# Similarity and Projections

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Suppose that

- $L_\theta$ is the line spanned by $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$,
- $R_\theta$ is the rotation by angle $\theta$.
- $A_\theta$ is the matrix of the orthogonal projection onto $L_\theta$.
- $B_\theta$ is the matrix of reflection across $L_\theta$.

**Your mission:**

1. Show that
$$A_\theta = R_\theta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R_{-\theta}.$$

2. Explain what this statement means in terms of two matrices being similar.

3. Conclude that all the $A_\theta$ matrices are similar to each other. (See Theorem 3.4.6.)

4. Either geometrically or by direct computation, verify that
$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = R_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = R_\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

5. By combining the formulas from parts (1) and (4), verify that
$$A_\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad A_\theta \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = 0.$$

   (You already knew geometrically that this should happen. The point of the question is to see how it relates to similarity.)

6. Now let's do reflections: Fill in the blank with a matrix:
$$B_\theta = R_\theta (??) R_{-\theta}.$$

7. Fill in the blanks with scalars:

$$B_\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = (??) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad B_\theta \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = (??) \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

8. Draw a coordinate grid starting with the basis vectors

$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

   What do the projection and reflection matrices *do* to this coordinate grid?

9. Generalize: Suppose that

$$M = S \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} S^{-1},$$

   and that $\vec{v}_1$ and $\vec{v}_2$ are the columns of $S$. What are $M\vec{v}_1$ and $M\vec{v}_2$?

**Problems for people with extra time:**

1. Show that $R_\theta\vec{v} \cdot R_\theta\vec{w} = \vec{v} \cdot \vec{w}$ and that $B_\theta\vec{v} \cdot B_\theta\vec{w} = \vec{v} \cdot \vec{w}$. What does this mean geometrically?

2. Show that $A_\theta\vec{v} \cdot \vec{w} = \vec{v} \cdot A_\theta\vec{w}$ and that $B_\theta\vec{v} \cdot \vec{w} = \vec{v} \cdot B_\theta\vec{w}$.

3. Suppose that $AB = A + B$. Let $A' = SAS^{-1}$ and $B' = SBS^{-1}$. Show that $A'B' = A' + B'$. (All these matrices are assumed to be square $n \times n$.)

4. Suppose that $A$ is invertible. Show that $AB$ and $BA$ are similar.

5. The matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

   represents reflection across some line. What is it? *Hint: It is not $L_\theta$.*