, $|\zeta\rangle\langle\xi|$, Y.

The mean of μ is b, the variance of μ is $\|\sigma\|$, and the moments of σ are known in non-commutative probability theory as *boolean cumulants* of μ.

Outline

Operator model for composition

We just described an operator model for the bijection $\mu \leftrightarrow (b, \sigma)$. Now we will describe how to build operators that realize the composition of two F-transforms F_{μ_1} and F_{μ_2}.

Proposition 2

Let μ_1 and μ_2 be compactly supported probability measures on $\mathbb{R}$. Suppose μ_j is realized by a self-adjoint X_j and vector ξ_j on the Hilbert space $\mathcal{H}_j$. Let $P_j = |\xi_j\rangle\langle\xi_j|$.

Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and $\xi = \xi_1 \otimes \xi_2$. Define

$$\widehat{X}_1 := X_1 \otimes P_2 \qquad \widehat{X}_2 := 1 \otimes X_2.$$

Let μ be the distribution of $X := \widehat{X}_1 + \widehat{X}_2$ with respect to ξ. Then

$$F_\mu = F_{\mu_1} \circ F_{\mu_2}.$$

Monotone independence

Note that the distribution of $\widehat{X}_j$ with respect to ξ is the same as the distribution of X_j with respect to ξ_j.

Moreover, the operators $\widehat{X}_1$ and $\widehat{X}_2$ are *monotone independent random variables* in the sense of Muraki [2].

The proper definition will be explained next time, but for our purposes today, monotone independence amounts to the following relations: First,

$$\widehat{X}_1 f(\widehat{X}_2) \widehat{X}_1 = \widehat{X}_1 \langle \xi, f(\widehat{X}_2) \xi \rangle \widehat{X}_1,$$

meaning that if a function of $\widehat{X}_2$ is sandwiched between two copies of $\widehat{X}_1$, then we can replace $f(\widehat{X}_2)$ by its "expectation"

$$\langle \xi, f(\widehat{X}_2) \xi \rangle = \int_{\mathbb{R}} f(x)\, d\mu_2(x).$$

Monotone independence

The reason for this relation is that

$$\begin{aligned}(X_1 \otimes P_2)(1 \otimes f(X_2))(X_1 \otimes P_2) &= (X_1^2 \otimes P_2 f(X_2) P_2) \\ &= (X_1 \otimes P_2)^2 \cdot \langle \xi_2, f(X_2)\xi_2 \rangle \\ &= \widehat{X}_1^2 \langle \xi, f(\widehat{X}_2)\xi \rangle.\end{aligned}$$

The other relation for monotone independence is that

$$\widehat{X}_1 f(\widehat{X}_2)\xi = \widehat{X}_1 \langle \xi, f(\widehat{X}_2)\xi \rangle \xi,$$

which is proved similarly.

Proof of Proposition 2

We need to show that $F_\mu = F_{\mu_1} \circ F_{\mu_2}$. But it will be easier to work with moment generating functions rather than F-transforms. Let

$$\tilde{G}_\mu(z) = G_\mu(1/z) = \sum_{k=0}^{\infty} z^{k+1} \langle \xi, X^k \xi \rangle.$$

Note that $\tilde{G}_\mu = \mathrm{inv} \circ F_\mu, \circ \mathrm{inv}^{-1}$, where inv denote the map $z \mapsto z^{-1}$. Thus, the equation $F_\mu = F_{\mu_1} \circ F_{\mu_2}$ that we want to prove is equivalent to

$$\tilde{G}_\mu = \tilde{G}_{\mu_1} \circ \tilde{G}_{\mu_2}.$$

Proof of Proposition 2

Let us write

$$\tilde{G}_\mu(z) = \langle \xi, (1/z - X)^{-1} \xi \rangle = \langle \xi, (1 - zX)^{-1} z \xi \rangle.$$

Note

$$1 - zX = 1 - z\widehat{X}_2 - z\widehat{X}_1 = (1 - z\widehat{X}_2)[1 - (1 - z\widehat{X}_2)^{-1} z\widehat{X}_1],$$

so

$$\begin{aligned}(1 - zX)^{-1} z &= [1 - (1 - z\widehat{X}_2)^{-1} z\widehat{X}_1]^{-1} (1 - z\widehat{X}_2)^{-1} z \\ &= \sum_{n=0}^{\infty} [(1 - z\widehat{X}_2)^{-1} z\widehat{X}_1]^n (1 - z\widehat{X}_2)^{-1} z\end{aligned}$$

Proof of Proposition 2

Thus, we get

$$\left\langle \xi, (1 - zX)^{-1} z\xi \right\rangle = \sum_{n=0}^{\infty} \left\langle \xi, [(1 - z\widehat{X}_2)^{-1} z\widehat{X}_1]^n (1 - z\widehat{X}_2)^{-1} z\xi \right\rangle.$$

Now $(1 - z\widehat{X}_2)^{-1}z$ is a function of $\widehat{X}_2$, and it is sandwiched between copies of $\widehat{X}_1$ and ξ. Thus, by monotone independence, we can replace each occurrence of $(1 - z\widehat{X}_2)^{-1}z$ by its expectation, which is

$$\left\langle \xi, (1 - z\widehat{X}_2)^{-1} z\xi \right\rangle = \left\langle \xi_2, (1 - zX_2)^{-1} z\xi_2 \right\rangle = \tilde{G}_{\mu_2}(z).$$

Proof of Proposition 2

Therefore,

$$\left\langle \xi, (1 - zX)^{-1} z\xi \right\rangle = \sum_{n=0}^{\infty} \left\langle \xi, [\tilde{G}_{\mu_2}(z)\widehat{X}_1]^n \tilde{G}_{\mu_2}(z)\xi \right\rangle$$
$$= \left\langle \xi, (1/\tilde{G}_{\mu_2}(z) - \widehat{X}_1)^{-1} \xi \right\rangle$$
$$= \tilde{G}_{\mu_1}(\tilde{G}_{\mu_2}(z)),$$

which is what we wanted to show.

Remark

If $F_\mu = F_{\mu_1} \circ F_{\mu_2}$, then μ is said to be the *monotone convolution* of μ_1 and μ_2, denoted $\mu = \mu_1 \rhd \mu_2$.

Outline

Background on Loewner chains

Definition

A *Loewner chain* is a family of functions $(F_t)_{t \geq 0}$ from $\mathbb{H}_+ \to \mathbb{H}_+$ such that

1. $F_t(z) - z$ is analytic in a neighborhood of ∞ with

$$F_t(z) = z - \frac{t}{z} + O\left(\frac{1}{z^2}\right)$$

2. For each $s \leq t$, there is another function $F_{s,t} : \mathbb{H}_+ \to \mathbb{H}_+$ with

$$F_s \circ F_{s,t} = F_t.$$

Background on Loewner chains

Some basic observations:

1. $F_{s,t}$ is uniquely determined by F_s and F_t.
2. Using Nevanlinna's theorem, F_t and $F_{s,t}$ can be written as the F-transforms of measures μ_t and $\mu_{s,t}$.
3. These measures all have mean zero. The variance of μ_t is t and the variance of $\mu_{s,t}$ is $t - s$.
4. In particular, $\mu_0 = \delta_0$ and $F_0 = \mathrm{id}$. Similarly, $F_{t,t} = \mathrm{id}$.
5. $\mu_s \rhd \mu_{s,t} = \mu_t$.

Background on Loewner chains

Theorem (Bauer [3])

If $(F_t)_{t \geq 0}$ is a Loewner chain, then $\partial_t F_t$ exists for all z for a.e. t, and there is a unique family of probability measures $(\sigma_t)_{t \geq 0}$ (which depend measurably on t and have uniformly bounded support for $t \leq T$) such that

$$\partial_t F_t(z) = -F_t'(z) G_{\sigma_t}(z).$$

Conversely, given such a family of measures $(\sigma_t)_{t \geq 0}$, there is a unique Loewner chain $(F_t)_{t \geq 0}$ satisfying the equation.

The measures $(\sigma_t)_{t \geq 0}$ are called the *driving measures* for the Loewner chain. The theorem thus describes a bijection between the driving measures $(\sigma_t)_{t \geq 0}$ and the measures $(\mu_t)_{t \geq 0}$ with $F_t = F_{\mu_t}$, in a similar spirit to the correspondence between σ and μ in Proposition 1 (restricting to the case where $b = 0$).

Remarks on the proof

Actually, Proposition 1 plays a role in obtaining the measures $(\sigma_t)_{t \geq 0}$ from F_t. By this proposition,

$$F_{s,t}(z) = z - G_{\tau_{s,t}}(z),$$

for some measure $\tau_{s,t}$ with total mass $t - s$. The measures σ_t are obtained as

$$\sigma_t = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \tau_{t,t+\epsilon} \text{for a.e. } t.$$

The differential equation derives from

$$\begin{aligned} F_{t+\epsilon}(z) - F_t(z) &= F_t(F_{t,t+\epsilon}(z)) - F_t(z) \\ &\approx F_t(z - \epsilon G_{\sigma_t}(z)) - F_t(z) \\ &\approx F_t'(z) \epsilon G_{\sigma_t}(z). \end{aligned}$$

Operator models for Loewner chains

How do we obtain the Loewner chain $(F_t)_{t \geq 0}$ from the measures $(\sigma_t)_{t \geq 0}$? Bauer did this by solving the equation through Picard iteration.

But Bauer also knew that $F_t = F_{\mu_t}$ can be represented in terms of the spectral measures of operators, so that Loewner chains should be connected to non-commutative probability theory [4]. Later, Schleißinger made the connection between Loewner chains and monotone independence explicit [5].

I looked for a natural way to build operators X_t out of the measures which produce the Loewner chain, that would also clearly show the relationship with monotone independence.

Operator models for Loewner chains

First, let's package all the measures σ_t into a single measure σ on $\mathbb{R} \times [0, +\infty)$ given by the disintegration

$$d\sigma(x, t) = d\sigma_t(x)\, dt,$$

that is, for $f \geq 0$,

$$\int_{\mathbb{R}\times[0,+\infty)} f(x, t)\, d\sigma(x, t) = \int_{[0,+\infty)} \int_{\mathbb{R}} f(x, t)\, d\sigma_t(x)\, dt.$$

Operator models for Loewner chains

Now let $\sigma^{\otimes n}$ be the product of n copies of σ on $(\mathbb{R} \times [0, +\infty))^{\times n}$. Then let σ_n be the restriction of $\sigma^{\otimes n}$ to

$$E_n = \{(x_1, \dots, x_n, t_1, \dots, t_n) : t_1 \geq t_2 \geq \dots \geq t_n \geq 0\}.$$

We set $\mathcal{H}_n = L^2(\sigma_n)$, and

$$\mathcal{H} = \mathbb{C}\xi \oplus \bigoplus_{n \geq 1} \mathcal{H}_n.$$

For simplicity, denote $\mathcal{H}_0 = \mathbb{C}\xi$. This $\mathcal{H}$ is a type of *Fock space*.

Operator models for Loewner chains

Like in Proposition 1, the operators X_t are built out of four pieces (well, actually only three since the mean of μ_t is zero).

First, for any function in $\zeta \in L^2(\sigma)$, there is a *creation operator* $\ell(\zeta) : \mathcal{H} \to \mathcal{H}$ which maps each $\mathcal{H}_n$ into $\mathcal{H}_{n+1}$ by

$$\ell(\zeta) f = (\zeta \otimes f)|_{E_{n+1}}.$$

Here $f \in L^2(\sigma^{\otimes n})$ supported in E_n and $\zeta \otimes f \in L^2(\sigma^{\otimes(n+1)})$, so we can restrict it to E_{n+1} to get a function in $L^2(\sigma_{n+1})$.

Similarly, for $\phi \in L^\infty(\sigma)$, we can define a *multiplication operator* $\mathfrak{m}(\phi)$ which multiplies a function f in $\mathcal{H}_n$ by $\phi \otimes 1^{\otimes(n-1)}$, so that

$$[\mathfrak{m}(\phi) f](x_1, \dots, x_n, t_1, \dots, t_n) = \phi(x_1, t_1) f(x_1, \dots, x_n, t_1, \dots, t_n).$$

The operator $\mathfrak{m}(\phi)$ is defined to act by zero on the subspace $\mathcal{H}_0$.

Operator models for Loewner chains

Theorem (J. 2017) [1]

For $t \geq 0$, define

$$X_t = \ell(1 \otimes \chi_{[0,t]}) + \ell(1 \otimes \chi_{[0,t]})^* + \mathfrak{m}(\mathrm{id}_{\mathbb{R}} \otimes \chi_{[0,t]}),$$

where $1 \otimes \chi_{[0,t]}$ is the function $(x,s) \mapsto \chi_{[0,t]}(s)$ in $L^2(\sigma)$ and $\mathrm{id}_{\mathbb{R}} \otimes \chi_{[0,t]}$ is the function $(x,s) \mapsto x\chi_{[0,t]}(s)$ in $L^2(\sigma)$.

Then the spectral measure μ_t associated to X_t and the vector ξ satisfies the Loewner equation

$$\partial_t F_{\mu_t} = -F'_{\mu_t} \cdot G_{\sigma_t}.$$

Sketch of proof

The first step is to show that F_{μ_t} forms a Loewner chain. To accomplish this, we use the operator

$$X_{s,t} = \ell(1 \otimes \chi_{[s,t]}) + \ell(1 \otimes \chi_{[s,t]})^* + \mathfrak{m}(\mathrm{id}_{\mathbb{R}} \otimes \chi_{[s,t]}).$$

One can show that X_s and $X_{s,t}$ are monotone independent — and in fact, this can be done by decomposing $\mathcal{H}$ into a tensor product as in Proposition 2, and expressing $X_s = Y \otimes P$ and $X_{s,t} = 1 \otimes Z$ for certain operators Y and Z.

Thus, letting $\mu_{s,t}$ be the measure associated to $X_{s,t}$, we get $\mu_t = \mu_s \rhd \mu_{s,t}$, or $F_{\mu_t} = F_{\mu_s} \circ F_{\mu_{s,t}}$.

Sketch of proof

Then we have to check that it satisfies the Loewner equation for the given driving measures $(\sigma_t)_{t \geq 0}$. Recall that $\tau_{s,t}$ is the measure given by

$$F_{s,t}(z) = z - G_{\tau_{s,t}}(z).$$

We need to show that $\sigma_t = \lim_{\epsilon \to 0^+} (1/\epsilon) \tau_{t,t+\epsilon}$.

Let $P = |\xi\rangle\langle\xi|$ and $Q = 1 - P$. By Proposition 1, $\tau_{t,t+\epsilon}$ is the spectral measure of $QX_{t,t+\epsilon}Q$ with respect to the vector $QX_{t,t+\epsilon}\xi$.

This vector is exactly $1 \otimes \chi_{[t,t+\epsilon]}$ in $L^2(\sigma) = \mathcal{H}_1 \subseteq \mathcal{H}$. The norm squared of this vector is ϵ.

Sketch of proof

One can check that the operator norm $\|\ell(\zeta)\| = \|\zeta\|$. This implies that $\|\ell(1 \otimes \chi_{[t,t+\epsilon]})\| = \epsilon^{1/2}$. Hence,

$$QX_{t,t+\epsilon}Q = Q\mathfrak{m}(\mathrm{id}_{\mathbb{R}} \otimes \chi_{[t,t+\epsilon)})Q + O(\epsilon^{1/2}).$$

We want the distribution of this operator with respect to a vector of norm squared ϵ. So up to an error of $O(\epsilon^{3/2})$, it is the same as the distribution of $Q\mathfrak{m}(\mathrm{id}_{\mathbb{R}} \otimes \chi_{[t,t+\epsilon)})Q$ for the vector $1 \otimes \chi_{[t,t+\epsilon]}$. That turns out to be $\int_t^{t+\epsilon} \sigma_s \, ds$, which will be asymptotically like $\epsilon\sigma_t$ as $\epsilon \to 0$ for almost every t.

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