Operator-Valued Non-Commutative Probability

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Contents

Preface

Historical Background

Non-commutative probability studies random variables which do not commute with each other. While classical probability takes measure theory as its foundation, non-commutative probability is formulated in terms of an algebra $\mathcal{B}$ of operators on a Hilbert space. The expectation is replaced by a state, a type of linear map $E : \mathcal{B} \to \mathbb{C}$.

In the non-commutative world, there are several types of independence. Free independence was discovered first in the 1980's and 1990's in the work of Dan Voiculescu. Free probability provided a surprising link between von Neumann algebras and random matrix theory. There is also a striking analogy between free probability and classical probability, including free versions of conditional expectations, the central limit theorem, non-commutative derivatives, a free Stein's method, free entropy, free Lèvy processes.

The role of the Fourier transform in manipulating the laws of random variables was played by certain complex-analytic functions related to the Cauchy-Stieltjes transform of a measure. In particular, the R-transform of a law μ is an analytic function near the origin which is additive when we add freely independent random variables. The power series coefficients of the R-transform are known as free cumulants, and the moments and cumulants of a law are related by a combinatorial formula involving non-crossing partitions, due to Roland Speicher.

Later, several other types of independence were discovered, notably Boolean and monotone independence (as well as its mirror image, anti-monotone independence). Many of the tools and results from free probability had Boolean and monotone analogues, including analytic transforms, moment-cumulant formulas, the central limit theorem, processes with independent increments. In particular, there is a bijection between Levy processes for classical, free, Boolean, monotone, and anti-monotone independence, due to Bercovici and Pata in the free/classical/Boolean case.

Operator-valued non-commutative probability is a further generalization of non-commutative probability, in which the expectation is not scalar-valued, but rather takes values in a C^*-algebra $\mathcal{A}$. One of the main motivations was that if $\mathcal{A}$ is a subalgebra of a tracial von Neumann algebra $(\mathcal{B}, \tau)$, then there is a conditional expectation $\mathcal{B} \to \mathcal{A}$, which can be thought of as an $\mathcal{A}$-valued expectation and has many of the same properties as the scalar-valued expectation τ. Furthermore, conditional independence can be thought of simply as an $\mathcal{A}$-valued version of independence.

Thus, in the operator-valued theory we take the additional complexity of conditioning and remove it at the cost of enlarging the algebra of scalars. Many other types of complexity can be absorbed into the algebra $\mathcal{A}$ in this way. For example, the law of a tuple $X_1, \dots, X_n$ over $\mathcal{A}$ can be represented as the $M_n(\mathcal{A})$-valued law of the diagonal matrix $X_1 \oplus \dots \oplus X_n$. A non-commutative polynomial over $\mathcal{A}$ can be reprsented as a monomial over $M_n(\mathcal{A})$. The resolvent of a polynomial in $X_1, \dots, X_n$ can be represented as the corner of a matrix-valued resolvent

$(z - \widehat{X})^{-1}$ where z is scalar matrix and $\widehat{X}$ is a matrix with entries that are affine in X_1, …, X_n.

Motivated by these examples, mathematicians began to develop operator-valued non-commutative probability along the same lines as scalar-valued non-commutative probability. They adapted each type of independence to the operator-valued setting as well as analytic transforms, cumulants, the central limit theorem, and Levy processes. A crucial difference is that in the $\mathcal{A}$-valued setting, the notions of positivity for laws and analyticity for the various transforms associated to a law need to take into account matrix amplification. This means, roughly speaking, that anything we write down should make sense in $M_n(\mathcal{A})$ just as well as it does in $\mathcal{A}$.

Scope and Approach

These notes will study the mathematical theory of operator-valued non-commutative probability. We will focus on the properties of independent random variables for four types of independence (free, Boolean, monotone, and anti-monotone). The end goals are the central limit theorem and the theory of processes with independent increments (including a generalized Bercovici-Pata bijection). We reach these goals using primarily three tools:

1. Construction of $\mathcal{A}$-valued Hilbert spaces on which the random variables act.
2. Analytic transforms associated to an $\mathcal{A}$-valued law.
3. Combinatorial formulas for moments.

Unlike a journal article that presents new results, notes have the ability to be self-contained and systematic. With the benefit of hindsight, we can distill the results of many articles into a unified framework, establish consistent notation, and optimize the proofs of fundamental results.

For example, here we will develop the analogy between the four types of independence systematically, proving results about all four types in parallel. This has not been done in a journal article due to length limitations and because the four types of independence studied here were discovered (and adapted to the operator-valued setting) at different times.

Moreover, a good analytic characterization of operator-valued Cauchy-Stieltjes transforms was not available until 2013 (due to Williams). This means that many of the papers that initially developed operator-valued independence did not have this result available at the time of writing, and thus they had to rely less on analytic tools. But now, using the newer analytic theory, we can offer a cleaner presentation of some of the older results.

In writing a self-contained exposition, it is necessary to restrict the scope. We have chosen to leave out other notions of independence (including classical independence) in favor of a stronger analogy between the four types included. Moreover, we will not explore the connections with physics or random matrix theory. And, apart from the introduction and exercises, we will work in the abstract setting of operator-valued probability, without much reference to the specific examples which motivated the theory. Rather, our emphasis will be on the analogies with classical probability theory and complex analysis.

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Chapter 1

Setup of the $\mathcal{A}$-Valued Theory

1.1 Background on C^*-algebras

As background, we recall some fundamentals of the theory of C^*-algebras. We do not give proofs for many of the statements. We refer to Blackadar [Bla06, Chapter II] for an encyclopedic list of results, proof sketches, and references.

C^*-algebras and $*$-homomorphisms

Definition 1.1.1. A $*$-*algebra* over $\mathbb{C}$ is an algebra over $\mathbb{C}$ together with a map $a \mapsto a^*$ such that $(a^*)^* = a$, the $*$ operation is conjugate-linear, and $(ab)^* = b^* a^*$. If $\mathcal{A}$ and $\mathcal{B}$ are $*$-algebras, then a $*$-homomorphism $\rho : \mathcal{A} \to \mathcal{B}$ is a homomorphism such that $\rho(a^*) = \rho(a)^*$.

Definition 1.1.2. A *(unital) C^*-algebra* is a unital $*$-algebra $\mathcal{A}$ over $\mathbb{C}$ together with a norm $\|\cdot\|$ such that

1. $(\mathcal{A}, \|\cdot\|)$ is a Banach space.
2. $\|ab\| \leq \|a\|\|b\|$.
3. $\|a^*a\| = \|a\|^2$.

Theorem 1.1.3. *Let $\mathcal{H}$ be a Hilbert space. If $\mathcal{A}$ is a subalgebra of $B(\mathcal{H})$ which is closed under adjoints and closed in operator norm, then $\mathcal{A}$ is a C^*-algebra, where the $*$-operation is the adjoint and the norm is the operator norm. Conversely, every C^*-algebra is isometrically $*$-isomorphic to such a C^*-algebra of operators on a Hilbert space.*

Proposition 1.1.4. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras.*

1. *If $\rho : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism, then $\|\rho(a)\| \leq \|a\|$ for every $a \in \mathcal{A}$.*
2. *If $\rho : \mathcal{A} \to \mathcal{B}$ is an injective $*$-homomorphism, then $\|\rho(a)\| = \|a\|$ for every $a \in \mathcal{A}$.*
3. *If $\mathcal{A}$ is a C^*-algebra, then there is only one norm on $\mathcal{A}$ which satisfies the C^*-algebra conditions.*

Positivity and States

Definition 1.1.5. An element a of a C^*-algebra $\mathcal{A}$ is said to be *positive* if a can be written as x^*x for some $x \in \mathcal{A}$. We also write this condition as $a \geq 0$. Furthermore, we write $a \geq b$ if $a - b \geq 0$.

Definition 1.1.6. A linear functional $\phi \in \mathcal{A}^*$ is *positive* if $a \geq 0$ implies $\phi(a) \geq 0$.

Definition 1.1.7. A *state* on a C^*-algebra $\mathcal{A}$ is a positive linear functional with $\phi(1) = 1$. We denote the set of states by $S(\mathcal{A})$.

Proposition 1.1.8. *Let $\mathcal{A}$ be a C^*-algebra.*

1. *Suppose that $\mathcal{A}$ is a C^*-algebra acting on a Hilbert space $\mathcal{H}$. An element $a \in \mathcal{A}$ is positive if and only if a is a positive operator on $\mathcal{H}$.*

2. *If ϕ is a positive linear functional, then $\|\phi\|_{\mathcal{A}^*} = |\phi(1)|$. In particular, the norm of a state is 1.*

3. *If $a \in \mathcal{A}$ is self-adjoint, then*
$$\|a\| = \sup_{\phi \in S(\mathcal{A})} |\phi(a)|.$$

4. *If $a \in \mathcal{A}$, then a is self-adjoint if and only if $\phi(a)$ is real for every state ϕ.*

5. *If $a \in \mathcal{A}$, then we have $a \geq 0$ if and only if $\phi(a) \geq 0$ for every state ϕ.*

The GNS Construction

Given a state ϕ on a C^*-algebra $\mathcal{A}$, one can define a sesquilinear form on $\mathcal{A}$ by $\langle a, b \rangle_\phi = \phi(a^*b)$. This form is nonnegative definite, and hence it satisfies the Cauchy-Schwarz inequality. If $\mathcal{K}_\phi = \{a : \phi(a^*a) = 0\}$, then the completion of $\mathcal{H}/\mathcal{K}_\phi$ with respect to $\|a\|_\phi = \phi(a^*a)^{1/2}$ is a Hilbert space, which we denote by $L^2(\mathcal{A}, \phi)$.

Moreover, every $a \in \mathcal{A}$ defines a bounded operator on $L^2(\mathcal{A}, \phi)$ by left multiplication. Indeed, because $a \mapsto \phi(b^*ab)$ is a positive functional and $\|a\|^2 - a^*a \geq 0$, we have $\|ab\|_\phi^2 = \phi(b^*a^*ab) \leq \|a\|^2\phi(b^*b) = \|a\|^2\|b\|_\phi^2$. Thus, the multiplication action of a is well-defined on the completed quotient $L^2(\mathcal{A}, \phi)$.

Therefore, there is a $*$-homomorphism $\pi_\phi : \mathcal{A} \to B(L^2(\mathcal{A}, \phi))$ given by $\pi_\alpha(a)[b] = [ab]$, where $[b]$ is the equivalence class of b in the completed quotient. This is called the *Gelfand-Naimark-Segal representation* of $\mathcal{A}$ on $L^2(\mathcal{A}, \phi)$. Furthermore, as a consequence of Proposition 1.1.8 (3), we have the following representation of $\mathcal{A}$.

Theorem 1.1.9. *Let $\mathcal{H} = \bigoplus_{\phi \in S(\mathcal{A})} L^2(\mathcal{A}, \phi)$, and let $\pi : \mathcal{A} \to B(\mathcal{H})$ be the direct sum of the GNS representations π_α. Then π is an isometric $*$-isomorphism.*

This construction is the basis of the fact that every C^*-algebra can be represented concretely on a Hilbert space.

Matrices over a C^*-algebra

Suppose that $\mathcal{A}$ is a C^*-algebra and let us realize $\mathcal{A}$ as an algebra of operators on the Hilbert space $\mathcal{H}$ as in Theorem 1.1.3. Then a matrix $x \in M_{n\times m}(\mathcal{A}) = \mathcal{A} \otimes M_{n\times m}(\mathbb{C})$ can be viewed as an operator $\mathcal{H}^n \to \mathcal{H}^m$, and we denote by $\|x\|$ its operator norm. Note that $M_{n\times m}(\mathcal{A})$ is already complete in the operator norm.

In particular, $M_n(\mathcal{A})$ is a C^*-algebra. Moreover, Proposition 1.1.4 (3) implies that $M_n(\mathcal{A})$ has a unique norm and thus the norm is independent of our choice of representation for $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. Furthermore, the norm on $M_{n\times m}(\mathcal{A})$ is also independent of the representation because if $x \in M_{n\times m}(\mathcal{A})$ then the operator norm satisfies $\|x\|^2 = \|x^*x\|$, and $x^*x \in M_m(\mathcal{A})$ hence $\|x^*x\|$ is independent of the choice of representation.

Furthermore, there is a coordinate-free characterization of positivity in $M_n(\mathcal{A})$ in terms of positivity in $\mathcal{A}$.

Lemma 1.1.10. *Let $A \in M_n(\mathcal{A})$. Then the following are equivalent:*

1. *$A \geq 0$ in $\mathcal{A}$.*
2. *For every $v \in M_{1\times n}(\mathcal{A})$, we have $v^*Av \geq 0$ in $\mathcal{A}$.*

Proof. As in Theorem 1.1.9, we can represent $\mathcal{A}$ as a concrete C^*-algebra of operators on $\mathcal{H} := \bigoplus_{\phi\in S(\mathcal{A})} \mathcal{H}_\phi$, where $\mathcal{H}_\phi = L^2(\mathcal{A}, \phi)$.

We can view A as an operator $\mathcal{H}^n \to \mathcal{H}^n$ and v as an operator $\mathcal{H} \to \mathcal{H}^n$. If $A \geq 0$, then v^*Av is positive by the basic theory of operators on Hilbert space, and hence $v^*Av \geq 0$ in $\mathcal{A}$.

Conversely, suppose that (2) holds. Observe that

$$\mathcal{H}^n = \bigoplus_{\phi\in S(\mathcal{A})} \mathcal{H}_\phi^n,$$

and the action of A on $\mathcal{H}^n$ is the direct sum of its actions on each $\mathcal{H}_\phi^n$. So it suffices to show that $A|_{\mathcal{H}_\phi^n}$ is positive for each state ϕ. We know that for each $v \in M_{1\times n}(\mathcal{A}) \cong \mathcal{A}^n$, we have

$$\phi(v^*Av) \geq 0.$$

Let $[v]$ denote the vector $([v_1], \dots, [v_n])$ as an equivalence class in $\mathcal{H}_\phi^n$. Then $\langle [v], A[v]\rangle \geq 0$. Such vectors $[v]$ are dense in $\mathcal{H}_\phi$ by construction and hence $A|_{\mathcal{H}_\phi} \geq 0$ as desired. □

1.2 Right Hilbert $\mathcal{A}$-modules

We begin with the $\mathcal{A}$-valued analogue of a Hilbert space. Right Hilbert $\mathcal{A}$-modules were introduced by Kaplansky [Kap53], Paschke [Pas73], and Rieffel [Rie74]. For further detail, see [Lan95]. A list of theorems and references can be found in [Bla06, §II.7].

Definition, Inner Products, Completions

Definition 1.2.1. Let $\mathcal{H}$ be a right $\mathcal{A}$-module. Then an *$\mathcal{A}$-valued pre-inner product* on $\mathcal{H}$ is a map $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$ such that

1. *Right $\mathcal{A}$-linearity:* We have

$$\langle \xi, \zeta_1 a_1 + \zeta_2 a_2\rangle = \langle \xi, \zeta_1\rangle a_1 + \langle \xi, \zeta_2\rangle a_2.$$

for ξ, ζ_1, $\zeta_2 \in \mathcal{H}$ and a_1, $a_2 \in \mathcal{A}$.

2. *Symmetry:* We have $\langle \xi, \zeta \rangle^* = \langle \zeta, \xi \rangle$.

3. *Nonnegativity:* $\langle \xi, \xi \rangle \geq 0$ in $\mathcal{A}$ for every $\xi \in \mathcal{H}$.

If in addition, $\langle \xi, \xi \rangle = 0$ implies that $\xi = 0$, then we say $\langle \cdot, \cdot \rangle$ is an *$\mathcal{A}$-valued inner product.*

Observation 1.2.2. *An $\mathcal{A}$-valued pre-inner product satisfies $\langle \zeta_1 a_1 + \zeta_2 a_2, \xi \rangle = a_1^* \langle \zeta_1, \xi \rangle + a_2^* \langle \zeta_2, \xi \rangle$.*

Lemma 1.2.3. *Let $\mathcal{H}$ be a right $\mathcal{A}$-module with an $\mathcal{A}$-valued pre-inner product, and denote $\|\xi\| = \|\langle \xi, \xi \rangle\|^{1/2}$.*

1. $\langle \zeta, \xi \rangle \langle \xi, \zeta \rangle \leq \|\xi\|^2 \langle \zeta, \zeta \rangle$.

2. $\|\langle \xi, \zeta \rangle\| \leq \|\xi\| \|\zeta\|$.

3. $\|\xi\| = \|\langle \xi, \xi \rangle\|^{1/2}$ *defines a semi-norm on $\mathcal{H}$.*

4. $\|\xi\| = \sup_{\|\zeta\| \leq 1} \|\langle \xi, \zeta \rangle\|$.

Proof. Suppose that $\phi \in S(\mathcal{A})$. Then $\phi(\langle \xi, \zeta \rangle)$ is a scalar-valued pre-inner product and therefore satisfies the Cauchy-Schwarz inequality. Thus, we have

$$\begin{aligned} \phi(\langle \zeta, \xi \rangle \langle \xi, \zeta \rangle) &= \phi(\langle \zeta, \xi \langle \xi, \zeta \rangle \rangle) \leq \phi(\langle \zeta, \zeta \rangle)^{1/2} \phi(\langle \xi \langle \xi, \zeta \rangle, \xi \langle \xi, \zeta \rangle \rangle)^{1/2} \\ &= \phi(\langle \zeta, \zeta \rangle)^{1/2} \phi(\langle \xi, \zeta \rangle^* \langle \xi, \xi \rangle \langle \xi, \zeta \rangle)^{1/2} \end{aligned}$$

Next, note that $a \mapsto \phi(\langle \xi, \zeta \rangle^* a \langle \xi, \zeta \rangle)$ is positive linear functional on $\mathcal{A}$ and therefore

$$|\phi(\langle \xi, \zeta \rangle^* \langle \xi, \xi \rangle \langle \xi, \zeta \rangle)| \leq \|\langle \xi, \xi \rangle\| \phi(\langle \xi, \zeta \rangle^* \langle \xi, \zeta \rangle) = \|\xi\|^2 \phi(\langle \xi, \zeta \rangle^* \langle \xi, \zeta \rangle).$$

Altogether,

$$\phi(\langle \zeta, \xi \rangle \langle \xi, \zeta \rangle) \leq \phi(\langle \zeta, \zeta \rangle)^{1/2} \|\xi\| \phi(\langle \zeta, \xi \rangle \langle \xi, \zeta \rangle)^{1/2}.$$

We cancel the term $\phi(\langle \zeta, \xi \rangle \langle \xi, \zeta \rangle)^{1/2}$ from both sides and then square the inequality to obtain

$$\phi(\langle \zeta, \xi \rangle \langle \xi, \zeta \rangle) \leq \|\xi\|^2 \phi(\langle \zeta, \zeta \rangle).$$

Because $\langle \zeta, \xi \rangle \langle \xi, \zeta \rangle$ and $\|\xi\|^2 \langle \zeta, \zeta \rangle$ are self-adjoint elements of $\mathcal{A}$ and this inequality holds for every state ϕ, we have

$$\langle \zeta, \xi \rangle \langle \xi, \zeta \rangle \leq \|\xi\|^2 \langle \zeta, \zeta \rangle,$$

so (1) is proved. Inequality (2) follows by taking the norm of both sides in $\mathcal{A}$ and then taking the square root.

The norm on $\mathcal{H}$ is clearly positive homogeneous. The triangle inequality holds because

$$\begin{aligned} \|\xi + \zeta\|^2 &= \|\langle \xi + \zeta, \xi + \zeta \rangle\| \\ &\leq \|\langle \xi, \xi \rangle\| + \|\langle \xi, \zeta \rangle\| + \|\langle \zeta, \xi \rangle\| + \|\langle \zeta, \zeta \rangle\| \\ &\leq (\|\xi\| + \|\zeta\|)^2. \end{aligned}$$

This proves (3). Moreover, (4) follows immediately from the Cauchy-Schwarz inequality (2). □

Definition 1.2.4. A *right Hilbert $\mathcal{A}$-module* is a right $\mathcal{A}$-module with an $\mathcal{A}$-valued pre-inner product such that $\mathcal{H}$ is a Banach space with respect to the semi-norm $\|\xi\| = \|\langle \xi, \xi \rangle\|^{1/2}$.

Lemma 1.2.5. *Let $\mathcal{H}$ be a right $\mathcal{A}$-module with an $\mathcal{A}$-valued pre-inner product. Define*

$$\mathcal{K} = \{\xi \in \mathcal{H} : \|\xi\| = 0\}.$$

Then $\langle \cdot, \cdot \rangle$ defines an inner product $\mathcal{H}/\mathcal{K}$, and the completion of $\mathcal{H}/\mathcal{K}$ with respect to the corresponding norm is a right Hilbert $\mathcal{A}$-module.

Proof. The Cauchy-Schwarz inequality implies that $\langle \cdot, \cdot \rangle$ yields a well-defined inner product on $\mathcal{H}/\mathcal{K}$. The right $\mathcal{A}$-action is bounded with respect to the norm of $\mathcal{H}$ since

$$\|\xi a\|^2 = \|\langle \xi a, \xi a \rangle\| = \|a^* \langle \xi, \xi \rangle a\| \leq \|\xi\|^2 \|a\|^2.$$

Thus, the right $\mathcal{A}$-action maps $\mathcal{K}$ into $\mathcal{K}$ and hence passes to a bounded action on the quotient. This in turn extends to the completion. The $\mathcal{A}$-valued inner product on $\mathcal{H}/\mathcal{K}$ extends to an $\mathcal{A}$-valued inner product on the completion because of the Cauchy-Schwarz inequality and the boundedness of the right $\mathcal{A}$-action. □

Orthogonality

Definition 1.2.6. If $\mathcal{H}$ is a right Hilbert $\mathcal{A}$-module, then we say that $\xi_1, \ldots, \xi_n \in \mathcal{H}$ are *orthogonal* if $\langle \xi_i, \xi_j \rangle = 0$ for $i \neq j$.

Unlike the scalar case, there is no reason why orthogonormal bases would exist in general. However, when we have orthogonal vectors, a version of the Pythagorean identity still holds

Observation 1.2.7. *If $\xi_1, \ldots, \xi_n$ are orthogonal, then*

$$\left\langle \sum_{j=1}^{n} \xi_j, \sum_{j=1}^{n} \xi_j \right\rangle = \sum_{j=1}^{n} \langle \xi_j, \xi_j \rangle,$$

and hence

$$\left\| \sum_{j=1}^{n} \xi_j \right\| \leq \left(\sum_{j=1} \|\xi_j\|^2 \right)^{1/2}.$$

Operators on Right Hilbert $\mathcal{A}$-modules

Definition 1.2.8. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be right Hilbert $\mathcal{A}$-modules. A linear map $T : \mathcal{H}_1 \to \mathcal{H}_2$ is *bounded* if

$$\|T\| := \sup_{\|h\| \leq 1} \|Th\| < +\infty.$$

We say that T is *right-$\mathcal{A}$-linear* if $(Th)a = T(ha)$ for each $a \in \mathcal{A}$.

The adjoint of a linear operator is defined the same way as in the scalar case, except that there is a no guarantee that an adjoint exists.

Definition 1.2.9. Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded right-$\mathcal{A}$-linear map between right Hilbert $\mathcal{A}$-modules. Then T is *adjointable* if there exists $T^* : \mathcal{H}_2 \to \mathcal{H}_1$ such that

$$\langle Th_1, h_2 \rangle_{\mathcal{H}_2} = \langle h_1, T^* h_2 \rangle_{\mathcal{H}_1}.$$

In this case, we say that T^* is an adjoint for T.

Proposition 1.2.10.

1. *If $T : \mathcal{H}_1 \to \mathcal{H}_2$ is adjointable, then the adjoint is unique.*
2. *If $T : \mathcal{H}_1 \to \mathcal{H}_2$ and $S : \mathcal{H}_2 \to \mathcal{H}_3$ are adjointable, then $(ST)^* = T^*S^*$.*
3. *If T is adjointable, then T^* is adjointable and $T^{**} = T$.*
4. $\|T^*T\| = \|T\|^2 = \|T^*\|^2$.

Proof. (1) Suppose that S and S' are two adjoints for T. Then for every h_1 and h_2, we have

$$\langle h_1, (S - S')h_2 \rangle = \langle Th_1, h_2 \rangle - \langle Th_1, h_2 \rangle = 0.$$

For each h_2, we can take $h_1 = (S - S')h_2$ to conclude that $Sh_2 = S'h_2$.

(2) Given that the adjoint is unique, this equality follows from the fact that

$$\langle STh_1, h_3 \rangle = \langle Th_1, S^*h_3 \rangle = \langle h_1, S^*T^*h_3 \rangle.$$

(3) Note that

$$\langle T^*h_2, h_1 \rangle = \langle h_1, T^*h_2 \rangle^* = \langle Th_1, h_2 \rangle^* = \langle h_2, Th_1 \rangle.$$

(4) Observe that

$$\begin{aligned} \|T\| = \sup_{\|h_1\| \le 1} \|Th_1\| &= \sup_{\|h_1\|, \|h_2\| \le 1} \|\langle Th_1, h_2 \rangle\| \\ &= \sup_{\|h_1\|, \|h_2\| \le 1} \|\langle h_1, T^*h_2 \rangle\| = \sup_{\|h_1\|, \|h_2\| \le 1} \|\langle T^*h_2, h_1 \rangle\| = \|T^*\|. \end{aligned}$$

Moreover, using the Cauchy-Schwarz inequality,

$$\|T^*T\| = \sup_{\|h_1\|, \|h_1'\| \le 1} \|\langle T^*Th_1, h_1' \rangle\| = \sup_{\|h_1\|, \|h_1'\| \le 1} \|\langle Th_1, Th_1' \rangle\| = \left(\sup_{\|h_1\| \le 1} \|Th_1\| \right)^2 = \|T\|^2.$$

□

Definition 1.2.11. We denote the $*$-algebra of bounded, adjointable, right-$\mathcal{A}$-linear operators $\mathcal{H} \to \mathcal{H}$ by $B(\mathcal{H})$.

1.3 Hilbert Bimodules

Now we introduce the $\mathcal{A}$-valued analogue of a representation of a C^*-algebra on a Hilbert space.

Definition 1.3.1. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. Then a *Hilbert $\mathcal{B}$-$\mathcal{A}$-bimodule* is a right Hilbert $\mathcal{A}$-module $\mathcal{H}$ together with $*$-homomorphism $\pi : \mathcal{B} \to B(\mathcal{H})$.

In this case, for $b \in \mathcal{B}$, we write $b\xi := \pi(b)\xi$, and thus view $\mathcal{H}$ as a $\mathcal{B}$-$\mathcal{A}$-bimodule. The left and right actions commute because by definition $B(\mathcal{H})$ consists of right-$\mathcal{A}$-linear operators.

A Hilbert $\mathcal{B}$-$\mathcal{A}$-bimodule can be thought of as a representation of a C^*-algebra $\mathcal{B}$ on an $\mathcal{A}$-valued Hilbert space. Of course, a Hilbert $\mathbb{C}$-$\mathcal{A}$-bimodule is equivalent to a right Hilbert $\mathcal{A}$-module.

Direct Sums

Given a family of Hilbert $\mathcal{B}$-$\mathcal{A}$-bimodules $\{\mathcal{H}_i\}_{i\in I}$, we define the *direct sum* $\bigoplus_{i\in I}\mathcal{H}_i$ as the completion of the algebraic direct sum with respect to the $\mathcal{A}$-valued inner product

$$\left\langle \sum_{i\in I}\xi_i, \sum_{i\in I}\zeta_i \right\rangle = \sum_{i\in I}\langle \xi_i, \zeta_i\rangle_{\mathcal{H}_i},$$

where $\sum_{i\in I}\xi_i$ and $\sum_{i\in I}\zeta_i$ are elements of the algebraic direct sum represented as sums of $\xi_i \in \mathcal{H}_i$ and $\zeta_i \in \mathcal{H}_i$ with only finitely many nonzero terms.

We must still verify that this definition makes sense. It is straightforward to check that this is an inner product, and therefore the completion is well-defined as a right Hilbert $\mathcal{A}$-module by Lemma 1.2.5. But it remains to show that left $\mathcal{B}$-action is bounded and extends to the completion. Let $b \in \mathcal{B}$ and let $\sum_{i\in I}\xi_i$ be in the algebraic direct sum. Then $\|b\|^2 - b^*b \geq 0$ in $\mathcal{B}$, and hence

$$\|b\|^2\langle \xi_i, \xi_i\rangle - \langle b\xi_i, b\xi_i\rangle = \langle \xi_i, (\|b\|^2 - b^*b)\xi_i\rangle \geq 0,$$

which implies that

$$\left\langle b\sum_{i\in I}\xi_i, b\sum_{i\in I}\xi_i \right\rangle = \sum_{i\in I}\langle b\xi_i, b\xi_i\rangle \leq \|b\|^2\sum_{i\in I}\langle \xi_i, \xi_i\rangle = \|b^2\|\left\langle \sum_{i\in I}\xi_i, \sum_{i\in I}\xi_i \right\rangle.$$

Therefore, the $\mathcal{B}$-action is bounded and so extends to the completion.

The direct sum operation is commutative and associative, up to natural isomorphism.

Tensor Products

Suppose we are given a Hilbert $\mathcal{C}$-$\mathcal{B}$-bimodule $\mathcal{K}$ and a Hilbert $\mathcal{B}$-$\mathcal{A}$-bimodule $\mathcal{H}$ and. Then we define the *tensor product* $\mathcal{K}\otimes_{\mathcal{B}}\mathcal{H}$ by equipping the algebraic tensor product with the pre-inner product

$$\langle \xi_1 \otimes \zeta_1, \xi_2 \otimes \zeta_2\rangle = \langle \zeta_1, \langle \xi_1, \xi_2\rangle\zeta_2\rangle$$

and then forming the completed quotient as in Lemma 1.2.5.

Let us expound the definition in more detail and verify that the construction makes sense. Let $\mathcal{V}$ be the algebraic tensor product of $\mathcal{K}$ and $\mathcal{H}$ over $\mathcal{B}$. That is, $\mathcal{V}$ is the vector space spanned by $\xi \otimes \zeta$, where $\xi \in \mathcal{K}$ and $\zeta \in \mathcal{H}$, modulo the span of vectors of the form

$$\xi \otimes (\zeta_1 + \zeta_2) - \xi \otimes \zeta_1 - \xi \otimes \zeta_2, \qquad (\xi_1 \otimes \xi_2) \otimes \zeta - \xi_1 \otimes \zeta - \xi_2 \otimes \zeta, \xi b \otimes \zeta - \xi \otimes b\zeta,$$

where $b \in \mathcal{B}$. Note that $\mathcal{V}$ is a $\mathcal{C}$-$\mathcal{A}$-bimodule with the actions given by

$$c(\xi \otimes \zeta) = c\xi \otimes \zeta, \qquad (\xi \otimes \zeta)a = \xi \otimes \zeta a.$$

We equip $\mathcal{V}$ with a $\mathcal{A}$-valued form $\langle\cdot,\cdot\rangle$ given by

$$\langle \xi_1 \otimes \zeta_1, \xi_2 \otimes \zeta_2\rangle = \langle \zeta_1, \langle \xi_1, \xi_2\rangle\zeta_2\rangle.$$

Observe that if we replace $\xi_j b \otimes \zeta_j$ with $\xi_j \otimes b\zeta_j$ for $j = 1$ or 2 and $b \in \mathcal{B}$, the result is unchanged due to the right $\mathcal{B}$-linearity of the inner product on $\mathcal{K}$; therefore, this $\mathcal{A}$-valeud form on $\mathcal{V}$ is well-defined. It is straighforward to check that this $\mathcal{A}$-valued form is right $\mathcal{A}$-linear and symmetric.

In order to check that this is nonnegative, consider a sum of simple tensors $\sum_{j=1}^n \xi_j \otimes \zeta_j$. Note that

$$\left\langle \sum_i \xi_i \otimes \zeta_i, \sum_j \xi_j \otimes \zeta_j \right\rangle = \sum_{i,j} \langle \zeta_i, \langle \xi_i, \xi_j \rangle \zeta_j \rangle = \langle \vec{\zeta}, X\vec{\zeta} \rangle_{\mathcal{H}^n},$$

where $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathcal{H}^n$ and X is the matrix $[\langle \xi_i, \xi_j \rangle]_{i,j}$ in $M_n(\mathcal{B})$. We claim that $X \geq 0$ in $M_n(\mathcal{B})$. This follows from Lemma 1.1.10 because for $v \in M_{n \times 1}(\mathcal{B})$, then

$$v^* X v = \sum_{i,j} \langle \xi_i v_i, \xi_j v_j \rangle = \left\langle \sum_i \xi_i v_i, \sum_j \xi_j v_j \right\rangle \geq 0.$$

Thus, X can be written as $B^* B$ for some $B \in M_n(\mathcal{B})$. Thus,

$$\langle \vec{\zeta}, X\vec{\zeta} \rangle_{\mathcal{H}^n} = \langle B\vec{\zeta}, B\vec{\zeta} \rangle_{\mathcal{H}^n} \geq 0.$$

This shows nonnegativity of the inner product.

Therefore, Lemma 1.2.5 shows that the completed quotient of $\mathcal{V}$ with respect to $\langle \cdot, \cdot \rangle$ is a well-defined right Hilbert $\mathcal{A}$-module $\mathcal{K} \otimes_{\mathcal{B}} \mathcal{H}$. Finally, we must verify that the left $\mathcal{C}$-action is well-defined. Let $c \in \mathcal{C}$. Then $\|c\|^2 - c^* c \geq 0$, so that $\|c\|^2 - c^* c = x^* x$ for some $x \in \mathcal{C}$. Thus, for a simple tensor $\sum_j \xi_j \otimes \zeta_j$, we have

$$\left\langle \sum_i \xi_i \otimes \zeta_i, (\|c\|^2 - c^* c) \sum_j \xi_j \otimes \zeta_j \right\rangle = \left\langle \sum_i x\xi_i \otimes \zeta_i, \sum_j x\xi_j \otimes \zeta_j \right\rangle \geq 0,$$

which implies that

$$\left\langle c \sum_i \xi_i \otimes \zeta_i, c \sum_j \xi_j \otimes \zeta_j \right\rangle \leq \|c\|^2 \left\langle \sum_i \xi_i \otimes \zeta_i, \sum_j \xi_j \otimes \zeta_j \right\rangle.$$

Hence, the action of c is bounded and thus passes to the completed quotient. Moreover, direct computation shows that the action of $\mathcal{C}$ is a $*$-homomorphism.

This shows that the tensor product is well-defined. Furthermore, it is straightforward to check that the tensor product is associative, that is, if $\mathcal{H}_j$ is an $\mathcal{A}_j$-$\mathcal{A}_{j-1}$-bimodule for $j = 1$, 2, 3, then

$$(\mathcal{H}_3 \otimes_{\mathcal{A}_2} \mathcal{H}_2) \otimes_{\mathcal{A}_1} \mathcal{H}_1 \cong \mathcal{H}_3 \otimes_{\mathcal{A}_2} (\mathcal{H}_2 \otimes_{\mathcal{A}_1} \mathcal{H}_1)$$

as a Hilbert $\mathcal{A}_3$-$\mathcal{A}_0$-bimodule. In particular, we can unambiguously write

$$\mathcal{H}_n \otimes_{\mathcal{A}_{n-1}} \cdots \otimes_{\mathcal{A}_1} \mathcal{H}_1$$

as a Hilbert $\mathcal{A}_n$-$\mathcal{A}_0$-bimodule when $\mathcal{H}_j$ is a HIlbert $\mathcal{A}_j$-$\mathcal{A}_{j-1}$ bimodule. Moreover, tensor products distribute over direct sums in the obvious way.

1.4 Completely Positive Maps and the GNS Construction

Now we will define the $\mathcal{A}$-valued analogue of positive linear functionals on an algebra $\mathcal{B}$ and the GNS construction. It turns out that positivity of a map $\sigma : \mathcal{B} \to \mathcal{A}$ is not a strong enough condition to make the GNS construction work. Rather, we need the notion of complete positivity. Complete positivity was first studied by Stinespring [Sti55], and the operator-valued GNS construction is closely related to the Stinespring dilation theorem and its extension by Kasparov [Kas80]. For further references, see [Bla06, §II.6.9-10, §II.7.5].

Definition 1.4.1. Let $\sigma : \mathcal{B} \to \mathcal{A}$ be a linear map. We denote by $\sigma^{(n)} : M_n(\mathcal{B}) \to M_n(\mathcal{A})$ the map given by applying σ entrywise. We say that σ is *completely positive* if $\sigma^{(n)}$ is positive for every n, that is, $\sigma^{(n)}(B^*B) \geq 0$ for every $B \in M_n(\mathcal{B})$.

Lemma 1.4.2. *Let $\mathcal{H}$ be an Hilbert $\mathcal{B}$-$\mathcal{A}$-bimodule and $\xi \in \mathcal{H}$. Then $\sigma(b) := \langle \xi, b\xi \rangle$ is a completely positive map $\mathcal{B} \to \mathcal{A}$.*

Proof. Choose a positive element B^*B in $M_n(\mathcal{B})$ and write $B = [b_{i,j}]$. By Lemma 1.1.10, to show that $\sigma^{(n)}(B^*B) \geq 0$, it suffices to show that for $v \in M_{n\times 1}(\mathcal{A})$, we have $v^*\sigma^{(n)}(B^*B)v \geq 0$. But

$$v^*\sigma^{(n)}(B^*B)v = \sum_{i,j} \langle \xi v_i, (B^*B)_{i,j}\xi v_j \rangle = \sum_{i,j,k} \langle B_{k,i}\xi v_i, B_{k,j}\xi v_j \rangle = \langle B(\xi v), B(\xi v) \rangle_{\mathcal{H}^n} \geq 0,$$

where $\xi v \in \mathcal{H}^n$ is the vector $(\xi v_1, \dots, \xi v_n)$ and B acts on $\mathcal{H}^n$ by matrix multiplication in the obvious way. □

Conversely, we will show that every completely positive map $\sigma : \mathcal{B} \to \mathcal{A}$ can be realized by a vector ξ in a Hilbert $\mathcal{B}$-$\mathcal{A}$-bimodule. We define a bimodule $\mathcal{B} \otimes_\sigma \mathcal{A}$ by equipping the algebraic tensor product $\mathcal{B} \otimes \mathcal{A}$ over $\mathbb{C}$ with the pre-inner product

$$\langle b_1 \otimes a_1, b_2 \otimes a_2 \rangle = a_1^*\sigma(b_1^*b_2)a_2.$$

This pre-inner product is clearly right $\mathcal{A}$-linear and symmetric. To show that it is nonnegative, consider a vector

$$\xi = \sum_{j=1}^n b_j \otimes a_j$$

and note that

$$\langle \xi, \xi \rangle = \sum_{i,j} a_i^*\sigma(b_i^*b_j)a_j.$$

The matrix $C = [b_i^*b_j]$ can be written in the form B^*B and hence is positive in $M_n(\mathcal{B})$. Therefore, by complete positivity of σ, the matrix $[\sigma(b_i^*b_j)]$ is positive in $M_n(\mathcal{A})$. Then by Lemma 1.1.10, $\sum_{i,j} a_i^*\sigma(b_i^*b_j)a_j \geq 0$ in $\mathcal{A}$. This shows nonnegativity of the pre-inner product.

Thus, by Lemma 1.2.5, we can define the completed quotient $\mathcal{B} \otimes_\sigma \mathcal{A}$ as a right Hilbert $\mathcal{A}$-module. Finally, we claim that the left multiplication action of $\mathcal{B}$ on $\mathcal{B} \otimes \mathcal{A}$ passes to the completed quotient. To do this, it suffices to show that this action is bounded with respect to $\langle \cdot, \cdot \rangle$.

The argument is the same as in the construction of the tensor product for bimodules. Given $b \in \mathcal{B}$, we have $\|b\|^2 - b^*b \geq 0$ and hence it can be written as x^*x for some $x \in \mathcal{B}$. Using complete positivity, one argues that $\langle c\xi, c\xi \rangle \geq 0$ whenever $\xi = \sum_{j=1}^n b_j \otimes a_j$. Thus, we conclude that $\|b\xi, b\xi\| \leq \|b\|^2 \langle \xi, \xi \rangle$. In summary, we have shown that the following definition makes sense.

Definition 1.4.3. Let $\sigma : \mathcal{B} \to \mathcal{A}$ be completely positive. We denote by $\mathcal{B} \otimes_\sigma \mathcal{A}$ the Hilbert $\mathcal{B}$-$\mathcal{A}$-bimodule defined as the completed quotient of the algebraic tensor product $\mathcal{B} \otimes \mathcal{A}$ over $\mathbb{C}$ with respect to the pre-inner product $\langle b_1 \otimes a_1, b_2 \otimes a_2 \rangle = a_1^*\sigma(b_1^*b_2)a_2$.

Moreover, a direct computation shows the following.

Lemma 1.4.4. *Let $\sigma : \mathcal{B} \to \mathcal{A}$ be completely positive. Let ξ be the vector $1 \otimes 1$ in $\mathcal{B} \otimes_\sigma \mathcal{A}$. Then $\sigma(b) = \langle \xi, b\xi \rangle$. In particular, a map σ is completely positive if and only if it can be expressed as $\sigma(b) = \langle \xi, b\xi \rangle$ for some ξ in a Hilbert $\mathcal{B}$-$\mathcal{A}$-bimodule $\mathcal{H}$.*

Finally, let us point out, that just as in the case of states, completely positive maps are automatically bounded (in fact, completely bounded).

Lemma 1.4.5. *Let $\sigma : \mathcal{B} \to \mathcal{A}$ be completely positive. If $B \in M_n(\mathcal{B})$, then $\|\sigma^{(n)}(B)\| \leq \|\sigma(1)\|\|B\|$.*

Proof. First, consider $b \in \mathcal{B}$ (for the case $n = 1$). Let $\mathcal{H} = \mathcal{B} \otimes_\sigma \mathcal{A}$ and $\xi = 1 \otimes 1$. Then by Cauchy-Schwarz,

$$\|\sigma(b)\| = \|\langle \xi, b\xi \rangle\| \leq \|\xi\|\|b\xi\| \leq \|b\|\|\xi\|^2 = \|b\|\|\sigma(1)\|.$$

For $n > 1$, note that if σ is completely positive, then $\sigma^{(n)}$ is also completely positive and hence by the preceding argument $\|\sigma^{(n)}(B)\| \leq \|B\|\|\sigma^{(n)}(1)\| = \|B\|\|\sigma(1)\|$. □

1.5 $\mathcal{A}$-valued Probability Spaces

Completely positive maps are the $\mathcal{A}$-valued analogue of positive linear functionals on C^*-algebras and measures on compact Hausdorff spaces. The analogue of a state or probability measure is a *$\mathcal{A}$-valued expectation*. While conditional expectations have a long history in C^*-algebra theory, the probabilistic point of view is due largely to Voiculescu [Voi95].

Definition 1.5.1. Let $\mathcal{A} \subseteq \mathcal{B}$ be unital C^*-algebras. An *$\mathcal{A}$-valued expectation* $E : \mathcal{B} \to \mathcal{A}$ is a unital positive $\mathcal{A}$-$\mathcal{A}$-bimodule map.

Remark 1.5.2. Complete positivity is automatic in this case. Indeed, if $\mathcal{A} \subseteq \mathcal{B}$ and $\Phi : \mathcal{B} \to \mathcal{A}$ is a positive $\mathcal{A}$-$\mathcal{A}$-bimodule map, then Φ is completely positive. To see this, consider a positive element $B^*B \in \mathcal{B}$ and given $v \in M_{1\times n}(\mathcal{A})$. Then we have

$$v^* E^{(n)}[B^*B]v = E[v^*B^*Bv] \geq 0$$

since $v^*B^*Bv \geq 0$ in $\mathcal{B}$ by Lemma 1.1.10. Since $v^*E^{(n)}[B^*B]v \geq 0$ for every v, Lemma 1.1.10 implies that $E^{(n)}[B^*B] \geq 0$.

This is also known in the literature as a conditional expectation $\mathcal{B} \to \mathcal{A}$. More explicitly, by "unital" we mean that $E[1] = 1$ and by "$\mathcal{A}$-$\mathcal{A}$-bimodule map," we mean that $E[ab] = aE[b]$ and $E[ba] = E[b]a$ for $b \in \mathcal{B}$ and $a \in \mathcal{A}$. The unital condition is the analogue of the normalization of a state or probability measure, and the $\mathcal{A}$-$\mathcal{A}$-bimodule property is the analogue of the property that $E[f(X)g(Y)|X] = f(X)E[g(Y)|X]$ in classical probability theory.

We have seen that a completely positive map $\mathcal{B} \to \mathcal{A}$ can always be represented $\langle \xi, b\xi \rangle$ for a vector ξ in a Hilbert $\mathcal{B}$-$\mathcal{A}$-bimodule. Let us now describe when $\langle \xi, b\xi \rangle$ is a conditional expectation.

Definition 1.5.3. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-$\mathcal{A}$-bimodule. A vector $\xi \in \mathcal{H}$ is said to be a *unit vector* if $\langle \xi, \xi \rangle = 1$. We say that ξ is *$\mathcal{A}$-central* if $a\xi = \xi a$ for $a \in \mathcal{A}$.

Lemma 1.5.4. *Let $\mathcal{A} \subseteq \mathcal{B}$ be a unital inclusion of C^*-algebras. If ξ is an $\mathcal{A}$-central vector in a Hilbert $\mathcal{B}$-$\mathcal{A}$-bimodule $\mathcal{H}$, then $E[b] := \langle \xi, b\xi \rangle$ is an $\mathcal{A}$-valued expectation. Conversely, if E is an $\mathcal{A}$-valued expectation, then the vector $\xi = 1 \otimes 1$ in $\mathcal{B} \otimes_E \mathcal{A}$ is an $\mathcal{A}$-central unit vector.*

Proof. Suppose that ξ is an $\mathcal{A}$-central unit vector in $\mathcal{H}$ and $E[b] = \langle \xi, b\xi \rangle$. We already know E is completely positive. Moreover, E is unital because ξ is a unit vector. Finally, to show that E is an $\mathcal{A}$-$\mathcal{A}$-bimodule map, observe that

$$E[ba] = \langle \xi, ba\xi \rangle = \langle \xi, b\xi a \rangle = \langle \xi, b\xi \rangle a = E[b]a.$$

On the other hand,

$$E[ab] = \langle \xi, ab\xi \rangle = \langle a^*\xi, b\xi \rangle = \langle \xi a^*, b\xi \rangle = a\langle \xi, b\xi \rangle = aE[b].$$

Conversely, suppose that E is an $\mathcal{A}$-valued expectation and let $\xi = 1 \otimes 1$ in $\mathcal{B} \otimes_E \mathcal{A}$. Clearly, ξ is unit vector because $\langle \xi, \xi \rangle = 1^* E[1^* 1] 1 = 1$. Moreover, we claim that $a \otimes 1 = 1 \otimes a$ in the completed quotient space $\mathcal{B} \otimes_E \mathcal{A}$. To see this, note that

$$\langle a \otimes 1 - 1 \otimes a, a \otimes 1 - 1 \otimes a \rangle = a^* E[1] a - a^* E[a] - E[a^*] a + E[a^* a] = 0.$$

It follows that

$$a\xi = a(1 \otimes 1) = a \otimes 1 = 1 \otimes a = (1 \otimes 1)a = \xi a,$$

so that ξ is $\mathcal{A}$-central. □

Remark 1.5.5. A careful examination of the proof for the converse direction shows that we did not use the full strength of the $\mathcal{A}$-$\mathcal{A}$-bimodule map assumption, only that $E|_{\mathcal{A}} = \mathrm{id}$. Therefore, we have shown that if E is completely positive and $E|_{\mathcal{A}} = \mathrm{id}$, then E is automatically an $\mathcal{A}$-$\mathcal{A}$-bimodule map.

Definition 1.5.6. An *$\mathcal{A}$-valued probability space* is a pair $(\mathcal{B}, E)$, where $\mathcal{B} \supseteq \mathcal{A}$ is a C^*-algebra and $E : \mathcal{B} \to \mathcal{A}$ is an $\mathcal{A}$-valued expectation, such that the representation of $\mathcal{B}$ on $\mathcal{B} \otimes_E \mathcal{A}$ is faithful, that is, the $*$-homomorphism $\mathcal{B} \to B(\mathcal{B} \otimes_E \mathcal{A})$ is injective.

This last condition that the representation is faithful is a type of non-degeneracy condition. For example, in the scalar-valued case where $\mathcal{B} = C(X)$ for a compact Hausdorff space and E is given by a probability measure P, the faithfulness condition says that (closed) support of P in X is all of X. In the operator-valued setting, this condition says intuitively that all information about the algebra $\mathcal{B}$ can be captured from the expectation E. This is a reasonable assumption because in non-commutative probability theory, we only care about aspects of the algebra that are observable from E.

1.6 $\mathcal{A}$-valued Laws and Generalized Laws

We now turn to the definition of $\mathcal{A}$-valued laws (and generalized laws). The results of this section are based on [Voi95], [PV13], [AW16].

As motivation, recall that if b is a real random variable on a probability space (X, P), then the *law of X* is the measure μ_b on $\mathbb{R}$ given by $\int f \, d\mu_b = E[f(b)]$. Similarly, if $\phi : \mathcal{B} \to \mathbb{C}$ is a state and $b \in \mathcal{B}$ is self-adjoint, then the *law of $\mathcal{B}$ with respect to ϕ* is the measure μ_b given by $\int f \, d\mu_b = \phi(f(b))$. In either case, if the measure μ_b is compactly supported, then it is uniquely specified by its moments $\int t^n \, d\mu_b(t) = \phi(b^n)$.

Now consider the case of a self-adjoint b in an $\mathcal{A}$-valued probability space $(\mathcal{B}, E)$. We want the law μ_b to encode the moments

$$E[a_0 b a_1 \dots b a_n]$$

for every $a_1, \dots, a_n \in \mathcal{A}$. Because there is no clear way to express these moments in terms of a measure, we will simply *define* the law of b as the sequence of moments, or equivalently a linear map from non-commutative polynomials in b with coefficients in $\mathcal{A}$ to the base algebra $\mathcal{A}$.

Definition 1.6.1. We denote by $\mathcal{A}\langle X\rangle$ the algebra of non-commutative polynomials in X with coefficients in $\mathcal{A}$, that is, $\mathcal{A}\langle X\rangle$ is the linear span of terms of the form $a_0 X a_1 X \dots a_{k-1} X a_k$. We endow $\mathcal{A}\langle X\rangle$ with the $*$-operation

$$(a_0 X a_1 X \dots a_{k-1} X a_k)^* = a_k^* X a_{k-1}^* \dots X a_1^* X a_0^*.$$

Definition 1.6.2. Let $(\mathcal{B}, E)$ be an $\mathcal{A}$-valued probability space and b a self-adjoint element of $\mathcal{B}$. The *law of* $\mathcal{B}$ is the map $\mu_b : \mathcal{A}\langle X\rangle \to \mathcal{A}$ given by $p(X) \mapsto E[p(b)]$.

In probability theory, it is a standard fact that every law μ (probability measure on $\mathbb{R}$) is the law of some random variable. Indeed, the random variable given by the x-coordinate on the probability space $(\mathbb{R}, \mu)$ will have the law μ. Thus, laws which arise from random variables are characterized abstractly as measures. In operator-valued non-commutative probability, there is also an abstract characterization of laws, and a way to explicitly construct a random variable which realizes a given law, a version of the GNS construction.

Definition 1.6.3. An $\mathcal{A}$-*valued law* is a linear map $\mu : \mathcal{A}\langle X\rangle \to \mathcal{A}$ such that

1. μ *is completely positive:* For any $P(X) \in M_n(\mathcal{A}\langle X\rangle)$ we have $\mu^{(n)}(P(X)^* P(X)) \geq 0$.
2. μ *is exponentially bounded:* There exist some $C > 0$ and $M > 0$ such that
$$\|\mu(a_0 X a_1 X \dots a_{k-1} X a_k)\| \leq C M^k \|a_0\| \dots \|a_k\| \text{ for all } a_1, \dots, a_k \in \mathcal{A}.$$
3. μ *is unital:* $\mu(1) = 1$.
4. μ *is an* $\mathcal{A}$-$\mathcal{A}$-*bimodule map:* $\mu(a p(X) a') = a \mu(p(X)) a'$ for $a, a' \in \mathcal{A}$.

Definition 1.6.4. Let $\mu : \mathcal{A}\langle X\rangle \to \mathcal{A}$. If $\|\mu(a_0 X a_1 X \dots a_{k-1} X a_k)\| \leq C M^k \|a_0\| \dots \|a_k\|$, then we say that M is an *exponential bound for* μ. Finally, we define the *radius of* μ as

$$\operatorname{rad}(\mu) := \inf\{M : M \text{ is an exponential bound for } \mu\}.$$

The following theorem is due to [Voi95] and is also proved in [PV13, Proposition 1.2].

Theorem 1.6.5. *For a map $\mu : \mathcal{A}\langle X\rangle \to \mathcal{A}$, the following are equivalent:*

1. *The map μ is an $\mathcal{A}$-valued law with exponential bound M.*
2. *There exists an $\mathcal{A}$-valued probability space $(\mathcal{B}, E)$ and a self-adjoint $\overline{X} \in \mathcal{B}$ such that $\mu = \mu_{\overline{X}}$ and $\|\overline{X}\| \leq M$.*

In particular, every law can be realized by an $\overline{X}$ with $\|\overline{X}\| = \operatorname{rad}(\mu)$.

To motivate the proof of (1) $\implies$ (2), let us rephrase the realization of a probability measure μ on $\mathbb{R}$ in terms of Hilbert spaces and operator algebras. We can construct the Hilbert space $L^2(\mathbb{R}, \mu)$ as the closure of $\mathbb{C}[x]$ with respect to the $L^2(\mu)$ norm. Let T be the operator of acts by multiplication on $L^2(\mathbb{R}, \mu)$, and let $C^*(T)$ be the C^*-algebra generated by T. Equip $C^*(T)$ with the expectation

$$E[f(T)] = \langle 1, f(T) 1\rangle_{L^2(\mathbb{R},\mu)} = \int_{\mathbb{R}} f(x)\, d\mu(x).$$

Then $(C^*(T), E)$ is a scalar-valued C^*-probability space and $T \in C^*(T)$ has the law μ.

Proof of Theorem 1.6.5. (1) $\implies$ (2). We define an $\mathcal{A}$-valued pre-inner-product on $\mathcal{A}\langle X\rangle \otimes \mathcal{A}$ by

$$\langle f_1(X) \otimes a_1, f_2(X) \otimes a_2 \rangle_\mu = a_1^* \mu(f_1(X)^* g_2(X)) a_2$$

and denote $\|f(X)\|_\mu = \|\langle f(X), f(X)\rangle_\mu\|^{1/2}$. Note that $\langle \cdot, \cdot \rangle_\mu$ is right $\mathcal{A}$-linear, is nonnegative, and satisfies $\langle g(X), f(X)\rangle = \langle f(X), g(X)\rangle^*$. Therefore, the Cauchy-Schwarz inequality holds and we can define the completed quotient with respect to this inner product. Denote this space by $\mathcal{A}\langle X\rangle \otimes_\mu \mathcal{A}$.

We want to define $\mathcal{B}$ as the C^*-algebra generated by left multiplication action of $\mathcal{A}\langle X\rangle$ on $\mathcal{A}\langle X\rangle \otimes_\mu \mathcal{A}$. If $f(X) \in \mathcal{A}\langle X\rangle$, then in order to show that the left multiplication action of $f(X)$ on $\mathcal{A}\langle X\rangle$ passes to a well-defined action on the completed quotient $\mathcal{A}\langle X\rangle \otimes_\mu \mathcal{A}$, it suffices to show that

$$\|f(X)g(X)\|_\mu \leq C\|g(X)\|_\mu$$

for some $C > 0$. In fact, because $\mathcal{A}\langle X\rangle$ is the algebra generated by $\mathcal{A}$ and X, it suffices to check the case where $f(X)$ is either some $a \in \mathcal{A}$ or X.

First, the argument that multiplication by a is well-defined on $\mathcal{A}\langle X\rangle \otimes_\mu \mathcal{A}$ is exactly the same as the argument for the GNS representation in Definition 1.4.3.

Second, we claim that if M is an exponential bound for μ, then left multiplication by X defines an operator $\rho(X)$ on $L^2(\mathcal{A}\langle X\rangle, \mu)$ with $\|\rho(X)\| \leq M$. The idea is to show that $R^2 - X^2$ is a positive operator for $R > M$. Unlike the case in Definition 1.4.3, we cannot express $R^2 - X^2$ as $\psi(X)^*\psi(X)$ in $\mathcal{A}\langle X\rangle$. We will fix this problem by looking at a certain analytic completion of $\mathcal{A}\langle X\rangle$ in which the power series representation of the function $\psi(t) = \sqrt{R^2 - t^2}$ will converge.

For a monomial $a_0 X a_1 \dots X a_k$, we denote

$$\mathfrak{p}(a_0 X a_1 \dots X a_k) = M^k \|a_0\| \dots \|a_k\|.$$

Then for $f(X) \in \mathcal{A}\langle X\rangle$, we define

$$\|f(X)\|_M = \inf \left\{ \sum_{j=1}^n \mathfrak{p}(f_j) : f_j \text{ monomials and } f = \sum_{j=1}^n f_j \right\}.$$

Let $\mathcal{A}\langle X\rangle_M$ be the completion of $\mathcal{A}\langle X\rangle$ in this norm. One checks easily that

$$\|f(X)g(X)\|_M \leq \|f(X)\|_{\mathfrak{p}} \|g(X)\|_M,$$

and this inequality extends to the completion, which makes $\mathcal{A}\langle X\rangle_M$ a Banach algebra. Similarly, the $*$-operation on $\mathcal{A}\langle X\rangle$ extends to the completion. By standard results from complex analysis, the function $\psi(t) = \sqrt{R^2 - t^2}$ has a power series expansion

$$\psi(t) = \sum_{j=0}^\infty \alpha_j t^j$$

which converges for $|t| < R$. In particular, the series converges absolutely for $t = M$, which means that

$$\psi(X) = \sum_{j=0}^\infty \alpha_j X^j$$

is a well-defined element of $\mathcal{A}\langle X\rangle_M$. Moreover, because of the absolute convergence, we can compute $\psi(X)^2$ by multiplying the series term by term and hence conclude that $\psi(X)^2 = R^2 - X^2$. Because M is an exponential bound for μ, we know that μ extends to the completion

$\mathcal{A}\langle X\rangle_M$ and remains completely positive on $\mathcal{A}\langle X\rangle_M$. Thus, using complete positivity, for $\xi\mathcal{A}\langle X\rangle \otimes \mathcal{A}$, we can show that

$$\langle \xi, (R^2 - X^2)\xi\rangle_\mu \geq 0,$$

which implies that $\|X\xi\|_\mu \leq R\|\xi\|_\mu$. By taking $R \searrow M$, we have $\|X\xi\|_\mu \leq M\|\xi\|_\mu$, which means that the multiplication operator $\rho(X)$ is well-defined and bounded by M as desired.

This shows that the left multiplication action of $\mathcal{A}\langle X\rangle$ on $L^2(\mathcal{A}\langle X\rangle, \mu)$ is well-defined and bounded. Let $\rho(f(X))$ denote the multiplication operator by $f(X)$. Then $\rho(f(X))$ is adjointable with adjoint given by $\rho(f(X)^*)$. Therefore, in light of Proposition 1.2.10, the closure of $\rho(\mathcal{A}\langle X\rangle)$ in the operator norm is a C^*-algebra, which we will denote by $\mathcal{B}$.

By linearity of E, we obtain that $a\otimes 1 = 1\otimes a$ for $a \in \mathcal{A}$, hence $\xi = 1\otimes 1$ is an $\mathcal{A}$-central unit vector in $\mathcal{A}\langle X\rangle \otimes_\mu \mathcal{A}$, which implies that $E[b] = \langle \xi, b\xi\rangle$ is an $\mathcal{A}$-valued expectation on $\mathcal{B}$. The representation of $\mathcal{B}$ on $\mathcal{B}\otimes_E\mathcal{A}$ is faithful because $\mathcal{B}\otimes_E\mathcal{A} \cong \mathcal{A}\langle X\rangle\otimes_\mu\mathcal{A}$ as Hilbert $\mathcal{B}$-$\mathcal{A}$-bimodules. Finally, $\overline{X} = \rho(X)$ in $\mathcal{B}$ has the law μ since $\mu_{\overline{X}}[f(X)] = E[f(\overline{X})] = E[\rho(f(X))] = \mu[f(X)]$. Therefore, (1) holds.

(2) $\implies$ (1). Suppose that $\mu = \mu_{\overline{X}}$. Let $\rho : \mathcal{A}\langle X\rangle \to \mathcal{B}$ be given by $\rho(p(X)) = p(\overline{X})$; note that ρ is a $*$-homomorphism and in particular it is a completely positive, unital, $\mathcal{A}$-$\mathcal{A}$-bimodule map. Therefore, $\mu = \mu_b = E \circ \rho$ is the composition of two completely positive, unital, $\mathcal{A}$-$\mathcal{A}$-bimodule maps. Moreover, μ is exponentially bounded since

$$\|\mu(a_0Xa_1\dots Xa_k)\| = \left\|E(a_0\overline{X}a_1\dots\overline{X}a_k)\right\| \leq \|a_0\overline{X}a_1\dots\overline{X}a_k\| \leq \left\|\overline{X}\right\|^k\|a_0\|\dots\|a_k\|.$$

□

In fact, this characterization theorem does not require us to assume that μ is an $\mathcal{A}$-$\mathcal{A}$-bimodule map. The more general result where we drop this assumption will be needed later for the discussion of various analytic transforms associated to a law.

Definition 1.6.6. An *$\mathcal{A}$-valued generalized law* is a completely positive and exponentially bounded map $\sigma : \mathcal{A}\langle X\rangle \to \mathcal{A}$.

Theorem 1.6.7. *For a map $\sigma : \mathcal{A}\langle X\rangle \to \mathcal{A}$, the following are equivalent:*

1. *σ is a generalized law with exponential bound M.*
2. *There exists a C^* algebra $\mathcal{B}$, a $*$-homomorphism $\rho : \mathcal{A}\langle X\rangle \to \mathcal{B}$, and a completely positive map $\overline{\sigma} : \mathcal{B} \to \mathcal{A}$ such that $\sigma = \overline{\sigma}\circ\rho$. We also have $\|\rho(X)\| \leq M$.*

The proof is exactly the same as for the previous theorem. Namely, we let ρ be the left multiplication action of $\mathcal{A}\langle X\rangle$ on $\mathcal{A}\langle X\rangle \otimes_\sigma \mathcal{A}$, let $\mathcal{B}$ be the C^* algebra generated by $\mathcal{A}\langle X\rangle$, and let $\tilde{\sigma}(b) = \langle 1, b\cdot 1\rangle_\sigma$.

Corollary 1.6.8. *Let $\sigma : \mathcal{A}\langle X\rangle \to \mathcal{A}$ be a generalized law. Then σ is a law if and only if* $\sigma|_{\mathcal{A}} = \mathrm{id}$.

Proof. Clearly, if σ is a law, then $\sigma|_{\mathcal{A}} = \mathrm{id}$. Conversely, suppose that $\sigma|_{\mathcal{A}} = \mathrm{id}$. Then a direct computation shows that for $a \in \mathcal{A}$, we have $\langle a\otimes 1 - 1\otimes a, a\otimes 1 - 1\otimes a\rangle_\sigma = 0$ and therefore $a\otimes 1 = 1\otimes a$ in $\mathcal{A}\langle X\rangle\otimes_\sigma\mathcal{A}$. This means that the vector $\xi = 1\otimes 1$ is $\mathcal{A}$-central. It is also a unit vector because $\sigma(1) = 1$. It follows that $\overline{\sigma}[b] = \langle\xi, b\xi\rangle$ defines an $\mathcal{A}$-valued expectation on $\mathcal{B} = C^*(\mathcal{A}\langle X\rangle)$, and we have $\sigma(f(X)) = \overline{\sigma}[f(\rho(X))]$, so that σ is the law of X under this expectation. □

Remark 1.6.9. In the proof of Theorem 1.6.5, it is unnecessary to tensor $\mathcal{A}\langle X\rangle$ with $\mathcal{A}$. Indeed, we could have simply defined the inner product $\langle f(X), g(X)\rangle_\mu = \mu(f(X)^* g(X))$ on $\mathcal{A}\langle X\rangle$ rather than $\mathcal{A}\langle X\rangle \otimes \mathcal{A}$, and this would already be an $\mathcal{A}$-valued inner product. However, for the more general setting of Theorem 1.6.7, it *is* necessary to use $\mathcal{A}\langle X\rangle \otimes_\sigma \mathcal{A}$.

1.7 Problems and Further Reading

Problem 1.1 (States)**.** Prove that a state ϕ on a C^*-algebra $\mathcal{A}$ is a completely positive map $\mathcal{A} \to \mathbb{C}$. Show that $\mathcal{A} \otimes_\phi \mathbb{C}$ is isomorphic as a left $\mathcal{A}$-module to the space $L^2(\mathcal{A}, \phi)$ in the GNS construction.

Problem 1.2 (Conditional expectations on von Neumann algebras)**.** Suppose that $(\mathcal{B}, \tau)$ is a tracial von Neumann algebra and $\mathcal{A} \subseteq \mathcal{B}$ is a von Neumann subalgebra. Let $\mathcal{H} = L^2(\mathcal{B}, \tau)$ and $\mathcal{K} = L^2(\mathcal{A}, \tau) \subseteq \mathcal{H}$.

1. Let $P_{\mathcal{K}} : \mathcal{H} \to \mathcal{K}$ be the orthogonal projection. Viewing $\mathcal{B}$ and $\mathcal{A}$ as subsets of $L^2(\mathcal{B})$ and $L^2(\mathcal{A})$, show that $P_{\mathcal{K}}$ restricts to a map $E : \mathcal{B} \to \mathcal{A}$.
2. Show that E is an $\mathcal{A}$-$\mathcal{A}$-bimodule map.
3. Viewing $\mathcal{B}$ and $\mathcal{A}$ as subsets of $B(\mathcal{H})$, show that $E[b] = P_{\mathcal{K}} b P_{\mathcal{K}}$.
4. Show that E is completely positive.

Problem 1.3.

1. Let $\eta : \mathcal{A} \to \mathcal{A}$ be completely positive and $\delta > 0$. Show that $\eta + \delta\,\mathrm{id}$ is invertible and $(\eta + \delta\,\mathrm{id})^{-1}$ is completely positive.
2. Let $\sigma : \mathcal{A}\langle X\rangle \to \mathcal{A}$ be a generalized law. Let $\eta = \sigma|_{\mathcal{A}}$ and suppose that $\eta - \delta\,\mathrm{id}$ is completely positive for some $\delta > 0$. Show that there exists an $\mathcal{A}$-valued law μ such that $\sigma = \eta \circ \mu$.

Problem 1.4. Let $X_1, \dots, X_n$ be random variables in the $\mathcal{A}$-valued probability space $(\mathcal{B}, E)$. Define the joint law of $X_1,\dots,X_n$ as a map $\mathcal{A}\langle X_1, \dots, X_n\rangle \to \mathcal{A}$ given by $f(X_1, \dots, X_n) \mapsto E[f(X_1, \dots, X_n)]$. Show that the $\mathcal{A}$-valued joint law of $X_1, \dots, X_n$ is equivalent information to the $M_n(\mathcal{A})$-valued law of the diagonal matrix $X_1 \oplus \cdots \oplus X_n \in M_n(\mathcal{B})$.

Chapter 2

Fully Matricial Functions

2.1 Introduction

One of the key tools in (scalar) non-commutative probability is the Cauchy-Stieltjes transform of a random variable X given by

$$G_X(z) = E[(z - X)^{-1}],$$

which is an complex-analytic function for z in the upper half-plane and in a neighborhood of ∞ (provided that X is bounded). The law of X can be recovered from the power series expansion of G_X at ∞ because

$$G_X(z^{-1}) = E[(z^{-1} - X)^{-1}] = \sum_{k=0}^{\infty} z^{k+1} E[X^k],$$

which is essentially the moment generating function for the law of X.

In this chapter, we describe an $\mathcal{A}$-valued analytic function theory suitable for $\mathcal{A}$-valued non-commutative probability, and in the next, we analyze the $\mathcal{A}$-valued Cauchy-Stieltjes transform.

It should not be surprising at this point that our notion of analyticity needs to take into account matrix amplifications. One concrete motivation for this is that, without taking matrix amplifications, the Cauchy-Stieltjes transform is insufficient to encode the $\mathcal{A}$-valued law of a random variable X.

One would naively define the Cauchy-Stieltjes transform G_X as a function an open subset of $\mathcal{A}$ given by $G_X(z) = E[(z - X)^{-1}]$. Looking at the power series of $G_X(z^{-1})$ at 0, we have

$$G_X(z^{-1}) = E[(z^{-1} - X)^{-1}] = E[(1 - zX)^{-1} z] = \sum_{k=0}^{\infty} E[(zX)^k z].$$

From this, we can recover all moments of the form $E[zXz \dots Xz]$. However, to know the law of X, we would need to consider all moments of the form $E[z_1 X z_2 \dots X z_k]$. Of course, for the Cauchy-Stieltjes transform *not* to encode the law of X would severely handicap analytic methods for operator-valued non-commutative probability.

But fortunately this problem is resolved by matrix amplification. We can consider the sequence of functions $G_X^{(n)}$ with domain in $M_n(\mathcal{A})$ given by $G_X^{(n)}(z) = E^{(n)}[(z - X^{(n)})^{-1}]$, where $X^{(n)}$ is the diagonal matrix with entries given by X. To recover the moment $E[z_1 X z_2 \dots X z_n]$

for $z_j \in \mathcal{A}$, we evaluate (the analytic extension of) $G_X^{(n+2)}(z^{-1})$ on the matrix

$$z = \begin{bmatrix} 0 & z_0 & 0 & \dots & 0 & 0 \\ 0 & 0 & z_1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & z_n \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

and obtain

$$\begin{aligned} G_X^{(n+2)}(z^{-1}) &= \sum_{k=0}^{\infty} E^{(n+2)}[(zX^{(n+2)})^k z] \\ &= \begin{bmatrix} 0 & z_0 & E[z_0Xz_1] & \dots & E[z_0Xz_1 \dots Xz_{n-1}] & E[z_0Xz_1 \dots Xz_n] \\ 0 & 0 & z_1 & \dots & E[z_1Xz_2 \dots Xz_{n-1}] & E[z_1Xz_2 \dots Xz_n] \\ 0 & 0 & 0 & \dots & E[z_2Xz_3 \dots Xz_{n-1}] & E[z_2Xz_3X \dots Xz_n] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & z_n \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \end{aligned}$$

where $E[z_0Xz_1 \dots Xz_n]$ can be recovered as the top right entry.

Thus, an analytic function F ought to be a sequence of functions $F^{(n)}$ defined on $n \times n$ matrices over $\mathcal{A}$. But we also need to guarantee that these functions "fit together consistently." More precisely, we will require that F respects direct sums and conjugation by invertible scalar matrices (see Definition 2.2.3).

Remarkably, these algebraic conditions, together with a local boundedness condition which is uniform in n, are sufficient to imply the existence of local power series expansions for the function $F^{(n)}$. The terms in these power series expansions are given by multilinear forms, much like the power series expansion for $G_X(z^{-1})$ is obtained from the multilinear forms $\mu(z_0Xz_1 \dots Xz_n)$. Moreover, just as in the case of $G_X(z^{-1})$, these multilinear forms are computed by evaluating $F^{(n)}$ on certain upper triangular matrices.

The study of such *non-commutative* or *fully matricial* functions originated in the 1970's with the work of Joseph Taylor [Tay72], [Tay73]. Dan Voiculescu studied fully matricial functions in the context of the free difference quotient and generalized resolvents [Voi00], [Voi04], [Voi10]. Mihai Popa and Victor Vinnikov clarified the connection between fully matricial function theory in the abstract and the various analytic transforms associated to non-commutative laws [PV13], which we will discuss in detail in the later chapters.

We have opted for a self-contained development of the theory of fully matricial functions, though somewhat restricted in scope. We are indebted to the systematic work of Kaliuzhnyi-Verbovetskyi and Vinnikov [KVV14], although we have not presented the proofs in exactly the same way. We write with the analogy to complex analysis always in mind, and with an eye towards the results of Williams and Anshelevich on the Cauchy-Stieltjes transform [Wil17], [AW16], which we will discuss in the next chapter. We follow Voiculescu in using the term "fully matricial" rather than "non-commutative" since it gives a more concrete description of the definition.

2.2 Fully Matricial Domains and Functions

In order to state the definition, we use the following notation.

1. We identify $M_n(\mathcal{A})$ with $\mathcal{A} \otimes M_n(\mathbb{C})$.

2. If $z \in \mathcal{A}^{(n)}$, then we denote

$$z^{(m)} = z \otimes 1_m = \begin{bmatrix} z & 0 & \dots & 0 & 0 \\ 0 & z & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & z & 0 \\ 0 & 0 & \dots & 0 & z \end{bmatrix} \in M_{nm}(\mathcal{A}).$$

3. If $z \in M_n(\mathcal{A})$ and $w \in M_m(\mathcal{A})$, then we denote

$$z \oplus w = \begin{bmatrix} z & 0 \\ 0 & w \end{bmatrix} \in M_{n+m}(\mathcal{A}).$$

4. If $z \in M_n(\mathcal{A})$, then we denote $B^{(n)}(z, r) = \{w \in M_n(\mathcal{A}) : \|z - w\| < r\}$.

Definition 2.2.1. A *fully matricial domain* Ω *over* $\mathcal{A}$ is a sequence of sets $\Omega^{(n)} \subseteq \mathcal{A}^{(n)}$ satisfying the following conditions.

1. Ω *respects direct sums:* If $z \in \Omega^{(n)}$ and $w \in \Omega^{(m)}$, then $z \oplus w \in \Omega^{(n+m)}$.

2. Ω *is uniformly open:* If $z \in \Omega^{(n)}$, then there exists $r > 0$ such that $B^{(nm)}(z^{(m)}, r) \subseteq \Omega^{(nm)}$ for all m.

3. Ω *is non-empty:* At least one $\Omega^{(n)}$ is non-empty.

Notation 2.2.2. We denote by $M_\bullet(\mathcal{A})$ the fully matricial domain $(M_n(\mathcal{A}))_{n \in \mathbb{N}}$.

Definition 2.2.3. Let Ω_1 and Ω_2 be fully matricial domains over $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. A fully matricial function $F : \Omega_1 \to \Omega_2$ is a sequence of functions $F^{(n)} : \Omega_1^{(n)} \to \Omega_2^{(n)}$ satisfying the following conditions.

1. F *respects intertwinings:* Suppose that $z \in \Omega_1^{(n)}$, $w \in \Omega_1^{(m)}$, $T \in M_{n \times m}(\mathbb{C})$. If $zT = Tw$, then $F^{(n)}(z)S = TF^{(m)}(w)$.

2. F *is uniformly locally bounded:* For each $x \in \Omega_1^{(n)}$, there exist r and $M > 0$ such that $B^{(nm)}(z^{(m)}, r) \subseteq \Omega_1^{(nm)}$ and $F^{(nm)}(B^{(nm)}(z^{(m)}, r)) \subseteq B^{(nm)}(0, M)$ for all m.

In order to check that a function F is fully matricial, it is often convenient to use the following equivalent characterization of the intertwining condition.

Lemma 2.2.4. *Let Ω_1 and Ω_2 be fully matricial domains and let $F : \Omega_1 \to \Omega_2$ be a sequence of functions. Then F respects intertwinings if and only if the following conditions hold.*

1. F respects direct sums: *If $z \in \Omega^{(n)}$ and $w \in \Omega^{(m)}$, then $F^{(n+m)}(z \oplus w) = F^{(n)}(z) \oplus F^{(m)}(w)$.*

2. F respects similarities: *Suppose that $z \in \Omega^{(n)}$, that $S \in M_n(\mathbb{C})$ is invertible, and that $SzS^{-1} \in \Omega^{(n)}$. Then $F^{(n)}(SzS^{-1}) = SF^{(n)}(z)S^{-1}$.*

Proof. First, assume that F respects intertwinings. To prove (1), fix $z \in \Omega^{(n)}$ and $w \in \Omega^{(m)}$. Then we have the block matrix equations

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & w \end{bmatrix} = z \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & w \end{bmatrix} = w \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Because F respects intertwinings, we have

$$\begin{bmatrix} 1 & 0 \end{bmatrix} F^{(n+m)}(z \oplus w) = F^{(n)}(z) \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 \end{bmatrix} F^{(n+m)}(z \oplus w) = F^{(m)}(w) \begin{bmatrix} 0 & 1 \end{bmatrix},$$

which together imply that

$$F^{(n+m)}(z \oplus w) = \begin{bmatrix} F^{(n)}(z) & 0 \\ 0 & F^{(m)}(w) \end{bmatrix} = F^{(n)}(z) \oplus F^{(m)}(w).$$

Next, fix z and S as in (2). Let $w = SzS^{-1}$. Then $Sz = wS$ and hence $SF^{(n)}(z) = F^{(n)}(w)S$, which means that $F^{(n)}(SzS^{-1}) = SF^{(n)}(z)S^{-1}$.

Conversely, suppose that (1) and (2) hold and consider an intertwining $zT = Tw$ where $z \in \Omega_1^{(n)}$, $w \in \Omega_1^{(m)}$, $T \in M_{n\times m}(\mathbb{C})$. Then observe that

$$\begin{bmatrix} z & 0 \\ 0 & w \end{bmatrix} \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & w \end{bmatrix},$$

and observe that

$$S = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

is invertible. Hence, $S(z \oplus w)S^{-1} = z \oplus w$ and therefore by assumptions (1) and (2), we have $S(F^{(n)}(z) \oplus F^{(m)}(w))S^{-1} = F^{(n)}(z) \oplus F^{(m)}(w)$. In other words,

$$\begin{bmatrix} F^{(n)}(z) & 0 \\ 0 & F^{(m)}(w) \end{bmatrix} \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F^{(n)}(z) & 0 \\ 0 & F^{(m)}(w) \end{bmatrix},$$

and hence, looking at the top right block, $F^{(n)}(z)T = TF^{(m)}(w)$. □

In order to reduce the number of superscripts cluttering up our paper, we introduce the following notation.

1. For $F : \Omega_1 \to \Omega_2$ and $z \in \Omega_1^{(n)}$, we will usually write $F(z)$ rather than $F^{(n)}(z)$, and the context will make clear the size of the matrix z.

2. If Ω_1 and Ω_2 are fully matricial domains, then we write $\Omega_1 \subseteq \Omega_2$ to mean that $\Omega_1^{(n)} \subseteq \Omega_2^{(n)}$ for every n.

3. We write $z \in \Omega$ to mean that $z \in \Omega^{(n)}$ for some n.

4. For $\Gamma \subseteq \Omega_1$ and $F : \Omega_1 \to \Omega_2$, we denote by $F(\Gamma)$ the sequence of sets $F(\Gamma)^{(n)} = F^{(n)}(\Gamma^{(n)})$. We define $F^{-1}(\Gamma)$ for $\Gamma \subseteq \Omega_2$ similarly.

5. For $z \in M_n(\mathcal{A})$, we denote by $B(z,r)$ the fully matricial domain

$$B^{(k)}(z,r) = \begin{cases} B^{(nm)}(z^{(m)},r), & k = nm \\ \varnothing, & \text{otherwise.} \end{cases}$$

In this notation, the uniform openness condition of Definition 2.2.1 states that for every $z \in \Omega$, there exists $r > 0$ such that $B(z,r) \subseteq \Omega$. Moreover, $F : \Omega_1 \to \Omega_2$ is uniformly locally bounded as in Definition 2.2.3 if and only if for every $z \in \Omega_1$, there exist R and M such that $F(B(z,R)) \subseteq B(0,M)$.

2.3 Difference-Differential Calculus

Definition 2.3.1. Let $F : \Omega_1 \to \Omega_2$ be fully matricial where Ω_j is fully matricial domain over $\mathcal{A}_j$. Suppose that $z_0 \in \Omega_1^{(n_0)}$, ..., $z_k \in \Omega_1^{(n_k)}$, suppose that $w_1 \in M_{n_0 \times n_1}(\mathcal{A})$, ..., $w_k \in M_{n_{k-1} \times n_k}(\mathcal{A}_1)$, and suppose that the block matrix

$$X := \begin{bmatrix} z_0 & w_1 & 0 & \dots & 0 & 0 \\ 0 & z_1 & w_2 & \dots & 0 & 0 \\ 0 & 0 & z_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & z_{k-1} & w_k \\ 0 & 0 & 0 & \dots & 0 & z_k \end{bmatrix}$$

is in $\Omega_1^{(n_0 + \dots + n_k)}$. Then we define

$$\Delta^k F(z_0, \dots, z_k)[w_1, \dots, w_k]$$

as the upper right $n_0 \times n_k$ block of $F(Z)$.

Lemma 2.3.2. *Let z_0, ..., z_k and w_1, ..., w_k and Z be as above, and assume that each of the submatrices*

$$Z_{i,j} := \begin{bmatrix} z_i & w_{i+1} & \dots & 0 & 0 \\ 0 & z_{i+1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & z_{j-1} & w_j \\ 0 & 0 & \dots & 0 & z_j \end{bmatrix}$$

is in the domain of F for each $i < j$. Then we have

$$F(Z) = \begin{bmatrix} F(z_0) & \Delta F(z_0,z_1)[w_1] & \Delta^2 F(z_0,z_1,z_2)[w_1,w_2] & \dots & \Delta^k F(z_0,\dots,z_k)[w_1,\dots,w_k] \\ 0 & F(z_1) & \Delta F(z_1,z_2)[w_2] & \dots & \Delta^{k-1} F(z_1,\dots,z_k)[w_2,\dots,w_k] \\ 0 & 0 & F(z_2) & \dots & \Delta^{k-2} F(z_2,\dots,z_k)[w_3,\dots,w_k] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & F(z_k) \end{bmatrix} \tag{2.3.1}$$

Proof. We proceed by induction on k with the base case $k = 0$ being trivial. Let $k \geq 1$. Let n_j be the size of the matrix x_j and let $N_{i,j} = n_i + \dots + n_j$. Then we have

$$Z \begin{bmatrix} 1_{N_{0,k-1} \times N_{0,k-1}} \\ 0_{n_k \times N_{0,k-1}} \end{bmatrix} = \begin{bmatrix} 1_{N_{0,k-1} \times N_{0,k-1}} \\ 0_{n_k \times N_{0,k-1}} \end{bmatrix} Z_{0,k-1}$$

and therefore

$$F(Z)\begin{bmatrix}1_{N_{0,k-1}\times N_{0,k-1}}\\ 0_{n_k\times N_{0,k-1}}\end{bmatrix} = \begin{bmatrix}1_{N_{0,k-1}\times N_{0,k-1}}\\ 0_{n_k\times N_{0,k-1}}\end{bmatrix} F(Z_{0,k-1}).$$

From this relation together with the induction hypothesis applied to $F(Z_{0,k-1})$ we deduce that

$$F(Z) = \begin{bmatrix} F(z_0) & \Delta F(z_0,z_1)[w_1] & \dots & \Delta^{k-1}F(z_0,\dots,z_{k-1})[w_1,\dots,w_{k-1}] & \\ 0 & F(z_1) & \dots & \Delta^{k-2}F(z_1,\dots,z_{k-1})[w_2,\dots,w_{k-1}] & \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & * \end{bmatrix}.$$

In other words, (2.3.1) is verified except in the last n_k columns. Similarly, by considering the intertwining

$$\begin{bmatrix}0_{N_{1,k}\times n_0} & 1_{N_{1,k}\times N_{1,k}}\end{bmatrix} Z = Z_{1,k}\begin{bmatrix}0_{N_{1,k}\times n_0} & 1_{N_{1,k}\times N_{1,k}}\end{bmatrix}$$

and applying the induction hypothesis to $F(Z_{1,k})$, we can verify (2.3.1) except in the first n_k rows. It remains to check (2.3.1) in the top right $n_0 \times n_k$ block; but this holds by definition of $\Delta^k F$. □

Lemma 2.3.3. *Suppose that* $z_0 \in \Omega_1^{(n_0)}$, ..., $z_k \in \Omega_1^{(n_k)}$, *suppose that* $w_1 \in M_{n_0\times n_1}(\mathcal{A})$, ..., $w_k \in M_{n_{k-1}\times n_k}(\mathcal{A})$, *and let* ζ_1, ..., $\zeta_k \in \mathbb{C}$. *Then we have*

$$\Delta^k F(z_0,\dots,z_k)[\zeta_1 w_1,\dots,\zeta_k w_k] = \zeta_1\dots\zeta_k F(z_0,\dots,z_k)[w_1,\dots,w_k]$$

provided that both quantities are defined under Definition 2.3.1.

Proof. We consider the intertwining

$$\begin{bmatrix} z_0 & \zeta_1 w_1 & \dots & 0 & 0 \\ 0 & z_1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & z_{k-1} & \zeta_k w_k \\ 0 & 0 & \dots & 0 & z_k \end{bmatrix}\begin{bmatrix} \zeta_1\dots\zeta_k & 0 & \dots & 0 & 0 \\ 0 & \zeta_2\dots\zeta_k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \zeta_k & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \zeta_1\dots\zeta_k & 0 & \dots & 0 & 0 \\ 0 & \zeta_2\dots\zeta_k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \zeta_k & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}\begin{bmatrix} z_0 & w_1 & \dots & 0 & 0 \\ 0 & z_1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & z_{k-1} & w_k \\ 0 & 0 & \dots & 0 & z_k \end{bmatrix},$$

apply the function F, and then examine the top right corner. □

Definition 2.3.4. If $z_j \in \Omega_1^{(n_j)}$ for $j = 0,\dots,k$, then we extend the definition of $\Delta^k F(z_0,\dots,z_k)[w_1,\dots,w_k]$ to arbitrary values of $w_1 \in M_{n_0\times n_1}(\mathcal{A})$, ..., $w_k \in M_{n_{k-1}\times n_k}(\mathcal{A})$ by setting

$$F(z_0,\dots,z_k)[w_1,\dots,w_k] = \frac{1}{\zeta_1\dots\zeta_k}\Delta^k F(z_0,\dots,z_k)[\zeta_1 w_1,\dots,\zeta_k w_k],$$

where ζ_1, ..., $\zeta_k \in \mathbb{C}\setminus\{0\}$ are chosen to be sufficiently small that

$$\begin{bmatrix} z_0 & \zeta_1 w_1 & \dots & 0 & 0 \\ 0 & z_1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & z_{k-1} & \zeta_k w_k \\ 0 & 0 & \dots & 0 & z_k \end{bmatrix} \in \Omega_1^{(n_0+\dots+n_k)}.$$

Such a choice of $\zeta_1, \ldots, \zeta_k$ is possible because $\Omega_1^{(n_0+\cdots+n_k)}$ is open. Lemma 2.3.3 guarantees that this definition of $F(z_0, \ldots, z_k)[w_1, \ldots, w_k]$ is independent of the choice of $\zeta_1, \ldots, \zeta_k$ and is consistent with the earlier Definition 2.3.1.

Lemma 2.3.5. $\Delta^k F(z_0, \ldots, z_k)[w_1, \ldots, w_k]$ *is multilinear in* $w_1, \ldots, w_k$.

Proof. We have already shown that $\Delta^k F(z_0, \ldots, z_k)$ behaves correctly when we multiply one of the w_j's by a scalar, so it remains to show that $\Delta^k F(z_0, \ldots, z_k)[w_1, \ldots, w_k]$ is additive in each variable y_j. First, consider the case $k = 1$ in which we must show that

$$\Delta F(z_0, z_1)[w + w'] = \Delta F(z_0, z_1)[w] + \Delta F(z_0, z_1)[w'].$$

Choose $\zeta \in \mathbb{C} \setminus \{0\}$ sufficiently small that the matrices

$$\begin{bmatrix} z_0 & \zeta(w+w') \\ 0 & z_1 \end{bmatrix}, \quad \begin{bmatrix} z_0 & \zeta w \\ 0 & z_1 \end{bmatrix}, \quad \begin{bmatrix} z_0 & \zeta w' \\ 0 & z_1 \end{bmatrix}, \quad \begin{bmatrix} z_0 & 0 & \zeta w \\ 0 & z_0 & \zeta w' \\ 0 & 0 & z_1 \end{bmatrix}$$

are all in the domain of F. From the intertwining

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_0 & 0 & \zeta w \\ 0 & z_0 & \zeta w' \\ 0 & 0 & z_1 \end{bmatrix} = \begin{bmatrix} z_0 & \zeta w \\ 0 & z_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we deduce that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} F\left(\begin{bmatrix} z_0 & 0 & \zeta w \\ 0 & z_0 & \zeta w' \\ 0 & 0 & z_1 \end{bmatrix} \right) = \begin{bmatrix} F(z_0) & \zeta \Delta F(z_0, z_1)[w] \\ 0 & F(z_1) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Similarly,

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} F\left(\begin{bmatrix} z_0 & 0 & \zeta w \\ 0 & z_0 & \zeta w' \\ 0 & 0 & z_1 \end{bmatrix} \right) = \begin{bmatrix} F(z_0) & \zeta \Delta F(z_0, z_1)[w'] \\ 0 & F(z_1) \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and thus altogether,

$$F\left(\begin{bmatrix} z_0 & 0 & \zeta w \\ 0 & z_0 & \zeta w' \\ 0 & 0 & z_1 \end{bmatrix} \right) = \begin{bmatrix} F(z_0) & 0 & \zeta \Delta F(z_0, z_1)[w] \\ 0 & z_0 & \zeta \Delta F(z_0, z_1)[w'] \\ 0 & 0 & z_1 \end{bmatrix}.$$

Finally, we use the intertwining

$$\begin{bmatrix} z_0 & \zeta(w+w') \\ 0 & z_1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_0 & 0 & \zeta w \\ 0 & z_0 & \zeta w' \\ 0 & 0 & z_1 \end{bmatrix}$$

to deduce that

$$\begin{bmatrix} F(z_0) & \zeta \Delta F(z_0, z_1)[w+w'] \\ 0 & F(z_1) \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F(z_0) & 0 & \zeta \Delta F(z_0, z_1)[w] \\ 0 & F(z_0) & \zeta \Delta F(z_0, z_1)[w'] \\ 0 & 0 & F(z_1) \end{bmatrix},$$

which shows that $\zeta \Delta F(z_0, z_1)[w + w'] = \zeta \Delta F(z_0, z_1)[w] + \zeta \Delta F(z_0, z_1)[w']$ as desired.

The argument in the general case is similar. To show linearity of $F(z_0,\dots,z_k)[w_1,\dots,w_n]$ in w_j, we consider replacing w_j by w_j+w_j'. The block 3×3 matrix used above is replaced by

$$\begin{bmatrix} z_0 & \zeta_1 w_1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & z_1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & z_{j-2} & \zeta_{j-1}w_{j-1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & z_{j-1} & 0 & \zeta_j w_j & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & z_{j-1} & \zeta_j w_j' & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & x_j & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & z_{k-1} & \zeta_k w_k \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & z_k \end{bmatrix}$$

and the intertwiners are replaced by

$$1_{n_0+\dots+n_{j-1}} \oplus \begin{bmatrix} \alpha & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \oplus 1_{n_{j+1}+\dots+n_k}$$

where α, $\beta = 0,1$ and where n_j is the size of the matrix $z_j j$. □

2.4 Taylor-Taylor Expansion

We have defined the derivative operators $\Delta^k F$ using the matricial structure of F. Now we will show that these same operators $\Delta^k F$ describe the differential and analytic properties of F, and in fact that F has local power series expansions in terms of $\Delta^k F$. We begin with the finite Taylor-Taylor expansion. This formula is named for Brook Taylor, who invented the original Taylor expansion, and Joseph L. Taylor, who pioneered the theory of non-commutative (fully matricial) functions.

Lemma 2.4.1 (Taylor-Taylor Formula). *Let $F:\Omega_1\to\Omega_2$ be a fully matricial function. Let $z_0\in\Omega_1^{(n)}$ and $m>1$ and suppose that $B^{(nm)}(z_0^{(m)},r)\subseteq\Omega_1^{(nm)}$. If $z\in B^{(n)}_{r/\sqrt{2}}(z_0)$, then*

$$F(z)=\sum_{k=0}^{m-2}\Delta^k F(z_*,\dots,z_*)[z-z_*,\dots,z-z_*]+\Delta^{m-1}F(z,z_*,\dots,z_*)[z-z_*,\dots,z-z_*],$$

where the $k=0$ term in the sum is to be interpreted as $F(z_*)$.

Proof. Observe that the $m\times m$ block matrix

$$Z=\begin{bmatrix} z & z-z_* & 0 & 0 & \dots & 0 & 0 \\ 0 & z_* & z-z_* & 0 & \dots & 0 & 0 \\ 0 & 0 & z_* & z-z_* & \dots & 0 & 0 \\ 0 & 0 & 0 & z_* & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & z_* & z-z_* \\ 0 & 0 & 0 & 0 & \dots & 0 & z_* \end{bmatrix}$$

is in $B^{(nm)}(z_*^{(m)}, r) \subseteq \Omega^{(nm)}$ provided that $\|z - z_*\| < r/\sqrt{2}$. We have the intertwining relation

$$\begin{bmatrix}1 & 1\dots & 1\end{bmatrix} Z = z \begin{bmatrix}1 & 1 & \dots & 1\end{bmatrix},$$

and therefore,

$$\begin{bmatrix}1 & 1\dots & 1\end{bmatrix} F(Z) = F(z) \begin{bmatrix}1 & 1\dots & 1\end{bmatrix}.$$

Looking at the rightmost block of $\begin{bmatrix}1 & 1\dots & 1\end{bmatrix} F^{(nm)}(Z)$ and applying the definition of $\Delta^k F$, we have

$$F(z_*) + \sum_{k=1}^{m-1} \Delta^k F(z_*, \dots, z_*)[z - z_*, \dots, z - z_*] + \Delta^m F(z, z_*, \dots, z_*)[z - z_*, \dots, z - z_*] = F^{(n)}(z).$$

□

Next, we give a non-commutative analog of the Cauchy estimates from complex analysis, which will help us prove convergence of the infinite Taylor-Taylor series. In the following, $\|\Delta^k F(z_0, \dots, z_k)\|$ denotes the norm of $\Delta^k F(z_0, \dots, z_k)$ as a multilinear form between Banach spaces, that is,

$$\|\Delta^k F(z_0, \dots, z_k)\| = \sup_{\|w_j\| \le 1} \big\|\Delta^k F(z_0, \dots, z_k)[w_1, \dots, w_k]\big\|.$$

Lemma 2.4.2. *Let $F : \Omega_1 \to \Omega_2$ be fully matricial. Let $Z = w_0 \oplus \dots \oplus w_k$ where $z_j \in \Omega_1^{(n_j)}$ and let $N = n_0 + \dots + n_k$. Suppose that $B^{(N)}(Z, R) \subseteq \Omega_1^{(N)}$ and $F(B^{(N)}(Z, R)) \subseteq B^{(N)}(0, M)$. Then*

$$\|\Delta^k F(z_0, \dots, z_k)\| \le \frac{M}{R^k} \text{ for } k \ge 1. \tag{2.4.1}$$

Proof. Suppose that $\|w_1\| \le 1$, …, $\|w_k\| \le 1$. For $r < R$, we have

$$W := \begin{bmatrix} z_0 & rw_1 & \dots & 0 & 0 \\ 0 & z_1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots\vdots & \\ 0 & 0 & \dots & z_{k-1} & rw_k \\ 0 & 0 & \dots & 0 & z_k \end{bmatrix} \in B^{(N)}(Z, R),$$

and hence $\|F(W) - F(Z)\| < M$. Looking at the top right block of $F(W) - F(Z)$, we obtain

$$\|F(z_0, \dots, z_k)[rw_1, \dots, rw_k]\| \le M.$$

Because this holds whenever $r < R$ and $\|w_j\| \le 1$, we have proven (2.4.1). □

Lemma 2.4.3. *Let $F : \Omega_1 \to \Omega_2$ be fully matricial. Let $z_* \in \Omega_1^{(n)}$, and suppose $B(z_*, R) \subseteq \Omega_1$ and $F(B(z_*, R)) \subseteq B(0, M)$. Then for $z \in B^{(n)}(z_*, R)$, we have*

$$F(z) = \sum_{k=0}^{\infty} \Delta^k F(z_*, \dots, z_*)[z - z_*, \dots, z - z_*]. \tag{2.4.2}$$

Proof. It follows from Lemma 2.4.2 that the power series on the right hand side of (2.4.2) converges when $\|y - x\| < R$. It remains to show that the sum of the series is $F(y)$. If we assume that $\|z - z_*\| < R/\sqrt{2}$, then by Lemma 2.4.1,

$$F(z) = \sum_{k=0}^{m-1} \Delta^k F(z_*, \ldots, z_*)[z - z_*, \ldots, z - z_*] + \Delta^m F(z, z_*, \ldots, z_*)[z - z_*, \ldots, z - z_*].$$

Now if $\|z-z_*\| \leq R/2$, then we have $B(z_*, R/2) \subseteq B(z_*, R)$ and hence $F(B(z, R/2)) \subseteq B(0, M)$. Hence, by Lemma 2.4.2,

$$\|\Delta^m F(z, z_*, \ldots, z_*)[z - z_*, \ldots, z - z_*]\| \leq \frac{2M\|z - z_*\|^m}{(R/2)^m}$$

which vanishes as $m \to \infty$. Therefore, (2.4.2) holds when $\|z - z_*\| < R/2$.

To extend (2.4.2) to $y \in B^{(n)}(z_*, R)$, we use complex analysis. Fix $z \in B^{(n)}(z_*, R)$. Note that for any state ϕ on $\mathcal{A}^{(n)}$ and for $|\zeta| < R/2\|z - z_*\|$, the function

$$g(\zeta) = \phi \circ F(z + \zeta(z - z_*)) = \sum_{k=0}^{\infty} \zeta^k \phi \circ \Delta^k F(z_*, \ldots, z_*)[z - z_*, \ldots, z - z_*]$$

is a scalar-valued analytic function. Now because F has also has a local power series expansion centered at $z + \zeta(z - z_*)$ whenever $z_* + \zeta(z - z_*)$ is in the domain of F, we see that g is actually analytic for $|\zeta| < R/\|z - z_*\|$. It follows that the power series expansion for g centered at 0 converges to g when $|\zeta| < R/\|z - z_*\|$. Thus, taking $\zeta = 1$, we obtain

$$\phi \circ F(z) = \sum_{k=0}^{\infty} \phi \circ \Delta^k F(z_*, \ldots, z_*)[z - z_*, \ldots, z - z_*],$$

and because this holds for arbitrary states ϕ, we have proved (2.4.2). □

2.5 Matricial Properties of $\Delta^k F$

We will now describe how $\Delta^k F(z_0, \ldots, z_k)$ behaves when we replace one of the z_j's with a direct sum. As a consequence, we will evaluate $\Delta^k F(z_0^{(n)}, \ldots, z_0^{(n)})$ as a type of matrix amplification of $\Delta^k F(z_0, \ldots, z_0)$, and hence derive a Taylor-Taylor expansion around a point z which will hold not only on a ball $B^{(n)}(z_0, r)$, but on a fully matricial ball $B(z_0, r)$. As a first step, we describe how the direct sum property of F carries over to $\Delta^k F$.

Lemma 2.5.1. *For $j = 1, \ldots, k-1$, we have*

$$\begin{aligned}\Delta^k F(z_0, \ldots, z_{j-1}, z_j \oplus z_j', z_{j+1}, \ldots, z_k) &\left[w_1, \ldots, w_{j-1}, \left[w_j, w_j'\right], \begin{bmatrix} w_{j+1} \\ w_{j+1}' \end{bmatrix}, w_{j+2}, \ldots, w_k\right] \\ = \Delta^k F(z_0, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_k) &\left[w_1, \ldots, w_{j-1}, w_j, w_{j+1}, w_{j+2}, \ldots, w_k\right] \\ + \Delta^k F(z_0, \ldots, z_{j-1}, z_j', z_{j+1}, \ldots, z_k) &\left[w_1, \ldots, w_{j-1}, w_j', w_{j+1}', w_{j+2}, \ldots, w_k\right].\end{aligned}$$

In the endpoint case $j = 0$, the same holds with the terms w_j and w_j' left out, and the endpoint case $j = k$, the same holds with the w_{j+1} and w_{j+1}' left out.

Proof. To simplify notation, first assume $k = 2$ and $j = 1$. Using the intertwining

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_0 & \zeta_1 w_1 & 0 & 0 \\ 0 & z_1 & 0 & \zeta_2 w_2 \\ 0 & 0 & z_1' & \zeta_2 w_2' \\ 0 & 0 & 0 & z_2 \end{bmatrix} = \begin{bmatrix} z_0 & \zeta_1 w_1 & 0 \\ 0 & z_1 & \zeta_2 w_2 \\ 0 & 0 & z_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

we deduce that

$$\Delta^2 F(z_0, z_1 \oplus z_1', z_2) \left[\begin{bmatrix} w_1 & 0 \end{bmatrix}, \begin{bmatrix} w_2 \\ w_2' \end{bmatrix} \right] = \Delta^2 F(z_0, z_1, z_2)[w_1, w_2].$$

A similar argument shows that

$$\Delta^2 F(z_0, z_1 \oplus z_1', z_2) \left[\begin{bmatrix} 0 & w_1' \end{bmatrix}, \begin{bmatrix} w_2 \\ w_2' \end{bmatrix} \right] = \Delta^2 F(z_0, z_1', z_2)[w_1', w_2'].$$

Then by linearity of $\Delta^2 F(z_0, z_1 \oplus z_1', z_2)$ in the first w coordinate, we get

$$\Delta^2 F(z_0, z_1 \oplus z_1', z_2) \left[\begin{bmatrix} w_1 & w_1' \end{bmatrix}, \begin{bmatrix} w_2 \\ w_2' \end{bmatrix} \right] = \Delta^2 F(z_0, z_1, z_2)[w_1, w_2] + \Delta^2 F(z_0, z_1', z_2)[w_1', w_2'].$$

The argument for the general case when $1 \leq j \leq k-1$ is the same except that we must augment our intertwining matrix by taking the direct sum with copies of the identity at the top left and bottom right (compare the general case of Lemma 2.3.5). The endpoint cases have a similar but simpler argument. □

Now we generalize the previous lemma to replace each z_j by an arbitrary direct sum.

Lemma 2.5.2. *Let Z_j be the $R_j \times R_j$ block diagonal matrix*

$$Z_j = z_{j,1} \oplus \cdots \oplus z_{j,R_j},$$

where the block $z_{j,r}$ is $n_{j,r} \times n_{j,r}$ and j runs from 0 to k. Let W_j be an $R_{j-1} \times R_j$ block matrix where the (r, s) block $w_{j,r,s}$ has dimensions $n_{j-1,r} \times n_{j,s}$. Then $\Delta^k F(Z_0, \ldots, Z_k)[W_1, \ldots, W_k]$ is an $R_0 \times R_k$ block matrix where the (r, s) block is given by

$$\sum_{r_1, \ldots, r_{k-1}} \Delta^k F(z_{0,r}, z_{1,r_1}, \ldots, z_{k-1,r_{k-1}}, z_{k,s})[w_{1,r,r_1}, w_{2,r_1,r_2}, \ldots, w_{k-1,r_{k-2},r_{k-1}}, w_{k,r_{k-1},s}].$$

Remark 2.5.3. In the last lemma, the conditions on the dimensions are such that it would make sense to multiply the matrices $Z_0 W_1 Z_1 \ldots W_k Z_k$ together. The lemma asserts that block entries of $\Delta^k F(Z_0, \ldots, Z_k)[W_1, \ldots, W_k]$ is computed from $\Delta^k F$ evaluated on the $z_{j,r}$'s and $w_{j,r,s}$'s in the same way as we would evaluate the matrix product $Z_0 W_1 Z_1 \ldots W_k Z_k$ in terms of products of the $z_{j,r}$'s and $w_{j,r,s}$'s.

Proof of Lemma 2.5.2. We fix k and proceed by induction on the total number of direct summands of the Z_j's. If some $Z_j j$ has more than one direct summand, we can break Z_j into the direct sum of $z_{j,1} \oplus \ldots z_{j,R_j - 1}$ and z_{j,R_j} and then apply Lemma 2.5.1, and thus reduce to an earlier stage of the induction. □

Now restricting our attention to the case where $z_{j,1} = \cdots = z_{j,R_j}$, we will write $\Delta^k F(z_0^{(m_0)}, \ldots, z_k^{(m_k)})$ as a matrix amplification of $\Delta^k F(z_0, \ldots, z_k)$, and in particular, we will be able to evaluate the derivative $\Delta^k F(z_*^{(m)}, \ldots, z_*^{(m)})$ used in the Taylor-Taylor expansion.

Definition 2.5.4. Let $\mathcal{V}_1, \ldots, \mathcal{V}_k$ and $\mathcal{V}$ be vector spaces and let $\Lambda : \mathcal{V}_1 \times \cdots \times \mathcal{V}_k \to V$ be a multilinear form. Choose indices $m_0, \ldots, m_k$. We then define the multilinear form

$$\Lambda^{(m_0,\ldots,m_k)} : M_{m_0 \times m_1}(\mathcal{V}_1) \times \cdots \times M_{m_{k-1}\times m_k}(\mathcal{V}_k) \to M_{m_0\times m_k}(\mathcal{V})$$

by

$$[\Lambda^{(m_0,\ldots,m_k)}(v_1,\ldots,v_k)]_{i,j} = \sum_{i=i_0,i_1,\ldots,i_{k-1},i_k=j} \Lambda[(v_1)_{i_0,i_1},\ldots,(v_k)_{m_{k-1},m_k}]$$

We will sometimes denote the matrix amplification $\Lambda^{(m_0,\ldots,m_k)}$ simply by $\Lambda^{\#}$ when we do not wish to specify the indices $m_0, \ldots, m_k$.

In particular, let $F : \Omega_1 \to \Omega_2$, where Ω_j is a fully matricial domain over $\mathcal{A}_j$. Let $z_j \in \Omega_1^{(n_j)}$ for $j = 0, \ldots, k$. Then we have a multilinear form

$$\Delta^k F(z_0,\ldots,z_k) : M_{n_0\times n_k}(\mathcal{A}_1) \times \cdots \times M_{n_{k-1}\times n_j}(\mathcal{A}_1) \to M_{n_0\times n_k}(\mathcal{A}_2).$$

If we choose indices $m_0, \ldots, m_k$ and identify $M_{m_{j-1}\times m_j}(M_{n_{j-1}\times n_j}(\mathcal{A}_1))$ with $M_{n_{j-1}m_{j-1}\times n_j m_j}(\mathcal{A}_1)$, then we have by Lemma 2.5.2 that

$$\Delta^k F(z_0^{(m_0)},\ldots,z_k^{(m_k)}) = \Delta^k F(z_0,\ldots,z_k)^{(m_0,\ldots,m_k)}.$$

We now state a version of the Cauchy estimates and Taylor-Taylor expansion that take into account matrix amplification, beginning with a norm for multilinear forms which is stable under matrix amplification.

Definition 2.5.5. Recall that the norm of a multilinear form on $M_{n_0\times n_1}(\mathcal{A})\times\cdots\times M_{n_{k-1}\times n_k}(\mathcal{A}) \to M_{n_0\times n_k}(\mathcal{A})$ is given by

$$\|\Lambda\| = \sup_{\|w_1\|,\ldots,\|w_k\|\le 1} \|\Lambda[w_1,\ldots,w_k]\|.$$

so we define the *completely bounded norm* as

$$\|\Lambda\|_{\#} = \sup_{m_0,\ldots,m_k} \|\Lambda^{(m_0,\ldots,m_k)}\|.$$

We say that Λ is *completely bounded* if $\|\Lambda\|_{\#} < +\infty$.

The next corollaries follow immediately from Lemma 2.5.2

Corollary 2.5.6. *We have* $\Delta^k F(z_*^{(m_0)},\ldots,z_*^{(m_k)}) = [\Delta^k F(z_*,\ldots,z_*)]^{(m_0,\ldots,m_k)}$.

Corollary 2.5.7. *Suppose that* $F : \Omega_1 \to \Omega_2$ *is fully matricial and* $B(z_*,R) \subseteq \Omega_1$ *and* $F(B(z_*,R)) \subseteq B(0,M)$. *Then* $\|\Delta^k F(z_*,\ldots,z_*)\|_{\#} \le M/R^k$.

Proof. Note that $B(z_*^{(m_0)} \oplus \cdots \oplus z_*^{(m_k)}, R) \subseteq B(z_*,R) \subseteq \Omega_1$, so it follows from Lemma 2.4.2 that

$$\|\Delta^k F(z_*^{(m_0)},\ldots,z_*^{(m_k)})\| \le \frac{M}{R^k}.$$ □

Corollary 2.5.8. *Let* $F : \Omega_1 \to \Omega_2$*, let* $z_* \in \Omega_1^{(n)}$*, and suppose that* $F(B(z_*,R)) \subseteq B(0,M)$*. Then for* $z \in B^{(nm)}(z_*^{(m)},R)$*, we have*

$$F(z) = \sum_{k=0}^{\infty} [\Delta^k F(z_*,\ldots,z_*)]^{(m,\ldots,m)}(z - z_*^{(m)},\ldots,z - z_*^{(m)}).$$

This amplified power series expansion will allow us to compute and to estimate the derivatives of F at points in $B(z_*, R)$.

Proposition 2.5.9. *Suppose that $F : \Omega_1 \to \Omega_2$ is fully matricial, $z_* \in \Omega^{(n)}$ and $B(z_*, R) \subseteq \Omega_1$ and $F(B(z_*, R)) \subseteq B(0, M)$. Let $z_0, \dots, z_k$ be points with $z_j \in B^{(nm_j)}(0, R)$. Then we have*

$$\Delta^k F(z_*^{(m_0)} + z_0, \dots, z_*^{(m_k)} + z_k)[w_1, \dots, w_k] = \sum_{m_0,\dots,m_k \geq 0} \Delta^{\ell_0+\dots+\ell_k+k} F(z_*, \dots, z_*)[\underbrace{z_0, \dots, z_0}_{\ell_0}, w_1, \underbrace{z_1, \dots, z_1}_{\ell_1}, \dots, w_k, \underbrace{z_k, \dots, z_k}_{\ell_k}]. \quad (2.5.1)$$

In particular,

$$\|\Delta^k F(z_*^{(m_0)} + z_0, \dots, z_*^{(m_k)} + z_k)\|_\# \leq \frac{M}{(R - \|z_0\|) \dots (R - \|z_k\|)} \quad (2.5.2)$$

and

$$\|\Delta^k F(z_*^{(m_0)} + z_0, \dots, z_*^{(m_k)} + z_k) - \Delta^k F(z_*^{(m_0)}, \dots, z_*^{(m_k)})\|_\# \leq \frac{M \sum_{j=1}^k \|z_j\|}{(R - \|z_0\|) \dots (R - \|z_k\|)}. \quad (2.5.3)$$

Proof. Note that to compute the derivatives, we may restrict our attention $F^{(mn)}$ for $m \in \mathbb{N}$. Since $F^{(mn)}(z - z_*^{(m)})$ is defined on $B(0^{(n)}, R)$, we may assume without loss of generality that $z_* = 0^{(n)}$. Furthermore, we denote

$$\Lambda_k = \Delta^k F(z_*, \dots, z^*),$$

so that

$$F(z) = \sum_{k=0}^\infty \Lambda_k^\#(z, \dots, z) \text{ for } z \in B(0^{(n)}, R).$$

Before performing the computation, we first show that the series converges absolutely and estimate it. Observe that

$$\begin{aligned} \sum_{\ell_0,\dots,\ell_k \geq 0} &\left\| \Lambda_{\ell_0+\dots+\ell_k+k}[\underbrace{z_0, \dots, z_0}_{\ell_0}, w_1, \underbrace{z_1, \dots, z_1}_{\ell_1}, \dots, w_k, \underbrace{z_k, \dots, z_k}_{\ell_k}] \right\| \\ &\leq \sum_{\ell_0,\dots,\ell_k \geq 0} \frac{M}{R^{\ell_0+\dots+\ell_k+k}} \|z_0\|^{\ell_0} \dots \|z_k\|^{\ell_k} \|w_1\| \dots \|w_k\| \\ &= \frac{M \|w_1\| \dots \|w_k\|}{(R - \|z_0\|) \dots (R - \|z_k\|)}, \end{aligned}$$

where the last equality follows from summing the geometric series.

Now let us show that sum converges to $\Delta^k F(z_0, \dots, z_k)[w_1, \dots, w_k]$. Consider the block upper triangular matrix Z with the entries $z_0, \dots, z_k$ on the diagonal, the entries $w_1, \dots, w_k$ just above the diagonal, and zeroes elsewhere. By rescaling, assume that $w_1, \dots, w_k$ are small enough that $\|Z\| < R$. Recall that $\Delta^k F(z_0, \dots, z_k)[w_1, \dots, w_k]$ is the upper right corner of $F(Z)$. The upper right block of $\Lambda_\ell^\#(Z, \dots, Z)$ is given by

$$\sum_{1=i_0,i_1,\dots,i_\ell=k+1} \Lambda_\ell^\#(Z_{i_0,i_1}, \dots, Z_{i_{\ell-1},i_\ell}).$$

Because Z is block upper triangular with entries on the diagonal and directly above it, the only nonzero terms are of the form

$$\Lambda_\ell^\#(\underbrace{z_0,\dots,z_0}_{m_0}, w_1, \underbrace{z_1,\dots,z_1}_{m_1}, \dots, w_k, \underbrace{z_k,\dots,z_k}_{m_k}).$$

Thus, we have

$$\Delta^k F(z_0,\dots,z_k)[w_1,\dots,w_k] = \sum_{\ell=0}^\infty \left(\sum_{\substack{m_0,\dots,m_k \geq 0 \\ m_0+\dots+m_k+k=\ell}} \Lambda_\ell^\#(\underbrace{z_0,\dots,z_0}_{m_0}, w_1, \underbrace{z_1,\dots,z_1}_{m_1}, \dots, w_k, \underbrace{z_k,\dots,z_k}_{m_k}) \right),$$

which is exactly (2.5.1) in the case $z_* = 0^{(n)}$.

We already showed that when $z_* = 0^{(n)}$,

$$\|\Delta^k F(z_0,\dots,z_k)\| \leq \frac{M}{(R-\|z_0\|)\dots(R-\|z_k\|)}.$$

Because the same reasoning applies to $\Delta^k(z_0^{(m_0)},\dots,z_k^{(m_k)})$ and yields the same estimate, we have bounded $\|\Delta^k F(z_0,\dots,z_k)\|_\#$ and proven (2.5.2).

To prove (2.5.3), observe that

$$\begin{aligned}
&\|\Delta^k F(z_0,\dots,z_k) - \Delta^k F(0^{(nm_0)},\dots,0^{(nm_k)})\| \\
&\leq \sum_{\substack{\ell_0,\dots,\ell_k \geq 0 \\ \ell_0+\dots+\ell_k \geq 1}} \left\| \Lambda_{\ell_0+\dots+\ell_k+k}[\underbrace{z_0,\dots,z_0}_{\ell_0}, w_1, \underbrace{z_1,\dots,z_1}_{\ell_1}, \dots, w_k, \underbrace{z_k,\dots,z_k}_{\ell_k}] \right\| \\
&\leq \sum_{\ell_0,\dots,\ell_k \geq 0} \frac{M}{R^{\ell_0+\dots+\ell_k+k}} \|z_0\|^{\ell_0} \dots \|z_k\|^{\ell_k} \|w_1\| \dots \|w_k\| \\
&= M\|w_1\|\dots\|w_k\| \left(\frac{1}{(R-\|z_0\|)\dots(R-\|z_k\|)} - \frac{1}{R^k} \right), \\
&\leq \frac{\sum_{j=0}^k \|z_j\|}{(R-\|z_0\|)\dots(R-\|z_k\|)}.
\end{aligned}$$

The same argument also applies to the matrix amplifications of $\Delta^k F(z_0,\dots,z_k)$ and hence we have bounded $\Delta^k F(z_0,\dots,z_k) - \Delta^k F(0^{(nm_0)},\dots,0^{nm_k})$ in $\|\cdot\|_\#$. □

Corollary 2.5.10. *$F(z)$ and $\Delta^k F(z,\dots,z)$ are uniformly locally Lipschitz functions z with respect to $\|\cdot\|_\#$. That is, for every z_*, there exists $r > 0$ such that $F(z)$ and $\Delta^k F(z,\dots,z)$ are Lipschitz on $B^{(n)}(z_*, r)$, with Lipschitz constants independent of n.*

Furthermore, the following lemma shows the multlinear forms in this amplified power series expansion are unique. That is, any other sequence of multilinear forms Λ_k which satisfies the equation in Corollary 2.5.8 must be equal to $\Delta^k F(z_0,\dots,z_0)$. This lemma justifies many ways of computing the derivatives of a fully matricial function F. As long as we obtain a power series that converges to F and the manipulation works for every size of matrices, then we must have the correct answer. For applications, see Problems 2.6 and 2.7 below.

Lemma 2.5.11. *Let $F : \Omega_1 \to \Omega_2$ and $z_* \in \Omega_1^{(n)}$. Let $\Lambda_k : M_n(\mathcal{A}_1)^k \to M_n(\mathcal{A}_2)$ be a sequence of multilinear forms. If for some $r > 0$, have*

$$F(z) = \sum_{k=0}^{\infty} \Lambda_k^{(m,\ldots,m)}(z - z_*^{(m)}, \ldots, z - z_*^{(m)})$$

for $z \in B^{(m)}(z_*, r)$ for all m, then $\Lambda_k = \Delta^k F(z_*, \ldots, z_*)$. In fact, we need only assume that the expansion holds when $z - z_*^{(m)}$ is strictly upper triangular and in $B^{(nm)}(z_*^{(m)}, r_m)$ for some $r_m > 0$.

Proof. Fix k. Fix $w_1, \ldots, w_k \in M_n(\mathcal{A})$ and let $\zeta_1, \ldots, \zeta_k$ be small scalars. Let

$$W = \begin{bmatrix} 0 & \zeta_1 w_1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \zeta_2 w_2 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots 0 & \zeta_k w_k & \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}]$$

Then we observe that

$$F(z_*^{(k+1)} + W) = \begin{bmatrix} 0 & \Lambda_1(\zeta_1 w_1) & \Lambda_2(\zeta_1 w_1, \zeta_2 w_2) & \ldots & \Lambda_k(\zeta_1 w_1, \ldots, \zeta_k w_k) & \\ 0 & 0 & \Lambda_1(\zeta_2 w_2) & \ldots & \Lambda_{k-1}(\zeta_2 w_2, \ldots, \zeta_k w_k) & \\ 0 & 0 & 0 & \ldots & \Lambda_{k-2}(\zeta_3 w_3, \ldots, \zeta_k w_k) & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \ldots \Lambda_1(\zeta_k w_k) & & \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}],$$

and by examining the upper right block, it follows that

$$\Delta^k F(z_*, \ldots, z_*)(\zeta_1 w_1, \ldots, \zeta_k w_k) = \Lambda_k(\zeta_1 w_1, \ldots, \zeta_k w_k).$$

Thus, $\Delta^k F(z_*, \ldots, z_*) = \Lambda_k$ as desired. □

2.6 Examples

Series of Multilinear Forms

Our first example is closely related to the material from the last section on the matrix amplifications of multilinear forms. We will characterize the fully matricial functions on the ball $B(0, R)$ (where 0 is the 1×1 zero matrix) as convergent series of multilinear forms. We remark that the corresponding notion of formal power series of multilinear forms was studied by Dykema [Dyk07].

Proposition 2.6.1. *Suppose that $\Lambda_k : \mathcal{A}_1^k \to \mathcal{A}_2$ is a completely bounded multilinear form and that* $\limsup_{k\to\infty} \|\Lambda_k\|_{\#}^{1/k} \leq 1/R$. *Then*

$$F^{(n)}(z) = \sum_{k=0}^{\infty} \Lambda_k^{(n)}[z, \ldots, z]$$

is a fully matricial function on $B(0,R)$ which satisfies $\Delta^k F(0,\dots,0) = \Lambda_k$. Moreover, F is uniformly bounded on $B(0,r)$ for each $r < R$. Conversely, if F is a fully matricial function on $B(0,R)$ which is uniformly bounded on $B(0,r)$ for each $r < R$, then F can be written in this form, where $\Lambda_k = \Delta^k F(0,\dots,0)$.

Proof. Let Λ_k be given with $\limsup_{k\to\infty} \|\Lambda_k\|_\#^{1/k} \leq 1/R$. Choose $r < R$ and let $r < r' < R$. Then for k greater than or equal to some N, we have $\|\Lambda_k\| \leq 1/r'$. This implies that for $\|z\| \leq r$, we have

$$\sum_{k=0}^\infty \|\Lambda_k^{(n,\dots,n)}(z,\dots,z)\| \leq \sum_{k=0}^{N-1} \|\Lambda_k\|_\# r^k + \sum_{k=N}^\infty \left(\frac{r}{r'}\right)^k < +\infty.$$

This shows that the series converges uniformly on $B(0,r)$ and defines a function F which is bounded on $B(0,r)$ for each $r < R$. To show that F is fully matricial, suppose $zT = Tw$ where $z \in B^{(n)}(0,R)$ and $w \in B^{(m)}(0,R)$ and $T \in M_{n\times m}(\mathbb{C})$. A direct computation from the definition of the matrix amplification of multilinear forms shows that

$$\Lambda_k^\#(z,\dots,z)T = \Lambda_k^\#(z,\dots,z,zT) = \Lambda_k^\#(z,\dots,z,Tw) = \Lambda_k^\#(z,\dots,z,zT,w) = \dots$$
$$\dots = \Lambda_k^\#(Tw,w,\dots,w) = T\Lambda_k^\#(w,\dots,w).$$

Therefore, $F(z)T = TF(w)$ as desired.

Now consider the converse direction. Suppose that $\|F(z)\| \leq M_r$ for $\|z\| \leq r < R$. By Corollary 2.5.7, we have $\|\Lambda_k\|_\# \leq M_r/r^k$, so that $\limsup_{k\to\infty} \|\Delta^k F(0,\dots,0)\|_\#^{1/k} \leq 1/r$. Thus holds for all $r < R$, and so $\limsup_{k\to\infty} \|\Delta^k F(0,\dots,0)\|_\#^{1/k} \leq 1/R$. Moreover, by Corollary 2.5.8, $F(z)$ is given as the sum of $\Delta^k F(0,\dots,0)^\#[z,\dots,z]$. □

Non-Commutative Polynomials

In particular, if $F(X) = a_0 X a_1 \dots X a_k$ is a monomial in $\mathcal{A}\langle X\rangle$, then there is a correpsonding multilinear form

$$\Lambda : (z_1,\dots,z_k) \mapsto a_0 z_1 a_1 \dots z_k a_k.$$

Note that

$$\Lambda^{(n)}(z_1,\dots,z_k) = a_0^{(n)} z_1 a_1^{(n)} \dots z_k a_k^{(n)}.$$

Thus, $\|\Lambda^{(n)}\| \leq \|a_0\| \dots \|a_k\|$, so that Λ is completely bounded. Thus, we can define a fully matricial function by

$$F^{(n)}(z) = \Lambda^{(n)}(z,\dots,z) = a_0^{(n)} z a_1^{(n)} \dots z a_k^{(n)}.$$

By linearity, for every non-commutative polynomial $F(X) \in \mathcal{A}\langle X\rangle$, the function $F(z)$ is fully matricial on $M_\bullet(\mathcal{A})$. Moreover, the derivatives $\Delta^k F$ are computed as in Lemma 2.5.9. For example, if $F(z) = a_0 z a_1 \dots z a_\ell$ and if $z_0,\dots, z_\ell \in \mathcal{A}$, we have

$$\Delta^k F(z_0,\dots,z_k)[w_1,\dots,w_k] =$$
$$\sum_{1\leq \ell_1 < \ell_2 < \dots < \ell_k \leq \ell} (a_0 z_0 a_1 \dots z_0 a_{\ell_1 - 1}) w_1 (a_{\ell_1} z_1 a_{\ell_1+1} \dots z_1 a_{\ell_2-1}) \dots w_k (a_{\ell_k} z_k a_{\ell_k+1} \dots z_k a_\ell).$$

2.7 Algebraic Operations

Proposition 2.7.1. *Suppose that $F, G : \Omega \to M_\bullet(\mathcal{A})$ are fully matricial. Then so are $F + G$ and FG.*

Proof. Note that if $zT = Tw$ for some scalar matrix T, then we have

$$(F+G)(z)T = F(z)T + G(z)T = TF(w) + TG(w) = T(F+G)(w),$$

and

$$(FG)(z)T = F(z)G(z)T = F(z)TG(w) = TF(w)G(w) = T(FG)(w),$$

so that $F+G$ and FG respect intertwinings. To show $F+G$ and FG are uniformly locally bounded, pick $z_0 \in \Omega^{(n)}$. Then F is bounded by M_1 on some ball $B(z_0, R_1)$ and G is bounded by M_2 on some ball $B(z_0, R_2)$. Letting $R = \min(R_1, R_2)$, we have

$$\|z - z_0^{(m)}\| \leq R \implies \|F(z) + G(z)\| \leq M_1 + M_2 \text{ and } \|F(z)G(z)\| \leq M_1 M_2.$$

□

Lemma 2.7.2. *The sequence of sets $\Omega^{(n)} = \{z \in \mathcal{A} : z \text{ is invertible}\}$ is a matricial domain and the function*

$$\operatorname{inv} : \Omega \to \Omega : z \mapsto z^{-1}$$

is fully matricial.

Proof. Note that Ω respects direct sums and is nonempty. To show that Ω is uniformly open, suppose that $z \in \Omega^{(n)}$. Then we claim that $B(z, 1/\|z^{-1}\|)$ is contained in Ω. To see this note that if $w \in B(z, 1/\|z^{-1}\|)$, then the series

$$w^{-1} = [z - (z-w)]^{-1} = z^{-1}[1 - (z-w)z^{-1}]^{-1} = \sum_{k=0}^{\infty} z^{-1}[(z-w)z^{-1}]^k$$

converges and we have

$$\|w^{-1}\| \leq \frac{\|z^{-1}\|}{1 - \|z^{-1}\|\|z-w\|}.$$

This same estimate shows that inv is uniformly locally bounded.

To show that inv respects intertwinings, suppose that $zT = Tw$. Multiplying by z^{-1} on the left and w^{-1} on the right yields $Tw^{-1} = z^{-1}T$ or $\operatorname{inv}(z)T = T\operatorname{inv}(w)$. □

Proposition 2.7.3. *Suppose that $F : \Omega_1 \to \Omega_2$ and $G : \Omega_2 \to \Omega_3$ are fully matricial. Then so is $G \circ F$.*

Proof. To show that $G \circ F$ respects intertwinings, suppose that $zT = Tw$. Then $F(z)T = TF(w)$ and hence $G(F(z))T = TG(F(w))$. To show uniform local boundedness, pick a point z_0. By uniform local boundedness of G, we can choose R and $M > 0$ such that $G(B(F(z_0), R)) \subseteq B(0, M)$. But by Corollary 2.5.10, there exists an R' such that $F(B(z_0, R')) \subseteq B(F(z_0), R)$. Thus, $G \circ F$ is uniformly bounded on $B(z_0, R')$. □

2.8 Inverse Function Theorem

We now present an inverse function theorem for fully matricial functions. For background and related results, see [Voi04, §11.5], [AKV13], [AKV15], [AM16]. In particular, the following result is a version of [AKV15, Theorem 1.4].

Theorem 2.8.1. *Let $z_* \in M_n(\mathcal{A})$ and $w_* \in M_n(\mathcal{B})$. Suppose that $F : B(z_*, R) \to B(w_*, M)$ is fully matricial with $F(z_*) = w_*$. Suppose that $\Lambda_1 = \Delta F(z_*, z_*)$ is invertible with $\|\Lambda_1^{-1}\|_\# \leq K$. Then there exist r_1 and r_2 such that the following holds.*

1. *For each $w \in B(w_*, r_2)$, there exists a unique $z \in B(z_*, r_1)$ with $F(z) = w$.*
2. *The inverse function $F^{-1} : B(w_*, r_2) \to B(z_*, r_1)$ is fully matricial.*

More precisely, we can take

$$r_1 = R\rho_1\left(\frac{MK}{R}\right), \qquad r_2 = \frac{R}{K}\rho_2\left(\frac{MK}{R}\right),$$

where

$$\rho_1(t) = 1 - \frac{t^{1/2}}{(1+t)^{1/2}} \qquad \rho_2(t) = 1 + 2t - 2t^{1/2}(1+t)^{1/2}.$$

Proof. First, consider the special case where $\mathcal{A} = \mathcal{B}$, $z_* = 0^{(n)}$, $w_* = 0^{(n)}$, $R = 1$, and $\Lambda_1 = \mathrm{id}$. Let $\Lambda_k = \Delta^k F(0^{(n)}, \dots, 0^{(n)})$. For $w \in M_{mn}(\mathcal{A})$, note that $F(z) = w$ if and only if z is a fixed point of the function

$$H_w(z) = w + z - F(z) = w - \sum_{k=2}^\infty \Lambda_k(z, \dots, z).$$

We want to show that if r and w are sufficiently small, then G_w defines a contraction $\overline{B}^{(nm)}(0,r) \to \overline{B}^{(nm)}(0,r)$ and hence has a unique fixed point in $\overline{B}^{(nm)}(0,r)$.

To determine when H_w is a contraction, we estimate $H_w(z) - H_w(z')$. Let $\Lambda_k = \Delta^k F(0^{(n)}, \dots, 0^{(n)})$. Then for $\|z'\|$ and $\|z\| \leq r$, we have

$$\begin{aligned}\|H_w(z) - H_w(z')\| &\leq \sum_{k=2}^\infty \|\Lambda_k(z, \dots, z) - \Lambda_k(z', \dots, z')\| \\ &\leq \sum_{k=2}^\infty \sum_{j=0}^{k-1} \|\Lambda_k(\underbrace{z, \dots, z}_{j}, z - z', \underbrace{z', \dots, z'}_{k-1-j})\| \\ &\leq M \sum_{k=2}^\infty k r^{k-1} \|z - z'\| \\ &= M\left(\frac{1}{(1-r)^2} - 1\right)\|z - z'\|.\end{aligned}$$

Therefore, H_w is a contraction provided that

$$M\left(\frac{1}{(1-r)^2} - 1\right) < 1$$

or equivalently $r < \rho_1(M)$.

To determine when $H_w(z)$ maps $B^{(mn)}(0,r)$ into itself, note that for $\|z\| \leq r$

$$\begin{aligned}\|H_w(z)\| &\leq \|w\| + \sum_{k=2}^{\infty} \|\Lambda_k(z,\dots,z)\| \\ &\leq \|w\| + \frac{Mr^2}{1-r}.\end{aligned}$$

Thus, we have $\|H_w(z)\| \leq r$ provided that

$$\|w\| \leq \psi(r) := r - \frac{Mr^2}{1-r}.$$

Altogether, we have shown that $r < \rho_1(M)$ and $\|w\| \leq \psi(r)$, then H_w is a strict contraction $B^{(mn)}(0,r) \to B^{(mn)}(0,r)$. Therefore, by the Banach fixed point theorem, H_w has a unique fixed point in $B^{(mn)}(0,r)$. We denote this fixed point by $G^{(mn)}(w)$. Thus, $G^{(mn)}$ is a function $\overline{B}^{(mn)}(0,\psi(r)) \to \overline{B}^{(mn)}(0,r)$ for $r < \rho_1(M)$. By uniqueness of the fixed point, the value of $G(w)$ is independent of the choice of r, so G defines a function on the union of the balls $\overline{B}^{(mn)}(0,\psi(r))$ for $r < \rho_1(M)$. But $\psi(\rho_1(M)) = \rho_2(M)$, and thus G defines a function $B(0^{(n)},\rho_2(M)) \to B(0^{(n)},\rho_1(M))$.

We claim that G is fully matricial. Consider a similarity $w' = SwS^{-1}$ where $z, z' \in B(0^{(n)},\rho_2(M))$ and $S \in GL_n(\mathbb{C})$. For r sufficiently close to $\rho_1(M)$, we have $\|w\|, \|w'\| \leq \psi(r)$. Note that $F(SG(w)S^{-1}) = SF(G(w))S^{-1} = SwS^{-1}$ and thus by uniqueness of the fixed point for $H_{SwS^{-1}}$ on $B^{(mn)}(0,r)$, we have $G(SwS^{-1}) = SG(w)S^{-1}$. The argument for direct sums is similar.

This completes the proof in the special case where $\mathcal{A} = \mathcal{B}$, $z_* = 0^{(n)}$, $w_* = 0^{(n)}$, $R = 1$, and $\Lambda_1 = \mathrm{id}$. Now consider a function F which satisfies the hypotheses of the theorem in the general case. Let

$$\widehat{F}^{(nm)}(z) = \frac{1}{R}(\Lambda_1^{-1})^{\#}[F^{(nm)}(Rz + z_*) - w_*].$$

Then $\widehat{F}$ is a fully matricial function $B(0^{(n)},1) \to B(0^{(n)}, MK/R)$. The previous argument yields an inverse function $\widehat{G} : B(0^{(n)},\rho_2(MK/R)) \to B(0^{(n)},\rho_1(MK/R))$. The inverse function to F is given by

$$G(w) = R\widehat{G}\left(\frac{1}{R}(\Lambda_1^{-1})^{\#}[w - w_*]\right) + z_*,$$

and this function is defined $B(w_*, (R/K)\rho_2(MK/R)) \to B(z_*, R\rho_1(MK/R))$. □

Remark 2.8.2. Curiously, $\psi(r)$ is maximized when $r = \rho_1(M)$. Thus, the choice of r which will guarantee that H_w maps $B^{(mn)}(0,r)$ into $B^{(mn)}(0,r)$ for the largest range of w is $r = \rho_1(M)$. This is the same as the largest choice of r which will guarantee that H_w is a contraction.

Remark 2.8.3. In fact, the proof never used directly the fact that $F(B(z_*,R)) \subseteq B(w_*,M)$. It only used the Cauchy esimate

$$\|\Delta^k F(z_*,\dots,z_*)\| \leq \frac{M}{R^k}.$$

Thus, the conclusion of the theorem holds when we replace the boundedness assumption by this Cauchy estimate.

Furthermore, the inverse function depends continuously on the original function F in the following sense.

Proposition 2.8.4. *Let $F, G : B(z_*, R) \to M_\bullet(\mathcal{B})$ be fully matricial. Suppose that*

$$F(B(z_*, R)) \subseteq B(F(z_*), M), \qquad G(B(z_*, R)) \subseteq B(G(z_*), M).$$

Suppose that $\Delta F(z_*, z_*)$ and $\Delta G(z_*, z_*)$ are invertible with

$$\|\Delta F(z_*, z_*)^{-1}\|_\# \leq K, \qquad \|\Delta G(z_*, z_*)^{-1}\|_\# \leq K.$$

Let r_1 and r_2 be as in Theorem 2.8.1 and let

$$F^{-1} : B(F(z_*), r_2) \to B(z_*, r_1), \qquad G^{-1} : B(G(z_*), r_2) \to B(z_*, r_1)$$

be the inverse functions given by that theorem. If we have

$$\sup_{z \in B(z_*, r_1)} \|F(z) - G(z)\| \leq \frac{r_2}{3},$$

then

$$\sup_{w \in B(G(w), r_2/3)} \|F^{-1}(w) - G^{-1}(w)\| \leq \frac{9r_1}{2r_2^2} \sup_{z \in B(z_*, r_1)} \|F(z) - G(z)\|.$$

Proof. Let $w \in B(G(z_*), r_2/3)$. Note that $B(G(z_*) r_2/3) \subseteq B(F(z_*), 2r_2/3)$ and hence $F^{-1}(w)$ is defined. Now let $w' = F \circ G^{-1}(w)$ and note that

$$F^{-1}(w) - G^{-1}(w) = F^{-1}(w) - F^{-1} \circ F \circ G^{-1}(w) = F^{-1}(w) - F^{-1}(w').$$

Moreover, we have

$$\|w - w'\| = \|G \circ G^{-1}(w) - F \circ G^{-1}(w)\| \leq \sup_{z \in B(z_*, r_1)} \|F(z) - G(z)\| \leq \frac{r_2}{3}.$$

Now because F^{-1} maps $B(z_*, r_2)$ into $B(F(z_*), r_1)$, we have by Lemma 2.4.1 and Proposition 2.5.9

$$\begin{aligned} \|F^{-1}(w) - F^{-1}(w')\| &= \left\|\Delta[F^{-1}](w, w')[w - w']\right\| \\ &\leq \frac{r_1}{(r_2 - \|w - F(z_*)^{(m)}\|)(r_2 - \|w' - F(z_*)^{(m)}\|)} \|w - w'\|. \end{aligned}$$

But $w \in B(F(z_*), r_2/3)$ and $w' \in B(F(z_*), 2r_2/3)$ and therefore

$$\begin{aligned} \|F^{-1}(w) - F^{-1}(w')\| &\leq \frac{r_1}{(r_2 - r_2/3)(r_2 - 2r_2/3)} \|w - w'\| \\ &\leq \frac{9r_1}{2r_2^2} \sup_{z \in B(z_*, r_1)} \|F(z) - G(z)\|. \end{aligned}$$

□

2.9 Uniformly Locally Bounded Families

In complex analysis, the identity theorem states that if two analytic functions on a connected domain Ω agree in a neighborhood of a point z_0, then they must agree on Ω. Another related result is that if a sequence of functions f_n is uniformly locally bounded, and if $f_n \to f$ in a neighborhood of a point, then $f_n \to f$ locally uniformly on Ω.

More generally, for a family of functions which is uniformly locally bounded, the topology of local uniform convergence on Ω is metrizable with the metric given by $\sup_{z \in B(z_0, r)} |f(z) - g(z)|$. In fact, for various choices of z_0 and r, we obtain equivalent metrics.

We will now describe the fully matricial analogues of these results.

Definition 2.9.1. A fully matricial domain Ω is *connected* if z and w are in $\Omega^{(n)}$, then there exists $m > 0$ such that $z^{(m)}$ and $w^{(m)}$ are in the same connected component of $\Omega^{(nm)}$.

Definition 2.9.2. We say that a family $\mathcal{F}$ of fully matricial functions $\Omega \to M_\bullet(\mathcal{B})$ is *uniformly locally bounded* if for every $z_* \in \Omega$, there exists $R > 0$ and $M > 0$ such that

$$\sup_{z \in B(z_*,R)} \|F(z)\| \leq M \text{ for all } F \in \mathcal{F}.$$

Definition 2.9.3. Let $\mathcal{F}$ be uniformly locally bounded family of fully matricial functions $\Omega \to M_\bullet(\mathcal{B})$. For $z_* \in \Omega$, we denote

$$\operatorname{rad}(z_*, \mathcal{F}) = \sup \left\{ R > 0 : \sup_{F \in \mathcal{F}} \sup_{z \in B(0,R)} \|F(z)\| < +\infty \right\},$$

and we call $\operatorname{rad}(z_*, \mathcal{F})$ the *radius of uniform local boundedness of $\mathcal{F}$ at z_*.*

Definition 2.9.4. Let $\mathcal{F}$ be uniformly locally bounded family of fully matricial functions $\Omega \to M_\bullet(\mathcal{B})$. For $z_* \in \Omega$ and $r < \operatorname{rad}(z_*, \mathcal{F})$, we define

$$d_{z_*,r}(F, G) = \sup_{z \in B(z_*,r)} \|F(z) - G(z)\|$$

and

$$d'_{z_*,r}(F, G) = \sum_{k=0}^{\infty} r^k \|\Delta^k F(z_*, \dots, z_*) - \Delta^k G(z_*, \dots, z_*)\|_\#.$$

Definition 2.9.5. Let d_1 and d_2 be metrics on a set $\mathcal{X}$. We say that $d_1 \lesssim d_2$ if the map $\operatorname{id}_X : (\mathcal{X}, d_2) \to (\mathcal{X}, d_1)$ is uniformly continuous. In other words, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_2(x, y) < \delta \implies d_2(x, y) < \epsilon.$$

We say that d_1 and d_2 are *uniformly equivalent* or $d_1 \sim d_2$ if we have $d_1 \lesssim d_2$ and $d_2 \lesssim d_1$. Note that $\lesssim$ is transitive and $\sim$ is an equivalence relation.

Theorem 2.9.6. *Let Ω be a connected fully matricial domain. Let $\mathcal{F}$ be a uniformly locally bounded family of fully matricial functions $\Omega_1 \to M_\bullet(\mathcal{A}_2)$.*

1. *For $z_* \in \Omega$ and $r < \operatorname{rad}(z_*, \mathcal{F})$, the functions $d_{z_*,r}$ and $d'_{z_*,r}$ are metrics on $\mathcal{F}$.*
2. *All the metrics in the collection $\{d_{z_*,r}, d_{z_*,r} : z_* \in \Omega, r < \operatorname{rad}(z_*, \mathcal{F})\}$ are uniformly equivalent to each other.*

Proof.

Step 1: From the definition $d_{z_*,r}$, we see that $d_{z_*,r}$ is finite, satisfies the triangle inequality, and satisfies $d_{z_*,r}(F, G) = d_{z_*,r}(G, F)$. To show that $d'_{z_*,r}$ is finite, choose R with $r < R < \operatorname{rad}(z_*, \mathcal{F})$. By applying the Cauchy estimates (2.5.2), we see that for some constant M, we have

$$\|\Delta^k (F - G)(z_*, \dots, z_*)\|_\# \leq \frac{2M}{R^k},$$

so for $r < R$, we have

$$\sum_{k=0}^{\infty} r^k \|\Delta^k (F - G)(z_*, \dots, z_*)\|_\# < +\infty.$$

It is also immediate that $d_{z_*,r}$ satisfies the symmetry and triangle inequality properties. In other words, $d_{z_*,r}$ and $d'_{z_*,r}$ are pseudometrics.

Step 2: We claim that $d_{z_*,r} \lesssim d'_{z_*,r}$. Note that for $z \in B^{(nm)}(z_*, r)$, we have

$$\begin{aligned}\|(F-G)(z)\| &\leq \sum_{k=1}^{\infty} \left\| \Delta^k (F-G)(z_*, \dots, z_*)^{\#}[z - z_*^{(m)}, \dots, z - z_*^{(m)}] \right\| \\ &\leq \sum_{k=1}^{\infty} \left\| \Delta^k (F-G)(z_*, \dots, z_*) \right\|_{\#} r^k,\end{aligned}$$

and hence $d_{z_*,r} \leq d'_{z_*,r}$.

Step 3: We claim that for $r_1, r_2 < \operatorname{rad}(z_*, \mathcal{F})$, we have $d'_{z_*,r_1} \lesssim d_{z_*,r_2}$. First, choose R with $r_1 < R < \operatorname{rad}(z_*, \mathcal{F})$, choose M such that

$$\sup_{z \in B(z_*, R)} \|F(z)\| \leq M$$

and note that by the Cauchy estimate (2.5.2), we have

$$\|\Delta^k (F-G)(z_*, \dots, z_*)\|_{\#} \leq \frac{2M}{R^k}.$$

By the same estimate we have

$$\|\Delta^k (F-G)(z_*, \dots, z_*)\|_{\#} \leq \frac{d_{z_*,r_2}(F,G)}{r_2^k}$$

Thus, we have

$$\begin{aligned}d'_{z_*,r_1}(F,G) &= \sum_{k=0}^{\infty} r_1^k \|\Delta^k (F-G)(z_*, \dots, z_*)\|_{\#} \\ &\leq \sum_{k=0}^{N-1} d_{z_*,r_2}(F,G) \left(\frac{r_1}{r_2} \right)^k + \sum_{k=N}^{\infty} 2M \left(\frac{r_1}{R} \right)^k \\ &= d_{z_*,r_2}(F,G) \frac{(r_1/r_2)^N - 1}{r_1/r_2 - 1} + \frac{2M(r_1/R)^N}{1 - r_1/R}.\end{aligned}$$

If $\epsilon > 0$, then by choosing N large enough, we can make the second term smaller than $\epsilon/2$. After we fix such an N, then if $d_{z_*,r_2}(F,G)$ is sufficiently small, then the first term will also be less than $\epsilon/2$. This shows that $d'_{z_*,r_1} \lesssim d_{z_*,r_2}$.

Step 4: Using Steps 2 and 3, we see that for $r_1, r_2 < \operatorname{rad}(z_*, \mathcal{F})$, we have

$$d_{z_*,r_1} \lesssim d'_{z_*,r_1} \lesssim d_{z_*,r_2},$$

so the distances $d_{z_*,r}$ are equivalent for different values of r. Similarly,

$$d'_{z_*,r_1} \lesssim d_{z_*,r_2} \lesssim d'_{z_*,r_2},$$

so the distances $d'_{z_*,r}$ are equivalent for different values of r. Finally, the distances $d_{z_*,r}$ and $d'_{z_*,r}$ are equivalent.

Step 5: Let us write $z \sim z'$ if the pseudometrics $d_{z,r}$ and $d_{z',r'}$ are equivalent for some r and r' (or equivalently for all r and r'). This defines an equivalence relation on Ω_1. We claim that each equivalence class is uniformly open. To see this, fix $z_* \in \Omega^{(n)}$. Choose an $R > 0$ and $M > 0$ such that

$$\sup_{F \in \mathcal{F}} \sup_{z \in B(0,R)} \|F(z)\| \leq M.$$

Suppose that $z \in B^{(nm)}(z_*, R/3)$. Then $B(z, 2R/3) \subseteq B(z_*, R)$ and hence $2R/3 < \operatorname{rad}(z, \mathcal{F})$. Also, since $B(z, 2R/3) \subseteq B(z_*, R)$, we have

$$d_{z,2R/3} \leq d_{z_*,R}.$$

On the other hand, we also have $\|z_*^{(m)} - z\| < R/3$ and hence $B(z_*^{(m)}, R/3) \subseteq B(z, 2R/3)$. Using the fact that F preserves direct sums, we have

$$\begin{aligned} d_{z_*,R/3}(F,G) &= \sup_{w \in B(z_*,R/3)} \|F(w) - G(w)\| \\ &= \sup_{w \in B(z_*,R/3)} \|F(w^{(m)}) - G(w^{(m)})\| \\ &\leq \sup_{w' \in B(z_*^{(m)},R/3)} \|F(w) - G(w)\| \\ &\leq d_{z,2R/3}(F,G). \end{aligned}$$

Therefore, we have $d_{z_*,R/3} \leq d_{z,2R/3} \leq d_{z_*,R}$ and hence $z \sim z_*$.

Step 6: Now we show that any two points $z_1 \in \Omega^{(n_1)}$ and $z_2 \in \Omega^{(n_2)}$ are equivalent. Note that because Ω is connected, there exists m such that $z_1^{(n_2 m)}$ and $z_2^{(n_1 m)}$ are in the same connected component of $\Omega^{(n_1 n_2 m)}$. As a consequence of Step 5, the equivalence classes of points in $\Omega^{(n_1 n_2 m)}$ are open subsets of $\Omega^{(n_1 n_2 m)}$. Each equivalence class in $\Omega^{(n_1 n_2 m)}$ is also relatively closed because its complement is the union of the other equivalence classes. Because $z_1^{(n_2 m)}$ and $z_2^{(n_1 m)}$ are in the same connected component, we must have $z_1^{(n_2 m)} \sim z_2^{(n_1 m)}$. As another consequence of Step 5, we have $z_1 \sim z_1^{(n_2 m)}$ and $z_2 \sim z_2^{(n_1 m)}$ and therefore $z_1 \sim z_2$.

Step 7: We have now shown that all the pseudometrics in claim (2) are uniformly equivalent. As a consequence if $d_{z_*,r}(F,G) = 0$ for some z_* and r, then this holds for all z_* and r which implies that $F = G$. Therefore, each $d_{z_*,r}$ is a metric. □

Corollary 2.9.7 (Identity Theorem)**.** *Let $\Omega \subseteq M_\bullet(\mathcal{A})$ be a connected fully matricial domain, and let $F, G : \Omega \to M_\bullet(\mathcal{B})$ be fully matricial functions and $z_* \in \Omega_1^{(n_0)}$. The following are equivalent:*

1. $\Delta^k F(z_*, \dots, z_*) = \Delta^k G(z_*, \dots, z_*)$ *for all* k.
2. $F = G$ *on* $B(z_*, r)$ *for some* $r > 0$.
3. $F = G$ *on* Ω.

Proof. Note that the family $\{F, G\}$ is uniformly locally bounded. Hence, this follows immediately from Theorem 2.9.6 □

Another consequence of the theorem is that if a sequence $\{F_n\}$ is uniformly locally bounded, and if F_n converges uniformly in a neighborhood of a point, then it converges on all of Ω in the following sense.

Definition 2.9.8. We say that a sequence F_n of fully matricial functions $\Omega \to M_\bullet(\mathcal{B})$ *converges uniformly locally to* F if for every $z_0 \in \Omega^{(n_0)}$, there exists $R > 0$ such that

$$\lim_{n\to\infty} \sup_{z \in B(z_0,R)} \|F_n(z) - F(z)\| = 0.$$

Lemma 2.9.9. *If F_n is fully matricial and $F_n \to F$ uniformly locally, then F is fully matricial.*

Proof. Note that F respects intertwinings because $zT = Tw$, then

$$F(z)T = \lim_{n\to\infty} F_n(z)T = \lim_{n\to\infty} TF_n(w) = TF(w).$$

To show that F is uniformly locally bounded, fix z_0. There exists $R > 0$ and n such that $\sup_{z\in B(z_0,R)} \|F_n(z) - F(z)\| \leq 1$. Since F_n is fully matricial, there exists r and M such $\sup_{z\in B(z_*,r)} \|F_n(z)\| \leq M$. This implies that $\|F(z)\| \leq M + 1$ for $z \in B(z_*, \min(r, R))$. □

Corollary 2.9.10. *Let Ω be a connected fully matricial domain and let $F_n : \Omega \to M_\bullet(\mathcal{B})$ be a sequence of fully matricial functions which is uniformly locally bounded. Let $z_* \in \Omega^{(n)}$. Then the following are equivalent:*

1. *For every k, the sequence $\Delta^k F_n(z_*, \dots, z_*)$ converges with respect to $\|\cdot\|_\#$.*
2. *For some $r > 0$, the sequence F_n converges uniformly on $B(z_*, r)$.*
3. *There exists some fully matricial function F such that $F_n \to F$ uniformly locally on Ω_1.*

Proof. Suppose that $R < \operatorname{rad}(z_*, \{F_n\})$. Using the Cauchy estimates, we see that

$$\sum_{k=0}^{\infty} R^k \|\Delta^k(F_n - F_m)(z_*, \dots, z_*)\|_\#$$

converges absolutely and the rate of convergence is independent of n and m. Therefore, (1) occurs if and only if $\{F_n\}$ is Cauchy in $d_{z_*,R}$. Because the metrics in Theorem 2.9.6 are uniformly equivalent, they preserve Cauchy sequences. Hence, $\{F_n\}$ is Cauchy in $d_{z_*,R}$ if and only if it is Cauchy in $d_{z,r}$ for every z and $r < \operatorname{rad}(z, \{F_n\})$. This is equivalent to (2) and equivalent to (3) in light of Lemma 2.9.9. □

2.10 Problems and Further Reading

Problem 2.1.

1. For a fully matricial domain Ω, define the similarity-invariant envelope $\operatorname{Sim}(\Omega)$ by

$$\operatorname{Sim}(\Omega)^{(n)} = \{SzS^{-1} : z \in \Omega^{(n)}, S \in GL_n(\mathbb{C})\}.$$

   Prove that $\operatorname{Sim}(\Omega)$ is a fully matricial domain.

2. Let $F : \Omega_1 \to \Omega_2$ be fully matricial. Show that F has a unique fully matricial extension to a function $\operatorname{Sim}(\Omega_1) \to \operatorname{Sim}(\Omega_2)$.

Problem 2.2. Suppose that F is fully matricial and $F(B(z_0, R)) \subseteq B(0, M)$. Show that the finite Taylor-Taylor expansion in Lemma 2.4.1 holds for all $z \in B(z_0, R)$, not just $z \in B(z_0, R/\sqrt{2})$.

Problem 2.3. Let $\Lambda : M_{n_0,n_1}(\mathcal{A}_\infty) \times \cdots \times M_{n_{k-1},n_k}(\mathcal{A}_1) \to M_{n_0,n_k}(\mathcal{A}_2)$ be a multilinear form. Show that

$$\|\Lambda\|_\# = \sup_m \|\Lambda^{(m,\dots,m)}\|.$$

Problem 2.4. Suppose that $F : \Omega_1 \to \Omega_2$ is fully matricial, let $z_j \in \Omega^{(n_j)}$, and suppose that $F(B(z_0 \oplus \cdots \oplus z_k, R)) \subseteq B(0, M)$. Prove that

$$\|\Delta^k F(z_0, \dots, z_k)\|_\# \leq \frac{M}{R^k}.$$

Hint: Use Lemma 2.5.2 and Problem 2.1.

Problem 2.5. State and prove a similarity-invariance property for $\Delta^k F(z_0, \dots, z_k)$.

Problem 2.6. Let $F, G : \Omega_1 \to M_\bullet(\mathcal{A}_2)$ be fully matricial.

1. Compute the power series expansion of FG by formally manipulating the power series of F and G and appealing to Lemma 2.5.11 for justification.
2. Compute $\Delta^k(FG)(z_0, \dots, z_k)$ directly using upper triangular matrices.

Problem 2.7. Let $F : \Omega_1 \to \Omega_2$ and $G : \Omega_2 \to \Omega_3$ be fully matricial. Compute the power series expansion of $G \circ F$ at z_0 by formally manipulating the power series expansions of F and G and appealing to Lemma 2.5.11. (Compare [Voi04, §13.10].

Problem 2.8. Define what it means for a sequence of fully matricial functions to be uniformly locally Cauchy. Show that a sequence is uniformly locally Cauchy if and only if it is uniformly locally convergent.

Problem 2.9. Suppose that $F_n : \Omega_1 \to M_\bullet(\mathcal{A}_2)$ is a uniformly locally bounded sequence of fully matricial functions. State and prove a version of Corollary 2.9.10 holds where uniform local convergence is replaced by:

1. Pointwise convergence in $\|\cdot\|_{M_n(\mathcal{A})}$.
2. Pointwise convergence in weak, strong, or σ-weak operator topology with respect to a given realization of $\mathcal{A}_2$ acting on some Hilbert space $\mathcal{H}$.
3. Convergence in $\|\cdot\|_2$ which is uniform on $B(z_0, R)$, where we assume that $(\mathcal{A}, \tau)$ is a tracial von Neumann algebra and $\|z\|_2 = [\tau \otimes \mathrm{Tr}(z^* z)]^{1/2}$ for $z \in M_n(\mathcal{A})$.

Further Reading

A systematic development of fully matricial function theory was given by Dmitry Kaliuzhnyi-Verbovetskyi and Victor Vinnikov [KVV14]. The authors work in much greater algebraic generality than we need here, replacing vector spaces over $\mathbb{C}$ with modules over a general commutative ring. In particular, they describe how to extend the domain of the function to a similarity-invariant envelope. They also show that the derivatives $\Delta^k F$ are higher-order matricial functions, characterized by a more complicated versions of the similarity and direct-sum conditions. We refer to their bibliography and introduction for a more complete list of references.

We remark that the free difference quotient has also been used in the study of operator modulus of continuity by Peller [Pel06], which will come up in our study of quantitative non-commutative central limit theorems.

Chapter 3

The Cauchy-Stieltjes Transform

3.1 Introduction

Recall that the Cauchy-Stieltjes transform of a finite measure on the real line is $g_\mu(\zeta) = \int_{\mathbb{R}} (\zeta - t)^{-1} \, d\mu(t)$. The Cauchy-Stieltjes transforms of spectral measures are an important tool for non-commutative probability both for computation and for analytic estimates. Some of its most useful properties are the following.

1. For a compactly supported measure μ, the power series coefficients of g_μ at ∞ are the moments of μ.
2. There are simple and sharp a priori estimates on g_μ and its derivatives; for instance, if $\operatorname{Im} \zeta \geq \epsilon$, then $|\partial_\zeta^n g_\mu(\zeta)| \leq \mu(\mathbb{R}) / \epsilon^{n+1}$.
3. There are straightforward analytic conditions that test whether a function g is the Cauchy-Stieltjes transform of some measure.

Properties (2) and (3) together mean that if an analytic function g satisfies some simple analytic conditions, then we obtain much more precise analytic information about g "for free."

This chapter will prove analogous properties to (1) - (3) above for the fully matricial Cauchy-Stieltjes transform of an $\mathcal{A}$-valued law. The main theorem will be the analytic characterization of Cauchy-Stieltjes transforms due to Williams [Wil17, Theorem 3.1]. As motivation for this result, and as an ingredient for the proof, we now state the analytic characterization of Cauchy-Stieltjes transforms in the scalar case. Here $\mathbb{H}_+ = \{\zeta \in \mathbb{C} : \operatorname{Im} \zeta > 0\}$ and $\mathbb{H}_- = \{\zeta \in \mathbb{C} : \operatorname{Im} \zeta < 0\}$ are the upper and lower half-planes.

Lemma 3.1.1. *Let $g : \mathbb{H}_+ \to \mathbb{C}$. The following are equivalent:*

1. *g is the Cauchy-Stieltjes transform of a measure μ supported in $[-M, M]$.*
2. *g is analytic, g maps $\mathbb{H}_+$ into $\mathbb{H}_-$, and $\tilde{g}(\zeta) = g(1/\zeta)$ has an analytic extension to $B(0, 1/M)$ satisfying $\tilde{g}(0) = 0$ and $g(\overline{\zeta}) = \overline{g(\zeta)}$.*

Proof. If $g(\zeta) = \int_{\mathbb{R}} (\zeta - t)^{-1} \, d\mu(t)$, then clearly g is an analytic function $\mathbb{H}_+ \to \mathbb{H}_-$. Moreover,

$$\tilde{g}(\zeta) = \int_{\mathbb{R}} \zeta (1 - t\zeta)^{-1} \, d\mu(t)$$

which is analytic on $B(0, 1/M)$, preserves complex conjugates, and vanishes at 0.

Conversely, suppose that g satisfies these analytic conditions. Recall that if u is bounded and continuous on $\overline{\mathbb{H}}_+$ and harmonic on $\mathbb{H}_+$, then

$$u(\zeta) = -\int_{\mathbb{R}} \frac{1}{\pi} \operatorname{Im}(\zeta - t)^{-1} u(t)\, dt;$$

this is because the integral on the right hand side is harmonic and bounded with the same limiting values as u on the boundary of $\mathbb{H}_+$. Letting $u_\delta(\zeta) = \operatorname{Im} g(t + i\delta)$, we have

$$\operatorname{Im} g(\zeta + i\delta) = -\int_{\mathbb{R}} \operatorname{Im}(\zeta - t)^{-1} \frac{1}{\pi} \operatorname{Im} g(t + i\delta)\, dt.$$

Now $-\int_{\mathbb{R}} (\zeta - t)^{-1} \pi^{-1} g(t + i\delta)\, dt$ is analytic on $\mathbb{H}_+$ and has the same imaginary part as $g(\zeta)$, so they must be equal up to adding a real constant. But both functions vanish as $\zeta \to \infty$ along the positive imaginary axis, and hence

$$g(\zeta + i\delta) = \int_{\mathbb{R}} (\zeta - t)^{-1}\, d\mu_\delta(t),$$

where

$$d\mu_\delta(t) = -\frac{1}{\pi} \operatorname{Im} g(t + i\delta)\, dt.$$

We want to define μ as a weak limit of μ_δ as $\delta \to 0$. To accomplish this, we first show that μ_δ does not have much mass outside $[-R, R]$ for $R > M$.

Because $\tilde{g}(\zeta)$ is analytic on $B(0, 1/M)$, we know that for $\epsilon > 0$, we have

$$|\zeta| < \frac{1}{R} \implies |\tilde{g}(\zeta)| \leq C_R$$

for some constant $C_R > 0$. Then by Schwarz's lemma for functions on the disk, we have

$$|\zeta| < \frac{1}{R} \implies |\tilde{g}(\zeta)| \leq C_R R |\zeta|.$$

Therefore,

$$|\zeta| > R \implies |g(\zeta)| \leq \frac{C_R'}{|\zeta|}.$$

Now $\operatorname{Im} g = 0$ on $\mathbb{R} \setminus [-M, M]$ and hence for $|t| > R$,

$$\begin{aligned} |\operatorname{Im} g(t + i\delta)| &= |\operatorname{Im} g(t + i\delta) - \operatorname{Im} g(t)| \\ &\leq |g(t + i\delta) - g(t)| \\ &\leq \delta \sup_{s \in [0,\delta]} |g'(t + is)|. \end{aligned}$$

Now $B(t+is, \frac{1}{2}(|t|-R)) \subseteq \{|\zeta| > R + \frac{1}{2}(|t|-R)\}$ where g is bounded by $C_R'/(R+(1/2)(|t|-R)) = 2C_R'/(|t| + R)$, and hence by the Cauchy estimates on derivatives,

$$|g'(t + is)| \leq \frac{2}{|t| - R} \frac{2C_R'}{|t| + R} = \frac{4C_R'}{|t|^2 - R^2}.$$

Thus,

$$|\operatorname{Im} g(t + i\delta)| \leq \frac{4C_R' \delta}{|t|^2 - R^2}.$$

In particular, letting $M < R' < R$, we have

$$\mu_\delta(\mathbb{R} \setminus [-R,R]) \leq \delta \int_{\mathbb{R}\setminus[-R,R]} \frac{4C'_{R'}}{|t|^2 - (R')^2}\, dt,$$

where the integral is finite. Therefore, for each point ζ, we have

$$g(\zeta) = g(\zeta + i\delta) + O(\delta) = \int_{-R}^{R} (\zeta - t)^{-1}\, d\mu_\delta(z) + O(\delta)$$

Moreover,

$$\mu_\delta([-R,R]) \leq \int_{-R}^{R} \frac{2R^2}{t^2 + R^2}\, d\mu_\delta(t) = -2R \operatorname{Im} g(iR + i\delta).$$

Thus, the measures $\mu_\delta|_{[-R,R]}$ have uniformly bounded mass, and hence this family of measures is compact. Therefore, for each R, there exists a sequence δ_n such that $\mu_{\delta_n}|_{[-R,R]}$ converges to some limit μ supported on $[-R,R]$ as $\delta \to 0$. In the limit, we have

$$g(\zeta) = \int (\zeta - t)^{-1}\, d\mu(t).$$

Thus, for each $R > M$, we have $g = g_\mu$ for some μ supported on $[-R,R]$. The moments of μ are uniquely determined by the power series expansion of g at ∞, hence μ is unique. Then μ is supported in $[-R,R]$ for every $R > M$, so that μ is supported in $[-M,M]$. □

3.2 Definition

We have seen in §2.1 that the Cauchy-Stieltjes transform of an $\mathcal{A}$-valued law should be viewed as a fully matricial function over $\mathcal{A}$ rather than simply an $\mathcal{A}$-valued function. To give the full definition, we must first define the natural domain for the Cauchy-Stieltjes transform, which consists of operators with positive imaginary part. Thus, we begin with the basic properties of real and imaginary parts of operators.

Notation 3.2.1. For $z \in M_n(\mathcal{A})$, we denote $\operatorname{Re}(z) = \frac{1}{2}(z + z^*)$ and $\operatorname{Im}(z) = \frac{1}{2i}(z - z^*)$.

Observation 3.2.2. *The operators* $\operatorname{Re}(z)$ *and* $\operatorname{Im}(z)$ *are self-adjoint and* $z = \operatorname{Re}(z) + i \operatorname{Im}(z)$. *Moreover, if* $\mathcal{H}$ *is a Hilbert space or a right Hilbert* $\mathcal{A}$*-module and* $z \in B(\mathcal{H})$ *and* $\xi \in \mathcal{H}$*, then*

$$\operatorname{Re}\langle \xi, z\xi \rangle = \langle \xi, \operatorname{Re}(z)\xi \rangle \qquad \operatorname{Im}\langle \xi, z\xi \rangle = \langle \xi, \operatorname{Im}(z)\xi \rangle.$$

Lemma 3.2.3. *Suppose that* $z \in M_n(\mathcal{A})$ *and* $\operatorname{Im} z \geq \epsilon > 0$*, where* ϵ *is a scalar and the inequality holds in* $\mathcal{A}$*. Then* z *is invertible with* $\|z^{-1}\| \leq 1/\epsilon$ *and* $\operatorname{Im}(z^{-1}) \leq -\epsilon/\|z\|^2$.

Proof. Note that $M_n(\mathcal{A})$ is a C^*-algebra and hence can be realized as a concrete C^*-algebra of operators on a Hilbert space $\mathcal{H}$. Then observe that for $\xi \in \mathcal{H}$, we have

$$\|\xi\|\|z\xi\| \geq |\langle \xi, z\xi \rangle| \geq \operatorname{Im}\langle \xi, z\xi \rangle = \langle \xi, (\operatorname{Im} z)\xi \rangle \geq \langle \xi, \epsilon\xi \rangle = \epsilon\|\xi\|^2,$$

which shows that $\|z\xi\| \geq \epsilon\|\xi\|$ and hence $\ker z = 0$ and $\operatorname{Ran} z$ is closed. On the other hand, we have $\operatorname{Im} z^* = -\operatorname{Im} z \leq -\epsilon$, so similar reasoning shows that $\|z^*\xi\| \geq \epsilon\|\xi\|$ which implies that $\ker z^* = 0$ and hence $\operatorname{Ran}(z) = \mathcal{H}$. Since $\ker z = 0$ and $\operatorname{Ran}(z) = \mathcal{H}$, it follows that z is invertible as a linear operator. Because of the estimate $\|z\xi\| \geq \epsilon\|\xi\|$, we know that z^{-1} is bounded with $\|z^{-1}\| \leq 1/\epsilon$.

Finally, to show that $\operatorname{Im}(z^{-1}) \leq -\epsilon/\|z\|^2$, note that for $\xi \in \mathcal{H}$, we have

$$\operatorname{Im}\langle \xi, z^{-1}\xi\rangle = \operatorname{Im}\langle zz^{-1}\xi, z^{-1}\xi\rangle = \operatorname{Im}\langle z^{-1}\xi, z^*(z^{-1}\xi)\rangle$$
$$= -\operatorname{Im}\langle z^{-1}\xi, z(z^{-1}\xi)\rangle \leq -\epsilon\|z^{-1}\xi\|^2 \leq \frac{-\epsilon}{\|z\|^2}\|\xi\|^2,$$

using the fact that $\|\xi\| = \|zz^{-1}\xi\| \leq \|z\|\|z^{-1}\xi\|$. □

Definition 3.2.4 (Fully Matricial Upper/Lower Half-plane)**.** We define $\mathbb{H}_{+,\epsilon}^{(n)}(\mathcal{A}) = \{z \in M_n(\mathcal{A}) : \operatorname{Im} z \geq \epsilon\}$ and define $\mathbb{H}_{+}^{(n)}(\mathcal{A}) = \bigcup_{\epsilon>0} \mathbb{H}_{+,\epsilon}^{(n)}(\mathcal{A})$. Finally, we define the *fully matricial upper half-plane* as $\mathbb{H}_+(\mathcal{A}) = (\mathbb{H}_+^{(n)}(\mathcal{A}))_{n\in\mathbb{N}}$.

Similarly, we define $\mathbb{H}_{-,\epsilon}^{(n)}(\mathcal{A}) = \{z \in M_n(\mathcal{A}) : \operatorname{Im} z \leq -\epsilon\}$ and $\mathbb{H}_{-}^{(n)}(\mathcal{A}) = \bigcup_{\epsilon>0} \mathbb{H}_{-}^{(n)}(\mathcal{A})$.

Finally, we define $\mathbb{H}_{\pm,0}^{(n)}(\mathcal{A}) = \{z \in M_n(\mathcal{A}) : \pm \operatorname{Im} z \geq 0\}$.

Observation 3.2.5. *$\mathbb{H}_+(\mathcal{A})$ and $\mathbb{H}_-(\mathcal{A})$ are connected fully matricial domains (although $\mathbb{H}_{+,0}(\mathcal{A})$ and $\mathbb{H}_{-,0}(\mathcal{A})$ are not because they fail to be open).*

Proof. To see that $\mathbb{H}_+(\mathcal{A})$ respect direct sums, suppose $z_1 \in \mathbb{H}_+^{(n_1)}(\mathcal{A})$ and $z_2 \in \mathbb{H}_+^{(n_2)}(\mathcal{A})$. Then $\operatorname{Im} z_1 \geq \epsilon_1$ and $\operatorname{Im} z_2 \geq \epsilon_2$ for some $\epsilon_1, \epsilon_2 > 0$. Then $\operatorname{Im}(z_1 \oplus z_2) = \operatorname{Im} z_1 \oplus \operatorname{Im} z_2 \geq \min(\epsilon_1, \epsilon_2)$, so that $z_1 \oplus z_2 \in \mathbb{H}_+^{(n_1+n_2)}(\mathcal{A})$.

To see that $\mathbb{H}_+(\mathcal{A})$ is uniformly open, suppose that $z \in \mathbb{H}_+^{(n)}(\mathcal{A})$. If $\operatorname{Im} z \geq \epsilon > 0$, then we have $B(z, \epsilon) \subseteq \mathbb{H}_+(\mathcal{A})$. Indeed, if $z' \in B^{(mn)}(z^{(n)}, \epsilon)$, then

$$\operatorname{Im} z' \geq \operatorname{Im}(z^{(n)}) - \|z^{(m)} - z'\| = (\operatorname{Im} z)^{(n)} - \|z^{(m)} - z'\| \geq \epsilon - \|z^{(m)} - z'\| > 0.$$

Furthermore, each $\mathbb{H}_+^{(n)}(\mathcal{A})$ is non-empty and connected (in fact, convex), and hence $\mathbb{H}_+(\mathcal{A})$ is non-empty and connected. The argument for $\mathbb{H}_-(\mathcal{A})$ is symmetrical. □

Definition 3.2.6 (Cauchy-Stieltjes Transform)**.** Let $\sigma : \mathcal{A}\langle X\rangle \to \mathcal{A}$ be a generalized law. We define the *Cauchy-Stieltjes transform* G_σ as the sequence of functions $G_\sigma^{(n)} : \mathbb{H}_+^{(n)}(\mathcal{A}) \to \mathbb{H}_{-,0}^{(n)}(\mathcal{A})$ given by

$$G_\sigma^{(n)}(z) = \overline{\sigma}^{(n)}[(z - \overline{X}^{(n)})^{-1}],$$

where $\overline{X}$ is the operator of left multiplication by X on $\mathcal{A}\langle X\rangle \otimes_\sigma \mathcal{A}$ and $\overline{\sigma}(b) = \langle 1 \otimes 1, b(1 \otimes 1)\rangle_\sigma$ (as in Theorem 1.6.7).

Note here that the definition makes sense because if $z \in \mathbb{H}_+^{(n)}$, then for some $\epsilon > 0$, we have

$$\operatorname{Im}(z - \overline{X}^{(n)}) = \operatorname{Im}(z) \geq \epsilon,$$

which implies that $z - \overline{X}^{(n)}$ is invertible.

Lemma 3.2.7. *For a generalized law σ, the Cauchy-Stieltjes transform G_σ is a fully matricial function. We also have*

$$z \in \mathbb{H}_{+,\epsilon}^{(n)}(\mathcal{A}) \implies \|G_\sigma(z)\| \leq \frac{\|\sigma(1)\|}{\epsilon}.$$

Proof. Let $\mathcal{B}$ be the C^*-algebra generated by $\mathcal{A}$ and $\overline{X}$. Note that the inclusion $\mathcal{A} \to \mathcal{B}$ is fully matricial. Moreover, the function $z \mapsto z - \overline{X}^{(n)}$ is the sum of two fully matricial functions, so it is fully matricial on $M_\bullet(\mathcal{A})$ and in particular on $\mathbb{H}_+(\mathcal{A})$. Since inv fully matricial, so is $(z - \overline{X}^{(n)})^{-1}$. Finally, $\overline{\sigma}$ is a completely bounded linear map and hence is fully matricial by Proposition 2.6.1, so $\overline{\sigma}^{(n)}[(z - X^{(n)})^{-1}]$ is fully matricial. ☐

In the future, we will simplify and slightly abuse notation by writing

$$G_\sigma^{(n)}(z) = \sigma^{(n)}[(z - X^{(n)})^{-1}],$$

that is, writing σ instead of $\overline{\sigma}$ even though σ is technically only defined on $\langle A \rangle \langle X \rangle$ and writing X for the multiplication operator $\overline{X}$.

3.3 Derivatives and Expansion at ∞

Lemma 3.3.1. *Let $z_j \in M_{n_j}(\mathcal{A})$ and $w_j \in M_{n_{j-1} \times n_j}(\mathcal{A})$. Then*

$$\Delta^k G_\sigma(z_0, \dots, z_k)[w_1, \dots, w_k] = (-1)^k \sigma^{(n_0 \times n_k)}[(z_0 - X^{(n_0)})^{-1} w_1 (z_1 - X^{(n_1)})^{-1} \dots w_k (z - X^{(n_k)})^{-1}]$$

and in particular if $\operatorname{Im} z_j \geq \epsilon_j$, *then*

$$\|\Delta^k G_\sigma(z_0, \dots, z_k)\| \leq \frac{\|\sigma(1)\|}{\epsilon_0 \dots \epsilon_k}.$$

Proof. Denote

$$Z = (z_0 - X^{(n_0)}) \oplus \dots \oplus (z_k - X^{n_k}).$$

Fix small scalars $\zeta_1, \dots, \zeta_k$ and define

$$W = \begin{bmatrix} 0 & \zeta_1 w_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \zeta_2 w_2 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots 0 & \zeta_k w_k & \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}]$$

Note that if $\zeta_1, \dots, \zeta_k$ are sufficiently small, then

$$\sigma^{(n)}[(Z + W - X^{(n)})^{-1}] = \sigma^{(n)}[Z^{-1}(1 + W(Z - X^{(n)})^{-1})^{-1}] = \sum_{j=0}^{k} \sigma^{(n)}[(Z - X^{(n)})^{-1}(W(Z - X^{(n)})^{-1})^j],$$

where the expansion is truncated because WZ^{-1} is nilpotent. By looking at the upper right block, we obtain the desired formula for $\Delta^k G_\sigma$, and the upper bound for $\|\Delta^k G_\sigma\|$ follows immediately using Lemma 3.2.3. ☐

Notation 3.3.2. We denote $\tilde{G}_\sigma(z) = G_\sigma(z^{-1})$ where defined.

Lemma 3.3.3. *Suppose that σ is a generalized law with* $\operatorname{rad}(\sigma) \leq M$. *Then $\tilde{G}_\sigma$ has a fully matricial extension to $B(0, 1/M)$ given by*

$$\tilde{G}_\sigma(z) = \sigma^{(n)}[z(1 - X^{(n)} z)^{-1}] = \sum_{k=0}^{\infty} \sigma^{(n)}[z(X^{(n)} z)^k].$$

Proof. First, we observe that if $z^{-1} \in \mathbb{H}_+^{(n)}(\mathcal{A})$, then

$$\begin{aligned}\tilde{G}_\sigma(z) &= \sigma^{(n)}[(z^{-1} - X^{(n)})^{-1}] \\ &= \sigma^{(n)}[(z^{-1} - X^{(n)})^{-1}] \\ &= \sigma^{(n)}[((1 - X^{(n)}z)z^{-1})^{-1}] \\ &= \sigma^{(n)}[z(1 - X^{(n)}z)^{-1}].\end{aligned}$$

However, the latter function is also defined whenever $\|z\| < 1/M$. Now we claim that this extension of $\tilde{G}_\sigma$ is fully matricial on the domain

$$\Omega^{(n)} := \{z : (1 - \overline{X}^{(n)}z)^{-1} \text{ is invertible}\} \supseteq B(0, 1/M) \cup \mathbb{H}_+(\mathcal{A}) \cup \mathbb{H}_-(\mathcal{A}),$$

where $\overline{X}$ is the multiplication operator on $\mathcal{A}\langle X\rangle \otimes_\sigma \mathcal{A}$. The argument that $\Omega^{(n)}$ is a matricial domain is similar to the argument that invertible elements of a C^*-algebra form a matricial domain. Moreover, $\overline{\sigma}^{(n)}[z(1 - \overline{X}^{(n)}z)^{-1}]$ is fully matricial on Ω because it a built out of the inclusion $\mathcal{A} \to \mathcal{B}$ by translation, inverse, products, and application of $\overline{\sigma}$. □

Lemma 3.3.4. *We have*

$$\Delta^k \tilde{G}_\sigma(0^{(n_0)}, \dots, 0^{(n_k)})[w_1, \dots, w_k] = \sigma^{(n_0 \times n_k)}[w_1 X^{(n_1)} w_2 \dots X^{(n_{k-1})} w_k].$$

Proof. From the geometric series expansion, we have for $z \in B^{(n)}(0, 1/M)$ that

$$\tilde{G}_\sigma(z) = \sum_{k=0}^\infty \sigma^{(n)}[z(X^{(n)}z)^k].$$

If we let Λ_k be the multilinear form

$$\Lambda_k(z_1, \dots, z_k) = \sigma[z_1 X z_2 \dots X z_k],$$

then for every n and every $z \in B^{(n)}(0^{(n)}, 1/M)$, we have

$$\tilde{G}_\sigma(z) = \sum_{k=0}^\infty \Lambda_k^{(n)}(z, \dots, z).$$

Therefore, by Lemma 2.5.11, we have $\Lambda_k = \Delta^k \tilde{G}_\sigma(0, \dots, 0)$. The general formula for $\Delta^k \tilde{G}_\sigma(0^{(n_0)}, \dots, 0^{(n_k)})$ follows from Lemma 2.5.2. □

Lemma 3.3.5. *If $\|z\| < 1/M$, then we have*

$$\left\|\tilde{G}_\sigma(z)\right\| \leq \frac{\|\sigma(1)\|\|z\|}{1 - M\|z\|}$$

In particular,

$$\|z\| < 1/(M + \epsilon) \implies \left\|\tilde{G}_\sigma(z)\right\| \leq \frac{\|\sigma(1)\|}{\epsilon}.$$

Proof. This follows by applying the triangle inequality to the geometric series expansion. □

3.4 Analytic Characterization

The following theorem is due to Williams [Wil17, Theorem 3.1] and Anshelevich-Williams [AW16, Theorem A.1].

Theorem 3.4.1. *Let $G^{(n)} : \mathbb{H}_+^{(n)}(\mathcal{A}) \to M_n(\mathcal{A})$. The following are equivalent:*

1. *$G = G_\sigma$ for some generalized law σ with $\operatorname{rad}(\sigma) \leq M$ if and only if the following conditions hold.*

2. *The following conditions hold:*

   (a) G is fully matricial.

   (b) G maps $\mathbb{H}_+^{(n)}(\mathcal{A})$ into ${}^{(n)}_{-,0}(\mathcal{A})$.

   (c) $\tilde{G}(z) = G(z^{-1})$ has a fully matricial extension to $B(0, 1/M)$.

   (d) This extension satisfies $\tilde{G}(0) = 0$ and $\tilde{G}(z^*) = \tilde{G}(z)^*$.

   (e) For every $\epsilon > 0$, there exists $C_\epsilon > 0$ such that $\|z\| \leq 1/(M + \epsilon)$ implies $\|\tilde{G}(z)\| \leq C_\epsilon$.

Proof of (1) $\implies$ (2). Assume that (1) holds. We have already shown that (a), (b), (c), and (e) hold in Lemmas 3.2.7, 3.3.3, 3.3.5. Moreover, (d) follows from power series expansion in Lemma 3.3.3. □

The proof of (2) $\implies$ (1) is more involved, so we will prove several lemmas before concluding the proof of the Theorem. First, we define the map $\sigma : \mathcal{A}\langle X\rangle \to \mathcal{A}$. The correct choice of σ is clear in light of Lemma 3.3.4.

Lemma 3.4.2. *Let G satisfy (2) of Theorem 3.4.1. Define $\sigma : \mathcal{A}\langle X\rangle \to \mathcal{A}$ by*

$$\sigma(z_0 X z_1 \dots X z_k) = \Delta^{k+1}\tilde{G}(0, \dots, 0)[z_0, \dots, z_k].$$

Then any $R > M$ is an exponential bound for $\sigma^{(n)}$.

Proof. Because $\|z\| \leq 1/R$ implies $\|\tilde{G}(z)\| \leq C_{R-M}$, we have by Lemma 2.4.2 that

$$\|\Delta^k \tilde{G}(0^{(n)}, \dots, 0^{(n)})\| \leq C_{R-M} R^k.$$

□

Next, we show that σ extends to the analytic completion of $\mathcal{A}\langle X\rangle$. Fix $R > M$. As in the proof of Theorem 1.6.5, we define a norm on $M_n(\mathcal{A}\langle X\rangle) = M_n(\mathcal{A})\langle X^{(n)}\rangle$ by

$$\|F(X^{(n)})\|_R = \inf\left\{\sum_{j=1}^n \mathfrak{p}(F_j) : F_j \text{ monomials and } f = \sum_{j=1}^n F_j\right\},$$

where $\mathfrak{p}(z_0 X^{(n)} z_1 \dots X^{(n)} z_k) = R^k \|z_0\| \dots \|z_k\|$ for $z_0, \dots, z_k \in M_n(\mathcal{A})$. We denote the completion by $\langle A\rangle\langle X\rangle_R^{(n)}$ and recall that this is a Banach $*$-algebra.

Lemma 3.4.3. *Fix $R > M$. Then the map $\sigma^{(n)}$ defined above extends to a bounded map $\mathcal{A}\langle X\rangle_R^{(n)} \to M_n(\mathcal{A})$. Moreover, if $\|z\|_R < 1/R$, then $1 - X^{(n)}z$ is invertible in $\mathcal{A}\langle X\rangle_R^{(n)}$ and we have*

$$\tilde{G}(z) = \sigma^{(n)}[z(1 - X^{(n)}z)^{-1}].$$

Proof. The first claim follows because $\|\sigma^{(n)}(F(X))\| \leq C_{R-M}\|F(X)\|_R$ since R is an exponential bound for σ. Next, suppose that $\|z\|_R \leq 1/R$. Then because the geometric series $(1 - X^{(n)}z)^{-1}$ converges in $\mathcal{A}\langle X\rangle_R^{(n)}$, we see that $1 - X^{(n)}z$ is invertible. Moreover, a direct power series computation shows that $\tilde{G}(z) = \sigma^{(n)}[z(1 - X^{(n)}z)^{-1}]$ after we invoke Lemma 2.5.8. □

With these preparations in order, we can begin to prove complete positivity of σ. We start out by proving that certain symmetric moments are positive.

Lemma 3.4.4. *Suppose that G satisfies (2) of Theorem 3.4.1 and define σ as in Lemma 3.4.2. Let A_0 and A_1 be self-adjoint elements of $M_n(\mathcal{A})$ with $A_1 \geq \epsilon > 0$. Then*

$$\sigma^{(n)}[(A_1(X^{(n)} + A_0))^{2k} A_1] \geq 0.$$

Proof. Fix A_0 and A_1 and let ϕ be a state on $M_n(\mathcal{A})$. Consider the scalar-valued function $g : \mathbb{H}_+ \to \overline{\mathbb{H}}_-$ given by

$$g(\zeta) = \phi \circ G(A_1^{-1}\zeta - A_0).$$

Now we analyze the behavior of g at ∞. Note that $\zeta^{-1}A_1^{-1} - A_0$ is invertible in $\mathcal{A}\langle X\rangle_R^{(n)}$ if ζ is small enough. In fact, for sufficiently small ζ, we have $\|(\zeta^{-1}A_1^{-1} - A_0)^{-1}\| < 1/R$. Thus, we have

$$\begin{aligned}
g(1/\zeta) &= \phi \circ \tilde{G}((A_1^{-1}\zeta^{-1} - A_0)^{-1}) \\
&= \phi \circ \sigma[(A_1^{-1}\zeta^{-1} - A_0)^{-1}(1 - X^{(n)}(A_1^{-1}\zeta^{-1} - A_0)^{-1})^{-1}] \\
&= \phi \circ \sigma[(A_1^{-1}\zeta^{-1} - A_0 - X^{(n)})^{-1}] \\
&= \phi \circ \sigma[A_1\zeta(1 - (X^{(n)} + A_0)A_1\zeta)^{-1}] \\
&= \sum_{k=0}^{\infty} \zeta^{k+1}\, \phi \circ \sigma[(A_1(X^{(n)} + A_0))^k A_1],
\end{aligned}$$

where the intermediate steps are performed in $\mathcal{A}\langle X\rangle_R^{(n)}$. In particular, $\tilde{g}(\zeta) = g(1/\zeta)$ extends to be analytic in a neighborhood of 0. Because $\tilde{G}$ preserves adjoints, we have $g(\overline{\zeta}) = \overline{g(\zeta)}$. Therefore, g is the Cauchy-Stieltjes transform of some compactly supported measure ρ on $\mathbb{R}$. Moreover, by examining the power series coefficients of $\tilde{g}$ at 0, we have

$$\phi \circ \sigma[(A_1(X^{(n)} + A_0))^{2k} A_1] = \int_{\mathbb{R}} t^{2k}\, d\rho(t) \geq 0.$$

Because this holds for every state ϕ, we have $\sigma[(A_1(X^{(n)} + A_0))^{2k} A_1] \geq 0$ by Proposition 1.1.8 (5). □

Lemma 3.4.5. *Let G satisfy (2) and let σ be as above. Let $F(Y) = C_0YC_1 \dots YC_k$ be a monomial in $M_n(\mathcal{A})\langle Y\rangle$ and let $A_0 \in M_n(\mathcal{A})$ be self-adjoint. Then*

$$\sigma^{(n)}(F(X^{(n)} + A_0)^* F(X^{(n)} + A_0)) \geq 0.$$

Proof. Let us denote $Y = X^{(n)} + A_0$. Consider the matrix

$$C_\delta = \begin{bmatrix} \delta & \delta^2 C_k^* & 0 & \dots & 0 & 0 & 0 \\ \delta^2 C_k & \delta & \delta^2 C_{k-1}^* & \dots & 0 & 0 & 0 \\ 0 & \delta^2 C_{k-1} & \delta & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \delta & \delta^2 C_2^* & 0 \\ 0 & 0 & 0 & \dots & \delta^2 C_2 & \delta & \delta^2 C_1^* \\ 0 & 0 & 0 & \dots & 0 & \delta^2 C_1 & \delta + \delta^{-4k} C_0^* C_0 \end{bmatrix}$$

Observe that if δ is sufficiently small, then $C_\delta \geq \epsilon$ for some $\epsilon > 0$. Indeed, the diagonal terms δ will be much larger than the off-diagonal terms, while the extra diagonal term $\delta^{-4k} C_k^* C_k$ is already positive. Therefore, by the previous lemma,

$$\sigma^{(n)}[(C_\delta Y^{(k+1)})^{2k} C_\delta] \geq 0.$$

We claim that the top left $n \times n$ block of $(C_\delta Y^{(k+1)})^{2k} C_\delta$ is equal to $F(Y)^* F(Y) + O(\delta)$. To see this, consider what happens when we multiply out $(C_\delta (X^{(n(k+1))} + A_0^{(k+1)}))^{2k} C_\delta$ using matrix multipication, treating each $n \times n$ block as a unit. The top left block of the product will be the sum of terms of the form

$$(C_\delta)_{1,i_1} Y (C_\delta)_{i_1,i_2} Y \dots (C_\delta)_{i_{k-2},i_{k-1}} Y (C_\delta)_{i_{k-1},1}$$

since $Y^{(k+1)}$ is a block diagonal matrix. Because C_δ is tridiagonal, the sequence of indices must have $|i_{j-1} - i_j| \leq 1$. We can picture such a sequence as a path in the graph with vertices $\{1, \dots, k+1\}$ and edges between j and $j+1$ and a self-loop at each vertex j.

All the entries in C_δ are $O(\delta)$ except the bottom right entry with the term $\delta^{-4k} C_k^* C_k$. Thus, any path which yields a term larger than $O(\delta)$ must reach the last vertex $k+1$ and use the self-loop at the vertex $k+1$. But if we travel along the path at a speed ≤ 1, the only way we can get from 1 to $k+1$, use the self-loop at $k+1$, and get then back to 1 in $2k+1$ steps is to follow the path

$$1, 2, \dots, k, k+1, k+1, k, \dots, 2, 1.$$

So the only term in the sum which is not $O(\delta)$ is the term

$$(\delta^2 C_k^*) Y \dots (\delta^2 C_1^*) Y (\delta + \delta^{-4k} C_0^* C_0) Y (\delta^2 C_1) \dots Y (\delta^2 C_k) = F(Y)^* F(Y) + O(\delta).$$

Hence, the upper left entry of $(C_\delta Y^{(k+1)})^{2k} C_\delta$ is $F(Y)^* F(Y) + O(\delta)$. As a consequence,

$$\sigma^{(n)}(F(Y)^* F(Y)) + O(\delta) \geq 0,$$

and thus by taking δ to zero, we have $\sigma^{(n)}(F(Y)^* F(Y)) \geq 0$. □

To finish the proof that $\sigma^{(n)}(P(X)^* P(X)) \geq 0$ for every P, we will use the following matrix amplification trick to reduce to the case of a monomial.

Lemma 3.4.6. *Let $P(X) \in M_n(\mathcal{A}\langle X \rangle)$ be a polynomial of degree d. Denote*

$$\widehat{X} = \begin{bmatrix} X & 1 \\ 1 & X \end{bmatrix}.$$

Then for some m, there exist matrices $C_0, \dots, C_d \in M_{2m}(\mathcal{A})$ such that

$$\begin{bmatrix} P(X) & 0 \\ 0 & 0 \end{bmatrix} = C_0 \widehat{X}^{(m)} C_1 \widehat{X}^{(m)} \dots C_{d-1} \widehat{X}^{(m)} C_d.$$

Proof. Fix d. Let $\Gamma^{(n)}$ be the set of all polynomials $\mathcal{A}\langle X\rangle^{(n)}$ of degree $\leq d$ which can be expressed as in the conclusion of the lemma.

First, we claim that $\Gamma^{(1)}$ contains the monomials in $\mathcal{A}\langle X\rangle$. Let $p(X) = a_0 X a_1 \dots X a_k$ be a monomial of degree $k \leq d$. Then we have

$$\begin{bmatrix} p(X) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X & 1 \\ 1 & X \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix} \dots \begin{bmatrix} X & 1 \\ 1 & X \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X & 1 \\ 1 & X \end{bmatrix} \begin{bmatrix} a_k & 0 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X & 1 \\ 1 & X \end{bmatrix} \right)^{d-k} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Next, we claim that if $P(X) \in \Gamma^{(n)}$ and $e_{i,j}$ is the (i,j) matrix unit in $M_k(\mathbb{C})$, then the matrix $P(X) \otimes e_{i,j}$ with $P(X)$ in the (i,j) block and zeroes elsewhere is in $\Gamma^{(nk)}$. Given such a $P(X)$, there exist $C_1, \dots, C_d$ in $M_{2m}(\mathcal{A})$ such that

$$\begin{bmatrix} P(X) & 0 \\ 0 & 0 \end{bmatrix} = C_0 \widehat{X}^{(m)} C_1 \widehat{X}^{(m)} \dots C_{d-1} \widehat{X}^{(m)} C_d.$$

Then observe the $2(m + n(k-1))$ by $2(m + n(k-1))$ matrix equation:

$$\begin{bmatrix} P(X) \otimes e_{i,j} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1_n \otimes e_{i,1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_0 & 0 \\ 0 & 0 \end{bmatrix} \widehat{X}^{(m+n(k-1))} \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} \dots \\ \dots \widehat{X}^{(m+n(k-1))} \begin{bmatrix} C_d & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1_n \otimes e_{1,j} & 0 \\ 0 & 0 \end{bmatrix}.$$

We caution the reader that the blocks C_j are $2m \times 2m$ while the blocks $P(X) \times e_{i,j}$ and $1_n \otimes e_{i,j}$ are $nk \times nk$.

Finally, we claim that $\Gamma^{(n)}$ is closed under addition. Suppose that $P(X)$ and $Q(X)$ are in $\Gamma^{(n)}$. Then there exist integers r and s and matrices $B_1, \dots, B_d \in M_{2r}(\mathcal{A})$ and $C_1, \dots, C_d \in M_{2s}(\mathcal{A})$ such that

$$\begin{bmatrix} P(X) & 0 \\ 0 & 0 \end{bmatrix} = B_0 \widehat{X}^{(r)} B_1 \widehat{X}^{(r)} \dots B_{d-1} \widehat{X}^{(r)} B_d$$

and

$$\begin{bmatrix} Q(X) & 0 \\ 0 & 0 \end{bmatrix} = C_0 \widehat{X}^{(s)} C_1 \widehat{X}^{(s)} \dots B_{d-1} \widehat{X}^{(s)} B_d.$$

Then observe that

$$\begin{bmatrix} P(X) + Q(X) & 0 \\ 0 & 0 \end{bmatrix} = S \begin{bmatrix} B_0 & 0 \\ 0 & C_0 \end{bmatrix} \widehat{X}^{(r+s)} \begin{bmatrix} B_1 & 0 \\ 0 & C_1 \end{bmatrix} \dots \widehat{X}^{(r+s)} \begin{bmatrix} B_d & 0 \\ 0 & C_d \end{bmatrix} S^*$$

Where

$$S = \begin{bmatrix} 1_{n \times n} & 0_{n \times (n-r)} & 1_{n \times n} & 0_{n \times (n-s)} \\ 0_{(r+s-n) \times n} & 0_{(r+s-n) \times (n-r)} & 0_{(r+s-n) \times n} & 0_{(r+s-n) \times (s-n)} \end{bmatrix}$$

Altogether, we have shown that $\Gamma = \bigcup_{n=1}^{\infty} \Gamma^{(n)}$ contains the 1×1 monomials of degree $\leq d$, is closed under $P \mapsto P \otimes E_{i,j}$, and is closed under addition. This implies that Γ contains all matrix polynomials of degree $\leq d$ as desired. □

Conclusion to the proof of Theorem 3.4.1. Suppose that G satisfies (2) of the theorem and let $\sigma : \mathcal{A}\langle X\rangle \to \mathcal{A}$ be given as in Lemma 3.4.2. To show that σ is completely positive, choose a polynomial $P(X) \in M_n(\mathcal{A}\langle X\rangle)$. Let

$$A_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in M_2(\mathcal{A}).$$

Then by Lemma 3.4.6, we can write $P(X)$ in the form

$$\begin{bmatrix} P(X) & 0 \\ 0 & 0 \end{bmatrix} = C_0(X^{(2m)} + A_0^{(m)})C_1(X^{(2m)} + A_0^{(m)}) \dots C_{d-1}(X^{(2m)} + A_0^{(m)})C_d,$$

where $C_j \in M_{2m}(\mathcal{A})$. Thus, by Lemma 3.4.5, we have

$$\sigma^{(2m)}[C_d^*(X^{(2m)}+A_0^{(m)})C_{d-1}^* \dots (X^{(2m)}+A_0^{(m)})C_0^*C_0(X^{(2m)}+A_0^{(m)}) \dots C_{d-1}(X^{(2m)}+A_0^{(m)})C_d] \geq 0,$$

which implies that $\sigma^{(n)}(P(X)^*P(X)) \geq 0$.

Next, we have shown in Lemma 3.4.2 that σ exponentially bounded by R whenever $R > M$. Therefore, σ is a generalized law with $\operatorname{rad}(\sigma) \leq M$.

It remains to show that the Cauchy transform of σ is the original function G. It follows from Lemma 3.4.3 that $\tilde{G}(z) = \tilde{G}_\sigma(z)$ when $\|z\| < 1/R$. If we let $z_0 = 2iR$, then we have $z \in B^{(n)}(z_0, R)$ implies that $\operatorname{Im} z \geq R+\epsilon$ for some $\epsilon > 0$ which implies that $z^{-1} \in B^{(n)}(0, 1/R)$. Hence, we have $G = G_\sigma$ on $B(z_0, R)$. So by the identity theorem (Theorem 2.9.7), we have $G = \tilde{G}_\sigma$ on the whole matricial upper half-plane. □

We now give an analytic characterization of when the generlized law σ is a law, and hence complete the analytic characterization of the Cauchy-Stieltjes transforms of $\mathcal{A}$-valued laws.

Lemma 3.4.7. *Let σ be an $\mathcal{A}$-valued generalized law. Then the following are equivalent.*

1. *σ is a law.*
2. *$\Delta\tilde{G}_\sigma(0,0)[z] = z$ for all $z \in \mathcal{A}$.*
3. *For each n, $\lim_{\|z\|\to 0} z^{-1}\tilde{G}_\sigma(z) = 1_n$, where the limit occurs in norm and is taken over all invertible z.*

Proof. We have $\Delta\tilde{G}_\sigma(0,0)[z] = \sigma(z)$ for $z \in \mathcal{A}$. We also know by Corollary 1.6.8 that σ is a law if and only if $\sigma|_{\mathcal{A}} = \operatorname{id}$. This implies that (1) ⇔ (2).

(1) $\implies$ (3). If σ is a law, then

$$z^{-1}\tilde{G}_\sigma(z) = z^{-1}\sigma^{(n)}[z(1 - X^{(n)}z)^{-1}] = \sigma^{(n)}[(1 - X^{(n)}z)^{-1}],$$

which is fully matricial in a neighborhood of zero, and hence (3) holds.

(3) $\implies$ (1). Fix an invertible operator $z \in \mathcal{A}$. Then we have for scalars ζ that

$$\lim_{\zeta\to 0} \frac{1}{\zeta}\tilde{G}_\sigma(\zeta z) = z \lim_{\zeta\to 0}(\zeta z)^{-1}\tilde{G}_\sigma(\zeta z) = z.$$

On the other hand,

$$\lim_{\zeta\to 0} \frac{1}{\zeta}\tilde{G}_\sigma(\zeta z) = \lim_{\zeta\to 0} \sum_{k=0}^\infty \zeta^k \sigma[z(Xz)^k] = \sigma[z].$$

Therefore, $\sigma[z] = z$. Any element of $\mathcal{A}$ can be written as a linear combination of invertible operators and hence $\sigma|_{\mathcal{A}} = \operatorname{id}$, which means that σ is a law. □

We also have the following corollary of Theorem 3.4.1 which is helpful for estimating the radius of generalized laws.

Corollary 3.4.8. *Suppose that σ and τ are $\mathcal{A}$-valued generalized laws and $\operatorname{Im} G_\sigma(z) \geq \operatorname{Im} G_\tau(z)$. Then*

1. *$G_\tau(z) - G_\sigma(z)$ is the Cauchy-Stieltjes transform of some generalized law ρ.*
2. $\operatorname{rad}(\sigma) \leq \operatorname{rad}(\tau)$.
3. *For $\operatorname{Im} z \geq \epsilon$, we have $\|G_\sigma(z) - G_\tau(z)\| \leq \|\sigma(1) - \tau(1)\|/\epsilon$.*

Proof. (1) Observe that $G_\tau - G_\sigma$ maps $\mathbb{H}_+(\mathcal{A})$ into $\overline{\mathbb{H}}_-(\mathcal{A})$. Moreover, $\tilde{G}_\tau - \tilde{G}_\sigma$ extends to be fully matricial in a neighborhood of 0 in a way which preserves adjoints. Therefore, there is a generalized law ρ such that $G_\tau - G_\sigma = G_\rho$. Now $\rho = \tau - \sigma$ as maps $\mathcal{A}\langle X \rangle \to \mathcal{A}$.

(2) Because ρ is completely positive, we see that

$$\sigma(p(X)^* p(X)) \leq \tau(p(X)^* p(X)).$$

In particular, if $p(X) = a_0 X a_1 \dots X a_n$, then

$$\|\sigma(p(X))\| \leq \|\sigma(p(X)^* p(X))\|^{1/2} \leq \|\tau(p(X)^* p(X))\|^{1/2} \leq \operatorname{rad}(\tau)^n \|a_0\| \dots \|a_n\|,$$

so that $\operatorname{rad}(\sigma) \leq \operatorname{rad}(\tau)$.

(3) This follows by applying the estimate for Lemma 3.2.7 to $G_\rho(z)$. □

3.5 The F-Transform

Definition 3.5.1. Let μ be an $\mathcal{A}$-valued law. We define the *F-transform*

$$F_\mu(z) = G_\mu(z)^{-1}.$$

Lemma 3.5.2. *F_μ is a fully matricial function $\mathbb{H}_+(\mathcal{A}) \to \mathbb{H}_+(\mathcal{A})$.*

Proof. Suppose that $z \in \mathbb{H}_+(\mathcal{A})$. If we have $\operatorname{Im} z \geq \epsilon$, then by Lemma 3.2.3, we have

$$\operatorname{Im}(z - X^{(n)})^{-1} \leq \frac{-\epsilon}{\|z - X^{(n)}\|^2}.$$

By complete positivity of μ and the fact that $\mu(1) = 1$, we have

$$\operatorname{Im} G_\mu(z) \leq \frac{-\epsilon}{\|z - X^{(n)}\|^2}.$$

This implies that $G_\mu(z) \in \mathbb{H}_-(\mathcal{A})$ and in particular $G_\mu(z)$ is invertible. Moreover, one checks from Lemma 2.7.2 that $\operatorname{inv} : \mathbb{H}_-(\mathcal{A}) \to \mathbb{H}_+(\mathcal{A})$ is fully matricial, and hence $F_\mu(z)$ is fully matricial. □

The following characterization of F-transforms will be useful in the later chapters for understanding the analytic transforms associated to non-commutative independence. It also serves as an example of the applications of Theorem 3.4.1. A related characterization of $z - F_\mu(z)$ as the self-energy of some law was given in [PV13, Theorem 5.6] and [Wil17, Corollary 3.3], while the statement that $z - F_\mu(z)$ is the Cauchy-Stieltjes transform of a generalized law was proved in [PV13, Remark 5.7]. Compare also [SW97, Proposition 3.1] (scalar case), [Jek17, Proposition 3.9] (by the present author).

Theorem 3.5.3.

1. *If μ is an $\mathcal{A}$-valued law, then there exists a self-adjoint $a_0 \in \mathcal{A}$ and a generalized law σ such that $F_\mu^{(n)}(z) = z - a_0^{(n)} - G_\sigma^{(n)}(z)$.*
2. *Conversely, if a_0 is a self-adjoint element of $\mathcal{A}$ and σ is a generalized law, then there exists an $\mathcal{A}$-valued law μ such that $F_\mu^{(n)}(z) = z - a_0^{(n)} - G_\sigma^{(n)}(z)$.*
3. *We have $a_0 = \mu(X)$ and $\sigma(a) = \mu(XaX) - \mu(X)a\mu(X)$ for $a \in \mathcal{A}$.*
4. *We have $\operatorname{rad}(\sigma) \leq 2\operatorname{rad}(\mu)$ and*

$$\operatorname{rad}(\mu) \leq \frac{1}{2}\left(\|a_0\| + M + \sqrt{|\|a_0\| - M|^2 + 4\|\sigma(1)\|}\right).$$

Proof. (1) We want to show that $B_\mu(z) = z - F_\mu(z)$ has the form $a_0 + G_\sigma(z)$ for some generalized law σ. The first step is to show that H maps $\mathbb{H}_+(\mathcal{A})$ into $\overline{\mathbb{H}}_-(\mathcal{A})$, which is equivalent to showing that $\operatorname{Im} F_\mu(z) \geq \operatorname{Im} z$.

Let $z \in \mathbb{H}_+^{(n)}(\mathcal{A})$. Let us write X for the left multiplication operator on $\mathcal{A}\langle X\rangle \otimes_\mu \mathcal{A}$ and $\sigma(b) = \langle (1 \otimes 1), b(1 \otimes 1)\rangle$ for b in the C^*-algebra $\mathcal{B}$ generated by $\mathcal{A}\langle X\rangle$. Using the identity $\operatorname{Im}(b^{-1}) = (b^{-1})^*(\operatorname{Im} b)b^{-1}$, we have

$$\begin{aligned} -\operatorname{Im}(z - X^{(n)})^{-1} &= -(z^* - X^{(n)})^{-1}[\operatorname{Im}(z^* - X^{(n)})](z - X^{(n)})^{-1} \\ &= (z^* - X^{(n)})^{-1}\operatorname{Im}(z)(z - X^{(n)})^{-1}. \end{aligned}$$

By complete positivity of μ, we have

$$-\operatorname{Im}\mu^{(n)}[(z - X^{(n)})^{-1}] \geq \mu^{(n)}[(z^* - X^{(n)})^{-1}\operatorname{Im}(z)(z - X^{(n)})^{-1}].$$

Note that if $A \in M_n(\mathcal{A})$ is positive and $B \in M_n(\mathcal{B})$, then $\mu^{(n)}[(B - \mu^{(n)}(B))^* A(B - \mu^{(n)}(B))] \geq 0$ which impiles that $\mu^{(n)}[B^*AB] \geq \mu^{(n)}(B)^* A\mu^{(n)}(B)$. In particular, we have

$$\mu^{(n)}[(z^* - X^{(n)})^{-1}\operatorname{Im}(z)(z - X^{(n)})^{-1}] \geq \mu^{(n)}[(z^* - X^{(n)})^{-1}]\operatorname{Im}(z)\mu^{(n)}[(z - X^{(n)})^{-1}].$$

Thus, we have shown that

$$-\operatorname{Im} G_\mu(z) \geq G_\mu(z)^*(\operatorname{Im} z)G_\mu(z).$$

Now observe that

$$\begin{aligned} \operatorname{Im} F_\mu(z) &= \operatorname{Im} G_\mu(z)^{-1} \\ &= -(G_\mu(z)^*)^{-1}[\operatorname{Im} G_\mu(z)]G_\mu(z)^{-1} \\ &\geq (G_\mu(z)^*)^{-1}[G_\mu(z)^*(\operatorname{Im} z)G_\mu(z)]G_\mu(z)^{-1} \\ &= \operatorname{Im} z. \end{aligned}$$

Therefore, B_μ maps $\mathbb{H}_+(\mathcal{A})$ into $\overline{\mathbb{H}}_-(\mathcal{A})$.

Next, let us analyze the behavior of $\tilde{B}_\mu(z) = B_\mu(z^{-1})$ near zero. Let $M = \operatorname{rad}(\mu)$. Note that the series

$$P^{(n)}(z) = \sum_{k=0}^\infty \mu^{(n)}[X^{(n)}(zX^{(n)})^k]$$

converges for $\|z\| < 1/M$ and satisfies

$$\|P(z)\| \leq \sum_{k=0}^{\infty} M(\|z\|M)^k = \frac{1}{1/M - \|z\|}$$

We also have

$$\tilde{G}_\mu(z) = z + zP(z)z.$$

and thus

$$\begin{aligned} \tilde{B}_\mu(z) &= z^{-1} - F_\mu(z) \\ &= z^{-1} - (z + zP(z)z)^{-1} \\ &= z^{-1} - (1 + P(z)z)^{-1} z^{-1} \\ &= \sum_{k=0}^{\infty} [-P(z)z]^k P(z). \end{aligned}$$

This series converges provided that $\|P(z)z\| < 1$ for which it is sufficient that

$$\frac{\|z\|}{1/M - \|z\|} < 1 \Leftrightarrow \|z\| < \frac{1}{2M}.$$

In this case, we have

$$\left\|\tilde{B}_\mu(z)\right\| \leq \frac{\|P(z)\|}{1 - \|P(z)z\|} \leq \frac{1}{1/2M - \|z\|}.$$

Thus, the function $\tilde{B}_\mu(z)$ has a fully matricial extension to $B(0, 1/2M)$. Because $\tilde{G}_\mu(z^*) = \tilde{G}_\mu(z)^*$, we have $\tilde{B}_\mu(z^*) = \tilde{B}_\mu(z)^*$. In particular, $a_0 = \tilde{B}_\mu(0)$ is self-adjoint. Also, $\tilde{B}_\mu(z) - \tilde{B}_\mu(0)$ satisfies all the properties of Theorem 3.4.1, so that $B_\mu(z) - a_0 = G_\sigma(z)$ for some generalized law σ with $\mathrm{rad}(\sigma) \leq 2M$. Thus, we have proved (1) as well as the first estimate in (4).

(2) Let $B^{(n)}(z) = G_\sigma^{(n)}(z) + a_0^{(n)}$. $F(z) = z - B(z)$ and $G(z) = F(z)^{-1}$. Note that $F : \mathbb{H}_+(\mathcal{A}) \to \mathbb{H}_+(\mathcal{A})$ and hence $G : \mathbb{H}_+(\mathcal{A}) \to \mathbb{H}_-(\mathcal{A})$.

Now consider the behavior of $\tilde{B}(z)$ and $\tilde{G}(z)$ near zero. Letting $M = \mathrm{rad}(\sigma)$, we have by Lemma 3.3.3 that

$$\|\tilde{B}(z)\| \leq \frac{\|\sigma(1)\|\|z\|}{1 - M\|z\|} + \|a_0\|.$$

Next, observe that

$$\begin{aligned} \tilde{G}(z) &= (z^{-1} - \tilde{B}(z))^{-1} \\ &= \sum_{k=0}^{\infty} (z\tilde{B}(z))^k z, \end{aligned}$$

where the series expansion makes sense provided that $\|B(z)\|\|z\| < 1$. In particular, given our estimate on $\|\tilde{B}(z)\|$, we see that $\tilde{G}(z)$ has a fully matricial extension to a neighborhood of 0 which satisfies all the properties in Theorem 3.4.1 and therefore $G(z)$ is the Cauchy transform of some generalized law μ. Moreover, from the power series expansion of $\tilde{G}(z)$, we see that $z^{-1}\tilde{G}(z) \to 1$ as $z \to 0$ and thus by Lemma 3.4.7, μ is a law.

To compute $\mathrm{rad}(\mu)$, observe that $\tilde{G}(z)$ is uniformly bounded on $B(0, r)$ provided that $r < 1/M$ and

$$r\left(\frac{\|\sigma(1)\|r}{1 - Mr} + \|a_0\|\right) < 1.$$

The function on the left hand side is strictly increasing on $(0, 1/M)$, and approaches 0 and $+\infty$ at the left and right endpoints of the interval. Hence, there is a unique $r^* \in (0, 1/M)$ satisfying

$$r^* \left(\frac{\|\sigma(1)\| r^*}{1 - M r^*} + \|a_0\| \right) = 1.$$

and $\operatorname{rad}(\mu) \leq 1/r^*$. To solve for r^*, we change variables to $R = 1/r^*$ and obtain a quadratic equation in R. Because r^* was the unique solution in $(0, 1/M)$, we know that R must be the unique solution in $(M, +\infty)$ for the quadratic equation, and therefore must equal the larger root of this quadratic. After some computation, we obtain

$$R = \frac{1}{2} \left(\|a_0\| + M + \sqrt{(\|a_0\| - M)^2 + 4\|\sigma(1)\|} \right).$$

This proves the second estimate in (4).

It remains to prove claim (3). For sufficiently small $z \in \mathcal{A}$, we have

$$\begin{aligned} \tilde{B}_\mu(z) &= P(z) - P(z) z P(z) + O(\|z\|^2) \\ &= \mu(X) + \mu(XzX) - \mu(X) z \mu(X) + O(\|z\|^2), \end{aligned}$$

while on the other hand

$$\begin{aligned} \tilde{B}_\mu(z) &= a_0 + \tilde{G}_\sigma(z) \\ &= a_0 + \sigma(z) + O(\|z\|^2). \end{aligned}$$

Thus, $a_0 = \mu(X)$ and $\sigma(z) = \mu(XzX) - \mu(X) z \mu(X)$ as desired. □

The quantity $\operatorname{Var}_\mu(a) := \mu(XaX) - \mu(X) a \mu(X)$ will be significant in the coming chapters. As a consequence of what we have just shown, Var_μ is a completely positive $\mathcal{A} \to \mathcal{A}$. This Var_μ is related to the variance in classical probability theory. Indeed, if $\mathcal{A} = \mathbb{C}$ and μ is a measure on the real line, then Var_μ is a map $\mathbb{C} \to \mathbb{C}$ which is simply multiplication by a positive scalar, and this positive scalar is the classical variance of μ.

3.6 Convergence in Moments

Definition 3.6.1. If σ is a generalized law, then we define the kth moment of σ as the multilinear form

$$\operatorname{Mom}_k(\sigma)[w_0, \dots, w_k] = \sigma(w_0 X w_1 \dots X w_k)$$

or equivalently

$$\operatorname{Mom}_k(\sigma) = \Delta^{k+1} \tilde{G}_\sigma(0, \dots, 0).$$

Definition 3.6.2. Let σ_n and σ be generalized laws. We say that $\sigma_n \to \sigma$ *in moments* if

$$\lim_{n \to \infty} \|\operatorname{Mom}_k(\sigma_n) - \operatorname{Mom}_k(\sigma)\|_\# = 0 \text{ for every } k,$$

where $\|\cdot\|_\#$ is the completely bounded norm for multilinear forms. Similarly, we say that $\{\sigma_n\}$ is *Cauchy in moments* if $\{\operatorname{Mom}_k(\sigma_n)\}$ is Cauchy for each k.

Definition 3.6.3. We denote by $\Sigma_M(\mathcal{A})$ the set of $\mathcal{A}$-valued laws with $\operatorname{rad}(\mu) \leq M$. We denote by $\Sigma_{M,K}^{\text{gen}}(\mathcal{A})$ the set of $\mathcal{A}$-valued generalized laws σ with $\operatorname{rad}(\sigma) \leq M$ and $\|\sigma(1)\| \leq K$.

Lemma 3.6.4. *If $\{\sigma_n\}$ in $\Sigma_{M,K}^{\text{gen}}(\mathcal{A})$ is Cauchy in moments, then it converges in moments. Also, $\Sigma_M(\mathcal{A})$ is a closed subset of $\Sigma_{M,1}^{\text{gen}}(\mathcal{A})$ with respect to the convergence in moments.*

Proof. Clearly, the multilinear forms $\operatorname{Mom}_k(\sigma_n)$ converge to some multilinear form Λ_k. We can define $\sigma : \mathcal{A}\langle X\rangle \to \mathcal{A}$ by $\sigma(w_0 X w_1 \dots X w_k) = \Lambda_k(w_0, \dots, w_k)$. Then $\sigma_n[f(X)] \to \sigma[f(X)]$ for each $f(X) \in \mathcal{A}\langle X\rangle$ and hence σ is completely positive and exponentially bounded by M. Also, $\|\sigma(1)\| \leq K$ since $\|\sigma_n(1)\| \leq K$. Therefore, σ is a generalized law in $\Sigma_{M,K}^{\text{gen}}(\mathcal{A})$ and $\sigma_n \to \sigma$ in moments.

To show that $\Sigma_M(\mathcal{A})$ is closed, note that the property of $\sigma : \mathcal{A}\langle X\rangle \to \mathcal{A}$ being a unital $\mathcal{A}$-$\mathcal{A}$-bimodule map is preserved under limits. □

Proposition 3.6.5. *If $r < 1/M$ and $\sigma, \tau \in \Sigma_{M,K}^{\text{gen}}(\mathcal{A})$, define*

$$d_r(\sigma, \tau) = \sum_{k=0}^{\infty} r^{k+1} \|\operatorname{Mom}_k(\sigma) - \operatorname{Mom}_k(\tau)\|_{\#} = d_{0,r}(\tilde{G}_\sigma, \tilde{G}_\tau).$$

Then we have the following.

1. *d_r is a metric.*
2. *The metrics d_r for different values of r are uniformly equivalent.*
3. *$\{\sigma_n\} \subseteq \Sigma_{M,K}^{\text{gen}}$ is convergent / Cauchy in moments if and only if it convergent / Cauchy in d_r.*
4. *$\Sigma_{M,K}^{\text{gen}}(\mathcal{A})$ is complete with respect to d_r.*

Proof. Observe that $d_r(\sigma, \tau) = d_{0,r}(\tilde{G}_\sigma, \tilde{G}_\tau)$. It follows from Lemma 3.3.5 that $\mathcal{F} = \{\tilde{G}_\sigma : \sigma \in \Sigma_{M,K}^{\text{gen}}(\mathcal{A})\}$ is a uniformly locally bounded family of fully matricial functions on $B(0, 1/M)$ and that $\operatorname{rad}(0, \mathcal{F}) = 1/M$. Therefore, claims (1) and (2) follow from Theorem 2.9.6.

(3) Note that

$$\|\operatorname{Mom}_k(\sigma) - \operatorname{Mom}_k(\tau)\|_{\#} \leq \frac{1}{r^k} d_r(\sigma, \tau).$$

Hence, convergence or Cauchyness in d_r implies convergence or Cauchyness in moments. Conversely, using standard geometric series estimates,

$$d_r(\sigma, \tau) \leq \sum_{k=0}^{N-1} r^k \|\operatorname{Mom}_k(\sigma) - \operatorname{Mom}_k(\tau)\|_{\#} + \frac{(rM)^N}{1 - rM}$$

and hence convergence or Cauchyness in moments implies convergence or Cauchyness in d_r.

(4) This follows from (3) and Lemma 3.6.4. □

Proposition 3.6.6.

1. *The collection $\mathcal{G}_{M,K} = \{G_\sigma : \sigma \in \Sigma_{M,K}^{\text{gen}}(\mathcal{A})\}$ is a uniformly locally bounded family of fully matricial functions on $\mathbb{H}_+(\mathcal{A})$.*
2. *For each $z \in \mathbb{H}_{+,\epsilon}(\mathcal{A})$ we have $\operatorname{rad}(z, \mathcal{G}) \geq \epsilon$.*
3. *The metrics $d_{z,r}(G_\sigma, G_\tau)$ on $\mathcal{G}$ are uniformly equivalent to the metrics $d_r(\sigma, \tau)$.*
4. *$\mathcal{G}_{M,K}$ with the topology of uniform local convergence is homeomorphic to $\Sigma_{M,K}^{\text{gen}}$ with the topology of convergence in moments.*

Proof. (1) and (2) follow from Lemma 3.2.7.

To prove (3), note that the metrics $d_{z,r}$ are all equivalent to each other by Theorem 2.9.6. Moreover, note that all the elements of $B(3iM, M)$ are invertible and $\operatorname{inv}(B(3iM, M)) \subseteq B(0, 1/2M)$, so that

$$d_{3iM,M}(G_\sigma, G_\tau) \leq d_{0,1/2M}(\tilde{G}_\sigma, \tilde{G}_\tau) = d_{1/2M}(\sigma, \tau).$$

Thus, $d_{3iM,M}(G_\sigma, G_\tau)$ can be estimated above by $d_{1/2M}(\sigma, \tau)$. For the converse direction, note that $\operatorname{inv}(B(1/2iM, 1/8M)) \subseteq B(2iM, 2M/3)$ and hence

$$d_{1/2iM,1/8M}(\tilde{G}_\sigma, \tilde{G}_\tau) \leq d_{2iM,2M/3}(G_\sigma, G_\tau).$$

By Theorem 2.9.6, the metric $d_{1/2iM,1/8M}$ is equivalent to $d_r(\sigma, \tau)$, and thus $d_r(\sigma, \tau)$ can be estimated from above by $d_{2iM,2M/3}(G_\sigma, G_\tau)$.

(4) is an immediate consequence of (3). □

3.7 Problems and Further Reading

Problem 3.1.

1. Let μ be a compactly supported measure on $\mathbb{R}$. Let $d\nu(t) = t^2\, d\mu(t)$. Show that

$$g_\mu(z) = \frac{1}{z} + \frac{1}{z^2} \int t\, d\mu(t) + \frac{1}{z^2} g_\nu(z).$$

2. Let μ be an $\mathcal{A}$-valued law. Show that there exists a generalized law ν such that

$$G_\mu^{(n)}(z) = z^{-1} + z^{-1}\mu(X)^{(n)} z^{-1} + z^{-1} G_\nu^{(n)}(z) z^{-1}.$$

Problem 3.2. Let $F_1, F_2, F_3 : \mathbb{H}_+(\mathcal{A}) \to \mathbb{H}_+(\mathcal{A})$ be fully matricial functions with $F_3 = F_1 \circ F_2$. Prove that if two out of the three F_j's are F-transforms of $\mathcal{A}$-valued laws, then so is the third. In particular, F-transforms form a semigroup under composition.

Problem 3.3. Suppose that $F_{\mu_3} = F_{\mu_1} \circ F_{\mu_2}$. Show that

$$\frac{1}{c} \operatorname{rad}(\mu_3) \leq \max(\operatorname{rad}(\mu_1), \operatorname{rad}(\mu_2)) \leq c \operatorname{rad}(\mu_3)$$

for some constant $c > 0$ independent of μ_j.

Problem 3.4. Prove that the metric $d_r(\sigma, \tau)$ on $\Sigma_{M,K}^{\mathrm{gen}}(\mathcal{A})$ is uniformly equivalent to the metric

$$\sup_{\operatorname{Im} z \geq \epsilon} \|G_\sigma(z) - G_\tau(z)\|.$$

Problem 3.5. Let σ_n and σ be in $\Sigma_{M,K}^{\mathrm{gen}}(\mathcal{A})$. Show that the following are equivalent:

1. $\sigma_n(f(X)) \to \sigma(f(X))$ in $\mathcal{A}$ for every polynomial $f(X) \in \mathcal{A}\langle X \rangle$.
2. $G_{\sigma_n}(z) \to G_\sigma(z)$ for every $z \in \mathbb{H}_+(\mathcal{A})$.

Problem 3.6. Let $(\mathcal{B}, \tau)$ be a tracial von Neumann algebra and $\mathcal{A}$ be a von Neumann subalgebra. Let $E : \mathcal{B} \to \mathcal{A}$ be the conditional expectation. Let $X \in \mathcal{B}$ be self-adjoint and let $G(z) = E[(z - X^{(n)})^{-1}]$ for $z \in \mathbb{H}_+^{(n)}(\mathcal{A})$.

1. If $z, z' \in \mathbb{H}_{+,\epsilon}^{(n)}(\mathcal{A})$, show that

$$\|G(z) - G(z')\|_2 \leq \frac{1}{\epsilon^2} \|z - z'\|_2.$$

2. Let $\mathcal{G}^{\mathrm{Tr}}$ be the set of Cauchy transforms obtained in this way. Show that $\mathcal{G}^{\mathrm{Tr}}$ is compact in the topology of pointwise σ-WOT convergence.

3. Show that this topology on $\mathcal{G}^{\mathrm{Tr}}$ is metrizable provided that $\mathcal{A}$ is separable in WOT.

Chapter 4

Non-Commutative Independences

4.1 Introduction

Free independence was discovered by Voiculescu [Voi86] and further developed by Speicher [Spe94]. His key insight was that the free product operation on groups and the corresponding operator algebras could be viewed as a non-commutative version of probabilistic independence. The analogy between the classical and free theories included the following elements:

1. *Rule for specifying mixed moments:* To say that algebras $\mathcal{B}_1, \dots, \mathcal{B}_n$ are freely independent specifies rule for determining the mixed moments of variables in the larger algebra which they generate.

2. *Product space construction:* Any two algebras could be joined in an independent way. In ordinary probability theory, this is the role of the product measure spaces, corresponding to a tensor product of the L^2 spaces, on which the two individual algebras act by multiplication on first and second coordinate. In free probability theory, products of algebras act on the free product of the underlying Hilbert spaces, a construction related to Fock spaces in physics.

3. *Convolution operation and analytic transforms:* In ordinary probability theory, the law of a sum of independent random variables is the convolution of the two individual laws, and the convolution can be computed using the Fourier transform of the measure. Given (1), the law of the sum of independent random variables is determined by the individual laws, and so "free convolution" is well-defined. Voiculescu found that the R-transform played a similar role in free probability theory; namely, the R transform of the "free convolution" of two laws is the sum of the R-transforms.

This theory was adapted to the operator-valued setting in [Voi95], [Spe98].

Another type of non-commutative independence, called Boolean independence, was introduced into non-commutative probability by Speicher and Woroudi [SW97], based on previous work by physicists. This independence had a rule for specifying mixed moments, a product space construction, and a convolution operation. For operator-valued Boolean independence, see [Pop09], [PV13, §2], [BPV13].

Finally, monotone independence was discovered by Muraki [Mur97], [Mur00], [Mur01], and adapted to the operator-valued setting by Popa [Pop08a] and Hasebe and Saigo [HS14]. There was a parallel theory of moment computations, product spaces, and analytic transforms. Unlike free and Boolean independence, monotone independence is sensitive to the order of algebras.

Thus, the monotone convolution operation is not commutative and it corresponds to composition rather than addition of analytic transforms.

After the discovery of several types of independence, Speicher formulated axioms for independences which lead to a natural commutative binary product operation, and he showed that tensor, free, and Boolean were the only three possilibities [Spe97]; Ben Ghorbal and Schürmann proved related results in the framework of category theory [BS02]. When the product is no longer required to be commutative, there are exactly two more possibilities, monotone independence and its mirror image anti-monotone independence, as proved by Muraki in 2003 [Mur03]. This in some sense classified the possible notions of independence. The analogous results in the operator-valued setting have not yet been studied.

Here we will focus on operator-valued free, Boolean, monotone, and anti-monotone independence. We exclude classical or tensor independence because it does not adapt well to the $\mathcal{A}$-valued setting if $\mathcal{A}$ is not commutative, and because the other types of independence have closer similarities with each other. As much as possible, we will present theories of these four types in parallel.

4.2 Independence of Algebras

Definition 4.2.1 (Free independence)**.** Let $(\mathcal{B}, E)$ be an $\mathcal{A}$-valued probability space. Then subalgebras $\mathcal{B}_1$, …, $\mathcal{B}_N$ containing $\mathcal{A}$ are said to be *freely independent* if we have

$$E[b_1 \dots b_k] = 0$$

whenever $b_j \in \mathcal{B}_{i_j}$ with $E[b_j] = 0$, provided that the consecutive indices i_j and i_{j+1} are distinct.

Definition 4.2.2. Let $\mathcal{B} \supseteq \mathcal{A}$ be C^*-algebras. We say that $\mathcal{C}$ is a (non-unital) $\mathcal{A}$-$*$-subalgebra of $\mathcal{B}$ if $\mathcal{B}$ if $\mathcal{B}$ is closed under addition, multiplication, and adjoints, and if $\mathcal{A}\mathcal{B} \subseteq \mathcal{B}$.

Definition 4.2.3 (Boolean independence)**.** Let $(\mathcal{B}, E)$ be an $\mathcal{A}$-valued probability space. Then $\mathcal{A}$-subalgebras $\mathcal{B}_1$, …, $\mathcal{B}_N$ are said to be *Boolean independent* if we have

$$E[b_1 \dots b_k] = E[b_1] \dots E[b_k]$$

whenever $b_j \in \mathcal{B}_{i_j}$, provided that the consecutive indices i_j and i_{j+1} are distinct.

Definition 4.2.4 (Monotone independence)**.** Let $(\mathcal{B}, E)$ be an $\mathcal{A}$-valued probability space. Then $\mathcal{A}$-subalgebras $\mathcal{B}_1$, …, $\mathcal{B}_N$ are said to be *monotone independent* if we have

$$E[b_1 \dots b_k] = E[b_1 \dots b_{r-1} E[b_r] b_{r+1} \dots b_k]$$

whenever $b_j \in \mathcal{B}_{i_j}$, provided that the index i_r is strictly greater than the consecutive indices i_{r-1} and i_{r+1} (if $r = 1$, we drop the condition on i_{r-1} and similarly for the case $r = k$).

Definition 4.2.5 (Anti-monotone independence)**.** Let $(\mathcal{B}, E)$ be an $\mathcal{A}$-valued probability space. Then $\mathcal{A}$-subalgebras $\mathcal{B}_1$, …, $\mathcal{B}_N$ are said to be *anti-monotone independent* if we have

$$E[b_1 \dots b_k] = E[b_1 \dots b_{r-1} E[b_r] b_{r+1} \dots b_k]$$

whenever $b_j \in \mathcal{B}_{i_j}$, provided that the index i_r is strictly less than the consecutive indices i_{r-1} and i_{r+1} (if $r = 1$, we drop the condition on i_{r-1} and similarly for the case $r = k$).

Remark 4.2.6. Free and Boolean independence are unchanged if we reorder the algebras $\mathcal{B}_1$, …, $\mathcal{B}_N$. However, monotone and anti-monotone independence are sensitive to order. Also, $\mathcal{B}_1$, …, $\mathcal{B}_N$ are anti-monotone independent if and only if $\mathcal{B}_n$, …, $\mathcal{B}_1$ are monotone independent.

The definition of independence provides enough information to evaluate the expectation of any element of the $\mathcal{A}$-algebra generated by $\mathcal{B}_1, \dots, \mathcal{B}_N$. Here and in the rest of this chapter, we state the result for all types of independence simultaneously. When we write "free / Boolean / monotone / anti-monotone," we mean that there are four versions of the theorem, one for each type of independence.

Lemma 4.2.7. *Suppose that $\mathcal{B}_1, \dots, \mathcal{B}_N$ are free / Boolean / monotone / anti-monotone independent $\mathcal{A}$-subalgebras, and assume in the free case that they are unital. If $b_j \in \mathcal{B}_{i_j}$ for $j = 1, \dots, k$, then $E[b_1 \dots b_k]$ is uniquely determined by $E|_{\mathcal{B}_1}, \dots, E|_{\mathcal{B}_N}$.*

Proof for the free case. Let $\mathcal{C}$ be the formal $\mathcal{A}$-algebra generated by $\mathcal{B}_1, \dots, \mathcal{B}_N$, that is, the span of all strings of the form $b_1 \dots b_k$ where b_j and b_{j+1} come from distinct algebras. Let

$$\mathcal{D} = \mathcal{A} + \operatorname{Span}\{b_1 \dots b_k : E[b_j] = 0, b_j \in \mathcal{B}_{i_j}, i_j \neq i_{j+1}\}.$$

We claim that $\mathcal{C} = \mathcal{D}$.

We must show that every string $b_1 \dots b_k$ can be represented as a linear combination of the terms in $\mathcal{D}$. We prove this by induction on k, the base case $k = 0$ being trivial. In the inductive step, let $k \geq 1$ and consider a string $b_1 \dots b_k$ where $b_j \in \mathcal{B}_{i_j}$ and $i_j \neq i_{j+1}$. We can write $b_j = c_j + a_j$ where $a_j = E[b_j]$ and $c_j = b_j - a_j$ has expectation zero. Then

$$b_1 \dots b_k = (c_1 + a_1) \dots (c_k + a_k).$$

We expand the right hand side into 2^k terms using the distributive property. The first term $c_1 \dots c_k$ has the desired form. We claim that each of the other terms can be expressed as a word in $\mathcal{C}$ with length less than k (so that we can apply the inductive hypothesis). Each term is a product of some c_j's and some a_j's, but we can group each a_j together with all the terms before or after until we reach one of the c_j's. Then if two adjacent elements come from the same algebra $\mathcal{B}_i$, then we can group them together into one term. After applying as many such regrouping operations as possible, we have expressed this term as a string of the form $b_1' \dots b_{k'}'$ with $k' < k$ and the terms b_j' coming from different $\mathcal{B}_{i_j}$'s with $i_{j+1} \neq i_j$. Then by the inductive hypothesis, this term is in $\mathcal{D}$.

This implies that every $c \in \mathcal{C}$ can be expressed as the sum of $a \in \mathcal{A}$ plus a linear combination of terms of the form $b_1 \dots b_k$, where $E[b_j] = 0$, $b_j \in \mathcal{B}_{i_j}$, and $i_j \neq i_{j+1}$. This decomposition was reached using purely algebraic operations and knowledge of $E|_{\mathcal{B}_i}$ for each i. Using freeness, each term of the form $b_1 \dots b_k$ has expectation zero. Thus, $E[c] = a$. □

Proof for the Boolean case. Starting with a string $b_1 \dots b_k$, we first group and relabel the terms so that any two consecutive terms come from different algebras. Then by definition of Boolean independence $E[b_1 \dots b_k] = E[b_1] \dots E[b_k]$. □

Proof for the (anti-)monotone case. In the monotone case, we proceed by induction on the length k of the string $b_1 \dots b_k$, where the base case $k = 1$ is trivial. By regrouping the terms if necessary, assume that consecutive terms come from different algebras. Then choose an index j such that i_j is maximal. By monotone independence,

$$E[b_1 \dots b_k] = E[b_1 \dots b_{j-1} E[b_j] b_{j+1} \dots b_k].$$

Since $E[b_j] \in \mathcal{A}$, this can be represented as a string of length $\leq k - 1$, to which we apply the induction hypothesis.

The anti-monotone case follows by symmetry from the monotone case. □

4.3 Construction of Product Spaces

In classical probability theory, one constructs the product $(\Omega, P) = (\Omega_1 \otimes \Omega_2, P_1 \otimes P_2)$ of two probability spaces (Ω_1, P_1) and (Ω_2, P_2). The algebras $\mathcal{B}_1 = L^\infty(\Omega_1, P_1)$ and $\mathcal{B}_2 = L^\infty(\Omega_2, P_2)$ embed into $\mathcal{B} = L^\infty(\Omega, P)$ as subalgebras which are classically independent, that is, $E[b_1 b_2] = E[b_1]E[b_2]$. The algebra $\mathcal{B}$ is thus a certain completed tensor product of $\mathcal{B}_1$ and $\mathcal{B}_2$ with the state $\int \cdot P$ being the tensor product of the two states P_1 and P_2. Moreover, the Hilbert space $L^2(\Omega, P)$ is the Hilbert-space tensor product of $L^2(\Omega_1, P_1)$ and $L^2(\Omega_2, P_2)$.

Similarly, in non-commutative probability, we seek to a way to independently join given algebras $\mathcal{B}_1, \dots, \mathcal{B}_N$. We construct the joint algebra by first constructing a joint Hilbert space, in the same way that classical independence arises from tensor products of Hilbert spaces.

More precisely, our goal in this section is to prove the following theorem. For the sake of brevity, we state the theorem only once for all four types of independence studied here. Rather than starting with initial algebras $\mathcal{B}_N$, we will prove independence for all of $B(\mathcal{H}_j)$ and comment on the case of other algebras in the next section.

Theorem 4.3.1. *Suppose that $(\mathcal{H}_1, \xi_1)$, …, $(\mathcal{H}_N, \xi_N)$ are Hilbert $\mathcal{A}$-$\mathcal{A}$-bimodules and ξ_j is an $\mathcal{A}$-central unit vector in $\mathcal{H}_j$. Denote $E_j[b] = \langle \xi_j, b\xi_j \rangle$ for $b \in B(\mathcal{H}_j)$.*

Then there exists a Hilbert $\mathcal{A}$-$\mathcal{A}$-bimodule $\mathcal{H}$, an $\mathcal{A}$-central unit vector ξ, and injective (non-unital) $*$-homomorphisms $\rho_j : B(\mathcal{H}_j) \to B(\mathcal{H})$ such that the following hold, where we denote $E[b] = \langle \xi, b\xi \rangle$:

1. *We have $E[\rho_j(b)] = E_j[b]$ for $b \in B(\mathcal{H}_j)$.*
2. *The algebras $\rho_1(B(\mathcal{H}_1))$, …, $\rho_n(B(\mathcal{H}_n))$ are free / Boolean / monotone / anti-monotone independent with respect to E.*
3. *If $b_j \in B(\mathcal{H}_j)$ with $E_j[b_j] = 0$, then*

$$\left\| \sum_{j=1}^N \rho_j(b_j) \right\| \leq 2 \left(\sum_{j=1}^N \|b_j\|^2 \right)^{1/2} + \max_{j \in [N]} \|b_j\|.$$

The space $(\mathcal{H}, \xi)$ and the maps ρ_j can be defined through the explicit constructions below (Definition 4.3.3, 4.3.4, 4.3.5, 4.3.6).

We begin with an elementary lemma before we divide the construction and proof into cases. References for Theorem 4.3.1 are included in the treatment of cases below.

Lemma 4.3.2. *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-$\mathcal{A}$-bimodule and let $\xi \in \mathcal{H}$ be an $\mathcal{A}$-central unit vector.*

1. *$\mathcal{A}\xi$ and $\mathcal{K} := \{\zeta : \langle \xi, \zeta \rangle = 0\}$ are $\mathcal{A}$-$\mathcal{A}$ Hilbert bimodules.*
2. *$\mathcal{H} = \mathcal{A}\xi \oplus \mathcal{K}$.*
3. *$\mathcal{A}\xi$ is isomorphic as a Hilbert $\mathcal{A}$-$\mathcal{A}$-bimodule to the bimodule $\mathcal{A}$ with the inner product given by $\langle a_1, a_2 \rangle = a_1^* a_2$.*

Proof. Note that $\mathcal{A}\xi$ is an $\mathcal{A}$-$\mathcal{A}$-bimodule because it is a left $\mathcal{A}$-module and $a\xi = \xi a$. Moreover, $\mathcal{K}$ is a $\mathcal{A}$-$\mathcal{A}$-bimodule because if $a \in \mathcal{K}$, then

$$\langle \xi, \zeta a \rangle = \langle \xi, \zeta \rangle a = 0$$

and

$$\langle \xi, a\zeta \rangle = \langle a^*\xi, \zeta \rangle = \langle \xi a^*, \zeta \rangle = a\langle \xi, \zeta \rangle = 0.$$

Moreover, any $\zeta \in \mathcal{H}$ can be written as

$$\zeta = \langle \xi, \zeta \rangle \xi + (\zeta - \langle \xi, \zeta \rangle \xi),$$

where the first term is in $\mathcal{A}\xi$ and the second term is in $\mathcal{K}$ because

$$\langle \xi, \zeta - \langle \xi, \zeta \rangle \xi \rangle = \langle \xi, \zeta - \xi \langle \xi, \zeta \rangle \rangle = \langle \xi, \zeta \rangle - \langle \xi, \xi \rangle \langle \xi, \zeta \rangle = 0.$$

Therefore, $\mathcal{H} = \mathcal{A}\xi \oplus \mathcal{K}$. Because this direct sum decomposition holds, the individual terms $\mathcal{A}\xi$ and $\mathcal{K}$ must be closed subspaces and hence are Hilbert $\mathcal{A}$-$\mathcal{A}$-bimodules.

Finally, we can define a map $\phi : \mathcal{A} \to \mathcal{A}\xi$ by $a \mapsto a\xi$. This map is clearly surjective. Using the fact that ξ is $\mathcal{A}$-central, one checks that this map preserves the inner product and is an isomorphism of Hilbert $\mathcal{A}$-$\mathcal{A}$-bimodules. □

The Free Case

The estimate 4.3.1 (3) in the scalar-valued free case is due to [Voi86, Lemma 3.2].

Definition 4.3.3. Let $\mathcal{H}_1$, …, $\mathcal{H}_N$ be Hilbert $\mathcal{A}$-$\mathcal{A}$-bimodules with a central unit vectors ξ_1, …, ξ_N, and write $\mathcal{K}_j = \{\zeta \in \mathcal{H}_j : \langle \xi_j, \zeta \rangle = 0\}$ and $\mathcal{H}_j = \mathcal{A}\xi_j \oplus \mathcal{K}_j$. We define the *free product* of $(\mathcal{H}_1, \xi_1), \dots, (\mathcal{H}_N, \xi_N)$ as the pair $(\mathcal{H}, \xi)$, where

$$\mathcal{H} = \mathcal{A}\xi \oplus \bigoplus_{k \geq 1} \bigoplus_{\substack{j_1, \dots, j_k \in [N] \\ j_r \neq j_{r+1}}} \mathcal{K}_{j_1} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{K}_{j_k},$$

as a Hilbert $\mathcal{A}$-$\mathcal{A}$-bimodule, where $\mathcal{A}\xi$ is a copy of the trivial $\mathcal{A}$-$\mathcal{A}$-bimodule $\mathcal{A}$ and $\xi = 1$.

In order to define $\rho_j : B(\mathcal{H}_j) \to B(\mathcal{H})$, we observe that by reindexing the terms in the direct sum and applying the distributive property of tensor products

$$\begin{aligned} \mathcal{H} &\cong (\mathcal{A}\xi \oplus \mathcal{K}_j) \otimes_{\mathcal{A}} \left(\mathcal{A} \oplus \bigoplus_{k \geq 1} \bigoplus_{\substack{j_1, \dots, j_k \in [N] \\ j_r \neq j_{r+1}; j_1 \neq j}} \mathcal{K}_{j_1} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{K}_{j_k} \right) \\ &\cong \mathcal{H}_j \otimes_{\mathcal{A}} \left(\mathcal{A} \oplus \bigoplus_{k \geq 1} \bigoplus_{\substack{j_1, \dots, j_k \in [N] \\ j_r \neq j_{r+1}; j_1 \neq j}} \mathcal{K}_{j_1} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{K}_{j_k} \right). \end{aligned}$$

One the right hand side, it is easy to see how $B(\mathcal{H}_j)$ acts by left-multiplication. This defines a $*$-homomorphism $\rho_j : B(\mathcal{H}_j) \to B(\mathcal{H})$ (which is unital in this case).

Proof of Theorem 4.3.1, free case.

(1) By construction $\rho_j(b)$ maps the direct summand $\mathcal{A}\xi \oplus \mathcal{K}_j$ into itself, and this subspace is isomorphic to $\mathcal{H}_j$ except that ξ_j is replaced by ξ.

(2) Let $k \geq 1$, and consider a product of terms $\rho_{j_1}(b_1)$, …, $\rho_{j_k}(b_k)$ where $E_{j_r}(b_r) = 0$ for each r. We claim that

$$\rho_{j_1}(b_1) \dots \rho_{j_k}(b_{j_k})\xi \in \mathcal{K}_{j_1} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{K}_{j_k},$$

which we will prove by induction on k. In the case $k = 1$, we express $b_{j_1}\xi_{j_1}$ in $\mathcal{H}_{j_1}$ as $a\xi_{j_1} + \zeta$, where $\zeta \in \mathcal{K}_{j_1}$ and the coefficient $a = \langle \xi_{j_1}, b_{j_1}\xi_{j_1} \rangle$. But by assumption $a = 0$, so that $b_{j_1}\xi_{j_1} \in \mathcal{K}_{j_1}$. For $k > 1$, we know by inductive hypothesis that

$$\zeta := \rho_{j_2}(b_2)\dots\rho_{j_k}(b_{j_k})\xi \in \mathcal{K}_{j_2} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{K}_{j_k}.$$

This sits inside the direct summand

$$\mathcal{K}_{j_2} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{K}_{j_k} \subseteq (\mathcal{A} \oplus \mathcal{K}_{j_1}) \otimes_{\mathcal{A}} \mathcal{K}_{j_2} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{K}_{j_k}. \subseteq \mathcal{H}$$

Because b_1 maps ξ_{j_1} into $\mathcal{K}_{j_1}$, we know that $\rho_{j_1}(b_1)$ maps ζ into $\mathcal{K}_{j_1} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{K}_{j_k}$ as desired.

Therefore, we have $\rho_{j_1}(b_1)\dots\rho_{j_k}(b_{j_k})\xi \in \mathcal{K}_{j_1} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{K}_{j_k}$, and hence

$$E[\rho_{j_1}(b_1)\dots\rho_{j_k}(b_{j_k})] = \langle \xi, \rho_{j_1}(b_1)\dots\rho_{j_k}(b_{j_k})\xi \rangle = 0,$$

which demonstrates free independence.

(3) Consider an operator of the form $\sum_{j=1}^n x_j$, where $x_j = \rho_j(b_j)$ and $E_j[b_j] = 0$. Let $\mathcal{M}_j \subseteq \mathcal{H}$ be the submodule consisting of tensor products where the first index is j, that is,

$$\mathcal{M}_j = \bigoplus_{k \geq 1} \bigoplus_{\substack{j_2,\dots,j_k \in [N] \\ j \neq j_2 \neq j_3 \neq \dots}} \mathcal{K}_j \otimes_{\mathcal{A}} \mathcal{K}_{j_2} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{K}_{j_k}.$$

Note that

$$\mathcal{H} = \mathcal{A}\xi \oplus \bigoplus_{j=1}^n \mathcal{M}_j.$$

Since $\mathcal{M}_j$ is a direct summand of $\mathcal{H}$, there is a projection $p_j \in B(\mathcal{H})$ onto $\mathcal{M}_j$. Let $x_j = \rho_j(b_j)$ and write

$$\sum_{j=1}^N x_j = \sum_{j=1}^N (1-p_j)x_j(1-p_j) + \sum_{j=1}^N p_j x_j (1-p_j) + \sum_{j=1}^N (1-p_j)x_j p_j + \sum_{j=1}^N p_j x_j p_j.$$

Observe that $(1-p_j)x_j(1-p_j) = 0$. Indeed, if ζ is in the range of $1-p_j$, then ζ only contains terms in the tensor products where the first term is not j. Then using similar reasoning as in part (2), since $E[x_j] = 0$, we have $x_j\zeta \in \mathcal{M}_j$, and hence $(1-p_j)x_j\zeta = 0$.

Next, because the elements $p_j x_j(1-p_j)$ have orthogonal ranges, we have as a consequence of Observation 1.2.7 that

$$\left\| \sum_{j=1}^N p_j x_j (1-p_j) \right\| \leq \left(\sum_{j=1}^N \|p_j x_j (1-p_j)\|^2 \right)^{1/2} \leq \left(\sum_{j=1}^N \|x_j\|^2 \right)^{1/2}.$$

Similarly,

$$\left\| \sum_{j=1}^N (1-p_j) x_j p_j \right\| = \left\| \sum_{j=1}^N p_j x_j^* (1-p_j) \right\| \leq \left(\sum_{j=1}^N \|x_j\|^2 \right)^{1/2}.$$

Finally, again using orthogonality of the p_j's, we have

$$\left\| \sum_{j=1}^N p_j x_j p_j \right\| = \max_{j \in [N]} \|p_j x_j p_j\| \leq \max_{j \in [N]} \|x_j\|.$$

Thus,

$$\begin{aligned}\left\|\sum_{j=1}^N x_j\right\| &= \left\|\sum_{j=1}^N p_j x_j (1-p_j)\right\| + \left\|\sum_{j=1}^N (1-p_j) x_j p_j\right\| + \left\|\sum_{j=1}^N p_j x_j p_j\right\| \\ &\leq 2\left(\sum_{j=1}^N \|x_j\|^2\right)^{1/2} + \max_{j\in[N]} \|x_j\|\end{aligned}$$

as desired. □

The Boolean Case

A version of the following construction was done in the scalar case by [Ber06, §2]. The operator-valued case was done in [PV13, Remark 2.3].

Definition 4.3.4. Let $\mathcal{H}_1$, ..., $\mathcal{H}_N$ be Hilbert $\mathcal{A}$-$\mathcal{A}$-bimodules with a central unit vectors ξ_1, ..., ξ_N, and write $\mathcal{K}_j = \{\zeta \in \mathcal{H}_j : \langle \xi_j, \zeta\rangle = 0\}$ and $\mathcal{H}_j = \mathcal{A}\xi_j \oplus \mathcal{K}_j$.

We define the *Boolean product* of $(\mathcal{H}_1, \xi_1)$, ..., $(\mathcal{H}_N, \xi_N)$ as the pair $(\mathcal{H}, \xi)$, where

$$\mathcal{H} = \mathcal{A}\xi \oplus \bigoplus_{j=1}^N \mathcal{K}_j.$$

To define the maps ρ_j, we write $\mathcal{H}$ in the form

$$\mathcal{H} = \mathcal{H}_j \oplus \bigoplus_{k\neq j} \mathcal{K}_k.$$

If $b \in B(\mathcal{H}_j)$, then we define $\rho_j(b)$ to act by b on the first summand and by zero on other summands.

Proof of Theorem 4.3.1, Boolean case.

(1) This is a direct computation from the construction.

(2) Let $k \geq 1$, and consider a product of terms $\rho_{j_1}(b_1)$, ..., $\rho_{j_k}(b_k)$. We claim that

$$\rho_{j_1}(b_1)\dots\rho_{j_k}(b_k)\xi = E_{j_1}[b_1]\dots E_{j_k}[b_k]\xi + \zeta,$$

where $\zeta \in \mathcal{K}_{j_1}$, and we will prove this by induction. The base case $k = 1$ is immediate. Now suppose $k > 1$ and note by induction hypothesis

$$\zeta := \rho_{j_2}(b_2)\dots\rho_{j_k}(b_k)\xi = E_{j_2}[b_2]\dots E_{j_k}[b_k]\xi + \zeta'$$

with $\zeta' \in \mathcal{K}_{j_2}$. Since $j_2 \neq j_1$, we have $\rho_{j_1}(b_1)\zeta' = 0$. Meanwhile, if we set $a = E_{j_2}[b_2]\dots E_{j_k}[b_k]$, then

$$\rho_{j_1}(b_1)\zeta = \rho_{j_1}(b_1)a\xi = \langle \xi_{j_1}, b_1 a \xi_{j_1}\rangle\xi + \zeta,$$

where $\zeta \in \mathcal{K}_{j_1}$ by virtue of the construction of $\rho_{j_1}(b_1)$ and the orthogonal decomposition of $\mathcal{H}_{j_1}$ into $\mathcal{A}\xi_{j_1}$ and $\mathcal{K}_{j_1}$. But note that

$$\langle \xi_{j_1}, b_1 a \xi_{j_1}\rangle\xi = \langle \xi_{j_1}, b_1 \xi_{j_1} a\rangle\xi = \langle \xi_{j_1}, b_1\xi_{j_1}\rangle a\xi = E_{j_1}[b_1]E_{j_2}[b_2]\dots E_{j_k}[b_k]\xi,$$

which completes the induction step. It follows from this claim that

$$\begin{aligned} E[\rho_{j_1}(b_1)\dots\rho_{j_k}(b_k)] &= \langle \xi, \rho_{j_1}(b_1)\dots\rho_{j_k}(b_k)\xi\rangle \\ &= \langle \xi, E_{j_1}[b_1]\dots E_{j_k}[b_k]\xi\rangle + \langle \xi, \zeta\rangle \\ &= E_{j_1}[b_1]\dots E_{j_k}[b_k], \end{aligned}$$

which demonstrates Boolean independence.

(3) Consider an operator of the form $\sum_{j=1}^N x_j$, where $x_j = \rho_j(b_j)$ and $E_j[b_j] = 0$. Let $p_j \in B(\mathcal{H})$ denote the projection onto $\mathcal{K}_j$. One checks using similar reasoning as in part (2) that $(1-p_j)x_j(1-p_j) = 0$ and hence that

$$\sum_{j=1}^n x_j = \sum_{j=1}^n p_j x_j (1-p_j) + \sum_{j=1}^n (1-p_j)x_j p_j + \sum_{j=1}^n p_j x_j p_j.$$

The proof then proceeds exactly as in the free case, using the fact that the p_j's have orthogonal ranges. □

The (Anti-)Monotone Case

The following construction is due to [Mur00, §2] in the scalar-valued case and a version be found in [Ber05]. The operator-valued case is due to [Pop08a, §4].

Definition 4.3.5. Let $\mathcal{H}_1$, ..., $\mathcal{H}_N$ be Hilbert $\mathcal{A}$-$\mathcal{A}$-bimodules with a central unit vectors ξ_1, ..., ξ_N, and write $\mathcal{K}_j = \{\zeta \in \mathcal{H}_j : \langle \xi_j, \zeta\rangle = 0\}$ and $\mathcal{H}_j = \mathcal{A}\xi_j \oplus \mathcal{K}_j$.

We define the *monotone product* as the pair $(\mathcal{H}, \xi)$, where

$$\mathcal{H} = \mathcal{A}\xi \oplus \bigoplus_{k=1}^N \bigoplus_{N \geq j_1 > j_2 > \dots > j_k \geq 1} \mathcal{K}_{j_1} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{K}_{j_k}.$$

To define the maps ρ_j, we write $\mathcal{H}$ in the form

$$\begin{aligned} \mathcal{H} = (\mathcal{A}\xi \oplus \mathcal{K}_j) \otimes_{\mathcal{A}} \left(\mathcal{A} \oplus \bigoplus_{k \geq 1} \bigoplus_{j > j_1 > \dots > j_k} \mathcal{K}_{j_1} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{K}_{j_k} \right) \\ \oplus \bigoplus_{k \geq 1} \bigoplus_{\substack{j_1 > j_2 > \dots > j_k \\ j_1 > j}} \mathcal{K}_{j_1} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{K}_{j_k}. \end{aligned}$$

Let $b \in B(\mathcal{H}_j)$. The summand in the top line is the tensor product of $\mathcal{H}_j$ with another $\mathcal{A}$-$\mathcal{A}$-bimodule, and we define the action of $\rho_j(b)$ by its left action on $\mathcal{H}_j$. On the bottom summand, we define the action of $\rho_j(b)$ to be zero.

Proof of Theorem 4.3.1, monotone case. (1) The space $\mathcal{H}$ contains $\mathcal{A}\xi \oplus \mathcal{K}_j \cong \mathcal{H}_j$ as a direct summand, and the action of $\rho_j(b)$ on this subspace is the same as the action of b on $\mathcal{H}_j$, with ξ corresponding to ξ_j.

(2) In order to show monotone independence, we must show that

$$E[\rho_{p_s}(b_s)\dots\rho_{p_1}(b_1)\rho_j(b)\rho_{q_1}(b_1')\dots\rho_{q_t}(b_t')] = E[\rho_{p_s}(b_s)\dots\rho_{p_1}(b_1)E_j[b]\rho_{q_1}(b_1')\dots\rho_{q_t}(b_t')],$$

provided that $j > p_1$ if $s > 0$ and $j > q_1$ if $t > 0$, where $b \in B(\mathcal{H}_j)$ and $b_i \in B(\mathcal{B}_{p_i})$ and $b_i' \in B(\mathcal{H}_{q_i})$, where $E_j[b]$ denote the multiplication operator on $\mathcal{H}$ by $E_j[b] \in \mathcal{A}$. This claim is equivalent to

$$\langle \rho_{p_1}(b_1^*)\dots\rho_{p_s}(b_s^*)\xi, (\rho_j(b) - E_j[b])\rho_{q_1}(b_1')\dots\rho_{q_t}(b_t')\xi\rangle = 0.$$

Now we write

$$\begin{aligned} a &= E_j[b] \\ \zeta &= \rho_{p_1}(b_1^*) \dots \rho_{p_s}(b_s^*)\xi \\ \zeta' &= \rho_{q_1}(b_1') \dots \rho_{q_t}(b_t')\xi, \end{aligned}$$

and our goal is to show that $\langle \zeta, \rho_j(b)\zeta' \rangle = \langle \zeta, a\zeta' \rangle$.

We claim first that

$$\zeta, \zeta' \in \mathcal{N} := \mathcal{A}\xi \oplus \bigoplus_{k \geq 1} \bigoplus_{j > j_1 > j_2 > \dots > j_k} \mathcal{K}_{j_1} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{K}_{j_k}.$$

This is clear for ζ if $s = 0$ and hence $\zeta = \xi$. On the other hand, if $s > 0$, this follows because the image of $\rho_1(b_1^*)$ is contained in

$$\mathcal{A} \oplus \bigoplus_{k \geq 1} \bigoplus_{p_1 \geq j_1 > j_2 > \dots > j_k} \mathcal{K}_{j_1} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{K}_{j_k} \subseteq \mathcal{N},$$

and $p_1 < j$. The argument for ζ' is identical.

By construction $\rho_j(y)$ maps $\mathcal{N}$ into $(\mathcal{A} \oplus \mathcal{K}_j) \otimes \mathcal{N}$. However, since $E_j[b - a] = 0$, $\rho_j(b - a)$ maps the space $\mathcal{N}$ into

$$\mathcal{K}_j \otimes \mathcal{N} = \bigoplus_{k \geq 0} \bigoplus_{j > j_1 > j_2 > \dots > j_k} \mathcal{K}_j \otimes_{\mathcal{A}} \mathcal{K}_{j_1} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{K}_{j_k}.$$

Moreover, $\mathcal{K}_j \otimes \mathcal{N}$ is orthogonal to $\mathcal{N}$ by construction and hence

$$\langle \zeta, \rho_j(b - a)\zeta' \rangle = 0.$$

Finally, note that $\rho_j(a)|_{\mathcal{N}} = a|_{\mathcal{N}}$ and hence $\langle \zeta, \rho_j(b)\zeta' \rangle = \langle \zeta, a\zeta' \rangle$ as desired.

(3) Consider an operator of the form $\sum_{j=1}^N x_j$, where $x_j = \rho_j(b_j)$ and $E_j[b_j] = 0$. Let $\mathcal{M}_j \subseteq \mathcal{H}$ be the submodule consisting of tensor products where the first index is j, that is,

$$\mathcal{M}_j = \bigoplus_{k \geq 1} \bigoplus_{j > j_2 > \dots > j_k} \mathcal{K}_j \otimes_{\mathcal{A}} \mathcal{K}_{j_2} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{K}_{j_k}.$$

Note that $\mathcal{H} = \mathcal{A}\xi \oplus \bigoplus_{j=1}^N \mathcal{M}_j$, and let p_j be the projection onto $\mathcal{M}_j$. Moreover, x_j maps the orthogonal complement of $\mathcal{M}_j$ into $\mathcal{M}_j$ and hence $(1 - p_j)x_j(1 - p_j) = 0$. The argument then proceeds exactly the same as in the free case. □

For the anti-monotone case, the construction and the proof of Theorem 4.3.1 are exactly symmetrical to the monotone case. For example, the construction of the Hilbert $\mathcal{A}$-$\mathcal{A}$-module is as follows.

Definition 4.3.6. With $(\mathcal{H}_j, \xi_j)$ as above, we define the *anti-monotone product* as the pair $(\mathcal{H}, \xi)$, where

$$\mathcal{H} = \mathcal{A}\xi \oplus \bigoplus_{k=1}^{N} \bigoplus_{1 \leq j_1 < j_2 > \dots > j_k \geq 1} \mathcal{K}_{j_1} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{K}_{j_k}.$$

4.4 Products of Algebras

To define the product of non-commutative probability spaces $(\mathcal{B}_1, E_1)$, ..., $(\mathcal{B}_N, E_N)$, we apply the above construction to $(\mathcal{H}_j, \xi_j) = (\mathcal{B}_j \otimes_{E_j} \mathcal{A}, 1 \otimes 1)$. Then we define the product algebra $\mathcal{B}$ to be the C^*-subalgebra of $B(\mathcal{H})$ generated by $\mathcal{A}$ and $\rho_j(\mathcal{B}_j)$ for $j = 1, \dots, N$, with the expectation E given by the inner product with $\xi \in \mathcal{H}$.

There are two technical points we should comment on here. The first is faithfulness of the representation. If the representation of $\mathcal{B}_j$ on $\mathcal{B}_j \times_{E_j} \mathcal{A}$ is faithful, then the maps $\mathcal{B}_j \to \mathcal{B}$ are injective and the representation of $\mathcal{B}$ on $\mathcal{H}$ is faithful. Moreover, one can show that $\mathcal{B}\xi$ is dense in $\mathcal{H}$, so that $\mathcal{H} \cong \mathcal{B} \otimes_E \mathcal{A}$.

The second issue is that the inclusions $\rho_j : \mathcal{B}_j \to \mathcal{B}$ are not unital (except in the free case). Some authors deal with this by framing non-commutative probability theory in terms of non-unital algebras from the outset. Another option is to consider unital algebras $\mathcal{B}_j$, but to assume that each $\mathcal{B}_j$ has a non-unital $\mathcal{A}$-subalgebra $\mathcal{B}_{j,0}$ with $\mathcal{B}_j = \mathcal{B}_{j,0} \oplus \mathcal{A}$ as $\mathcal{A}$-$\mathcal{A}$-bimodules (the direct sum is not necessarily orthogonal). We can then construct the non-unital product algebra $\mathcal{B}_0$ of the algebras $\mathcal{B}_{j,0}$ and let $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{A}$. Then one obtains unital inclusions of $\mathcal{B}_j$ into $\mathcal{B}$.

However, we will not be too concerned about these issues in the rest of these notes, since we will focus on the computation of moments rather than the properties of the product operations on C^* algebras.

4.5 Associativity

Lemma 4.5.1. *Each of the four product operations for pairs $(\mathcal{H}, \xi)$ described in Definitions 4.3.3, 4.3.4, 4.3.5, 4.3.6 is associative in that sense that if $\medstar$ is one of these product operations, then we have natural isomorphisms*

$$\begin{aligned}\medstar_{j=1}^{3}(\mathcal{H}_j, \xi_j) &\cong ((\mathcal{H}_1, \xi_1) \medstar (\mathcal{H}_2, \xi_2)) \medstar (\mathcal{H}_3, \xi_3) \\ &\cong (\mathcal{H}_1, \xi_1) \medstar ((\mathcal{H}_2, \xi_2) \medstar (\mathcal{H}_3, \xi_3)).\end{aligned}$$

Moreover, the inclusion maps $\rho_j : B(\mathcal{H}_j) \to B(\mathcal{H})$ are equivalent in all three ways of expressing the product space.

The argument here is a straighforward rearrangement of the summands in the product space, using the distributive and associative properties of tensor products. Let us describe the proof of the first equality in the monotone case and leave the others as exercises for the reader.

Proof for the monotone case. We can write

$$(\mathcal{H}_1, \xi_1) \medstar (\mathcal{H}_2, \xi_2) = \mathcal{A}\xi' \oplus \mathcal{K}',$$

where

$$\begin{aligned}\mathcal{K}' &= \bigoplus_{k \geq 1} \bigoplus_{2 \geq j_1 \geq \dots \geq j_k \geq 1} \mathcal{K}_{j_1} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{K}_{j_k} \\ &= \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_2 \otimes \mathcal{K}_1.\end{aligned}$$

Note that

$$\begin{aligned}((\mathcal{H}_1,\xi_1)\varoast(\mathcal{H}_2,\xi_2))\#(\mathcal{H}_3,\xi_3) &= \mathcal{A}\xi \oplus \mathcal{K}' \oplus \mathcal{K}_3 \oplus \mathcal{K}_3 \otimes_{\mathcal{A}} \mathcal{K}' \\ &= \mathcal{A}\xi \oplus \bigoplus_{k\geq 1} \bigoplus_{3\geq j_1 > \dots > j_k \geq 1} \mathcal{K}_{j_1} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{K}_{j_k} \\ &= \varoast_{j=1}^3 (\mathcal{H}_j,\xi_j).\end{aligned}$$

It is easy to verify that the inclusions $B(\mathcal{H}_j) \to B(\mathcal{H})$ are the same for either decomposition of the product space. □

As a corollary, we have the following method for checking independence of subalgebras.

Lemma 4.5.2. *For free, Boolean, monotone, and anti-monotone independence, the following holds. Let $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ be $\mathcal{A}$-subalgebras of $(\mathcal{B}, E)$, and assume in the free case that they are unital. The following are equivalent:*

1. *$\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ are independent.*
2. *$\mathcal{B}_1$ and $\mathcal{B}_2$ are independent, and $\mathcal{B}_1 \vee \mathcal{B}_2$ and $\mathcal{B}_3$ are independent.*

Here $\mathcal{B}_1 \vee \mathcal{B}_2$ denotes the $\mathcal{A}$-subalgebra generated by $\mathcal{B}_1$ and $\mathcal{B}_2$.

Proof. Suppose (1) holds. By Lemma 4.2.7, (1) uniquely determines the expectation of elements of $\mathcal{B}_1 \vee \mathcal{B}_2 \vee \mathcal{B}_3$ in terms of $E|_{\mathcal{B}_j}$. Thus, rather than examining all possible algebras $\mathcal{B}$ containing independent copies of $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$, we may assume that $\mathcal{B} = B(\mathcal{H})$ where $(\mathcal{H}, \xi)$ is the independent product of the spaces $(\mathcal{H}_j, \xi_j) := (\mathcal{B}_j \vee \mathcal{A}) \otimes_E \mathcal{A}$ for $j = 1, 2, 3$. But by the previous lemma,

$$(\mathcal{H}, \xi) \cong ((\mathcal{H}_j, \xi_j)\varoast(\mathcal{H}_2, \xi_2))\varoast(\mathcal{H}_3, \xi_3).$$

This implies that $\mathcal{B}_1$ and $\mathcal{B}_2$ are independent and $\mathcal{B}_1 \vee \mathcal{B}_2$ and $\mathcal{B}_3$ are indpendent.

The argument for (2) $\implies$ (1) is similar. By applying Lemma 4.2.7 twice, we see that (2) uniquely determines the expectations of elements of $\mathcal{B}_1 \vee \mathcal{B}_2 \vee \mathcal{B}_3$ given $E|_{\mathcal{B}_j}$. Thus, we can assume that $\mathcal{B}_1$, ..., $\mathcal{B}_3$ are represented on the product space $((\mathcal{H}_j, \xi_j)\varoast(\mathcal{H}_2, \xi_2))\varoast(\mathcal{H}_3, \xi_3)$, which is isomorphic to $\#_{j=1}^3(\mathcal{H}_j, \xi_j)$. □

4.6 Independent Random Variables and Convolution of Laws

Next, we define what it means for random variables to be independent. In the following, for a self-adjoint X in $\mathcal{B} \supseteq \mathcal{A}$, it will be convenient to denote by $\mathcal{A}\langle X\rangle$ the subalgebra of $\mathcal{B}$ generated by $\mathcal{A}$ and X. This object is strictly speaking not the same thing as the formal polynomial algebra $\mathcal{A}\langle X\rangle$, but this abuse of notation is already entrenched in algebra. We also denote by $\mathcal{A}\langle X\rangle_0$ the polynomials with no constant term, that is,

$$\mathcal{A}\langle X\rangle_0 = \operatorname{Span}\{a_0 X a_1 \dots X a_k : a_j \in \mathcal{X}, k \geq 1\}.$$

Definition 4.6.1. Self-adjoint random variables X_1, ..., X_N in $(\mathcal{B}, E)$ are said to be *freely independent* if the algebra $\mathcal{A}\langle X_1\rangle$, ..., $\mathcal{A}\langle X_N\rangle$ are freely independent. Random variables X_1, ..., X_N are said to be *Boolean / monotone / anti-monotone independent* if the algebras $\mathcal{A}\langle X_1\rangle_0$, ..., $\mathcal{A}\langle X_n\rangle_0$ are Boolean / monotone / anti-monotone independent.

Lemma 4.6.2. *Given $\mathcal{A}$-valued laws μ_1, ..., μ_N, there exists an $\mathcal{A}$-valued probability space $(\mathcal{B}, E)$ with self-adjoint elements X_1, ..., X_N which are free / Boolean / monotone / anti-monotone independent.*

Proof. By Theorem 1.6.5, the law μ_j is realized by the random variable X_j acting on the Hilbert bimodule $\mathcal{H}_j = \mathcal{A}\langle X_j \rangle \otimes_{\mu_j} \mathcal{A}$ with the expectation given by the vector $\xi_j = 1 \otimes 1$ in $\mathcal{H}_j$. Let $(\mathcal{H}, \xi)$ be the independent product of the spaces $(\mathcal{H}_1, \xi_1)$, ..., $(\mathcal{H}_N, \xi_N)$. Let $\mathcal{B}$ be the C^*-algebra generated by $\mathcal{A}$ and the variables X_1, ..., X_N acting on $\mathcal{H}$ and define $E[b] = \langle \xi, b\xi \rangle$. Then X_1, ..., X_N are independent with laws μ_1, ..., μ_N. □

Definition 4.6.3 (Convolution)**.** Let μ and ν be laws, and let X and Y be free / Boolean / monotone / anti-monotone independent random variables with laws μ and ν respectively. We define the *free / Boolean / monotone / anti-monotone convolution* of μ and ν to be the law of $X + Y$. This is well-defined by Lemma 4.2.7. We denote this law by

$$\begin{aligned} &\mu \boxplus \nu \quad \text{(free case)} \\ &\mu \uplus \nu \quad \text{(Boolean case)} \\ &\mu \rhd \nu \quad \text{(monotone case)} \\ &\mu \lhd \nu \quad \text{(anti-monotone case).} \end{aligned}$$

Lemma 4.6.4.

1. *The operations* $\boxplus$, $\uplus$, $\rhd$, *and* $\lhd$ *are associative.*

2. *The operations* $\boxplus$ *and* $\uplus$ *are commutative.*

3. *We have* $\mu \rhd \nu = \nu \lhd \mu$.

Proof. These follow from the corresponding properties of independence. □

4.7 Analytic Transforms

Our next task to develop analytic tools for computing the independent convolution of two laws. In the classical case, this role is played by characteristic function (Fourier transform) of a law given by $\mathcal{F}\mu(\xi) = \int e^{ix\xi}\, d\mu(x)$, since addition of independent random variables or classical convolution of laws corresponds to multiplication of the Fourier transforms. In the non-commutative setting, this role is played by various fully matricial functions related to the Cauchy-Stieltjes transform.

The Free Case

The following analytic transforms were defined by Voiculescu [Voi86]. In the operator-valued case, the definition was developed by Dykema [Dyk07, §6].

Definition 4.7.1. For an $\mathcal{A}$-valued law μ, we define $F_\mu(z) = G_\mu(z)^{-1}$ and

$$\Phi_\mu(z) := F_\mu^{-1}(z) - z,$$

where $F_\mu^{-1}(z)$ is the functional inverse and z is in the image of F_μ. We define the R-transform $R_\mu(z) = \tilde{\Phi}_\mu(z) = \Phi_\mu(z^{-1})$.

We caution that some authors have slightly different conventions for the definition of R-transform. We will show that Φ_μ and R_μ are additive under free convolution, but first we must say on what domain Φ_μ and R_μ are defined.

Theorem 4.7.2. *The function* Φ_μ *is a well-defined fully matricial function*

$$\bigcup_{\delta > \|\mathrm{Var}_\mu(1)\|^{1/2}} \mathbb{H}_{2\delta,+}(\mathcal{A}) \to \mathbb{H}_-(\mathcal{A}).$$

Moreover, $R_\mu(z) = \tilde{\Phi}_\mu(z)$ *has a fully matricial extension to* $B(0, (3 - 2\sqrt{2})/\operatorname{rad}(\mu))$ *satisfying*

$$R_\mu(0) = \mu(X), \qquad R_\mu(z^*) = R_\mu(z)^*$$

and

$$\|R_\mu(z) - \mu(X)^{(n)}\| \leq \frac{2\|\mathrm{Var}_\mu(1)\| \operatorname{rad}(\mu)}{\sqrt{2} - 1}.$$

Proof. By Theorem 3.5.3, there exists a self-adjoint a_0 and a generalized law σ with $\operatorname{rad}(\sigma) \leq 2\operatorname{rad}(\mu)$ such that

$$F_\mu(z) = z - a_0 - G_\sigma(z).$$

Let $\delta > \|\sigma(1)\|^{1/2} = \|\mathrm{Var}_\mu(1)\|^{1/2}$. Then we claim that F_μ has an inverse function $\Psi : \mathbb{H}_{+,2\delta}(\mathcal{A}) \to \mathbb{H}_{+,\delta}(\mathcal{A})$. We will construct Ψ by a contraction mapping principle just as in the inverse function theorem. We want to solve the equation

$$w = \Psi(w) - a_0 - G_\sigma(\Psi(w)),$$

so that $\Psi(w)$ satisfies the fixed point equation

$$\Psi^{(n)}(w) = w + a_0^{(n)} + G_\sigma^{(n)}(\Psi(w)).$$

Let $H_w(z) = w + a_0 + G_\sigma(z)$. Note that by Lemma 3.3.1, if $z, z' \in \mathbb{H}_{\delta,+}^{(n)}(\mathcal{A})$, then

$$\|H_w(z) - H_w(z')\| = \|G_\sigma(z) - G_\sigma(z')\| \leq \frac{\|\sigma(1)\|}{\delta^2}\|z - z'\|.$$

Therefore, H_w is a contraction provided that $\delta > \|\sigma(1)\|^{1/2}$. Moreover, if $\operatorname{Im} w \geq 2\delta$, then H_w maps $\mathbb{H}_{\delta,+}(\mathcal{A})$ into itself because

$$\operatorname{Im} H_w(z) = \operatorname{Im} w + \operatorname{Im} G_\sigma(z) \geq 2\delta - \frac{\|\sigma(1)\|}{\delta} \geq \delta.$$

Therefore, by the Banach fixed point theorem, H_w has a unique fixed point $\Psi(w)$ in $\mathbb{H}_{+,\delta}(\mathcal{A})$. We also have

$$\|\Psi(w) - w\| = \|a_0^{(n)} + G_\sigma(\Psi(w))\| \leq \|a_0\| + \frac{\|\sigma(1)\|}{\delta} \leq \|a_0\| + \delta.$$

Therefore, if we define

$$\Psi_0(w) = w, \qquad \Psi_{k+1}(w) = H_w(\Psi_k(w)),$$

then for $\operatorname{Im} w \geq 2\delta$,

$$\|\Psi_k(w) - \Psi(w)\| \leq \left(1 - \frac{\|\sigma(1)\|}{\delta^2}\right)^k (\|a_0\| + \delta).$$

In particular, Ψ_k converges uniformly locally to Ψ on $\bigcup_{\delta > \|\sigma(1)\|^{1/2}} \mathbb{H}_{+,2\delta}(\mathcal{A})$. It follows that $\Psi(w)$ is fully matricial.

By definition,

$$\Phi_\mu(z) = \Psi(z) - z = a_0 + G_\sigma(\Psi(z)).$$

Therefore, Φ_μ is a fully matricial function $\bigcup_{\delta > \|\sigma(1)\|^{1/2}} \mathbb{H}_{+,2\delta}(\mathcal{A}) \to \overline{\mathbb{H}}_-(\mathcal{A})$. Now consider the behavior of $\tilde{\Phi}_\mu = R_\mu$ near zero. We have

$$\begin{aligned} R_\mu(z) &= a_0^{(n)} + G_\sigma(\Psi(z^{-1})) \\ &= a_0^{(n)} + G_\sigma(F_\mu^{-1}(z^{-1})) \\ &= a_0^{(n)} + \tilde{G}_\sigma(\tilde{G}_\mu^{-1}(z)). \end{aligned}$$

By the inverse function theorem, since $D\tilde{G}_\mu(0) = \mathrm{id}$, we know that $\tilde{G}_\mu$ has a inverse function in a neighborhood of zero, and hence R_μ is defined in a neighborhood of 0 and $R_\mu(0) = a_0 = \mu(X)$.

To get a more precise estimate on the size of the neighborhood, observe that for $R = 1/\operatorname{rad}(\mu)$, we have

$$\|\Delta^k \tilde{G}_\mu(0, \dots, 0)\|_\# \leq \frac{1}{R^{k-1}},$$

and therefore, we are in the setting of the inverse function theorem with $M = R$ and $K = 1$. Thus, by Theorem 2.8.1, $\tilde{G}_\mu^{-1}$ maps $B(0, R(3 - 2\sqrt{2})) \to B(0, R(1 - 1/\sqrt{2}))$. But note that

$$\left(1 - \frac{1}{\sqrt{2}}\right) R < \frac{1}{2} R = \frac{1}{2 \operatorname{rad}(\mu)} \leq \frac{1}{\operatorname{rad}(\sigma)},$$

and hence $B(0, R(1 - 1/\sqrt{2}))$ is within the ball where $\tilde{G}_\sigma$ is defined, so that $R_\mu = \tilde{G}_\sigma \circ \tilde{G}_\mu + a_0^{(n)}$ is defined on $B(0, R(3 - 2\sqrt{2}))$. Futhermore, $\tilde{G}_\sigma$ is bounded by

$$\frac{\|\sigma(1)\|}{(1/2)R - (1 - 1/\sqrt{2})R} = \frac{\|\sigma(1)\| \operatorname{rad}(\mu)}{1/2 + 1/\sqrt{2} - 1} = \frac{2\|\operatorname{Var}_\mu(1)\| \operatorname{rad}(\mu)}{\sqrt{2} - 1}.$$

□

The following result on the additivity of the R-transform was discovered in the scalar-valued case by Voiculescu [Voi86]. The original proof by Voiculescu used canonical realizations of a law μ by (non-self-adjoint) random variables on a Fock space, and this was adapted to the operator-valued setting by Dykema [Dyk07, §6]. This theorem can also be proved through the combinatorial apparatus of free cumulants due to Speicher [Spe94] [Spe98], which we describe in the next chapter. The analytic proof presented here is due (in the scalar-valued setting) to Lehner [Leh01, Theorem 3.1] and can also be found in [Tao, §4].

Theorem 4.7.3. *For* $\operatorname{Im} z \geq 2\delta > 2\|\operatorname{Var}_\mu(1) + \operatorname{Var}_\nu(1)\|^{1/2}$, *we have*

$$\Phi_{\mu \boxplus \nu}(z) = \Phi_\mu(z) + \Phi_\nu(z).$$

Also, for z *in a fully matricial neighborhood of* 0, *we have*

$$R_{\mu \boxplus \nu}(z) = R_\mu(z) + R_\nu(z)$$

Proof. Let X and Y be freely independent random variables in $(\mathcal{B}, E)$ which realize the laws μ and ν.

We begin by analyzing $R_\mu(z)$ in a neighborhood of the origin. Now $z^{-1} + R_\mu(z)$ is the functional inverse of $G_\mu(z)$ in a neighborhood of 0 which means that

$$E[(z^{-1} + R_\mu(z) - X)^{-1}] = z.$$

Multiplying by z^{-1} on the right, we can write rewrite this as

$$E[(1 + zR_\mu(z) - zX)^{-1}] = 1,$$

or in other words, the $\mathcal{B}$-valued fully matricial function

$$U_X(z) = (1 + zR_\mu(z) - zX)^{-1} - 1$$

has expectation zero (where in the definition of $U_X^{(n)}(z)$, X denotes $X^{(n)}$). The same holds for the analogously-defined function $U_Y(z)$. We want to show that $z^{-1} - R_\mu(z) - R_\nu(z)$ is the functional inverse of $G_{\mu\boxplus\nu}$, which means that

$$G_{\mu\boxplus\nu}(z^{-1} + R_\mu(z) + R_\nu(z)) = z,$$

which after multiplying by z^{-1} on the right is equivalent to

$$E[(1 + zR_\mu(z) + zR_\nu(z) - zX - zY)^{-1}] = 1.$$

We will rewrite the left hand side in terms of $U_X(z)$ and $U_Y(z)$ so that we can apply freeness together with the fact that $U_X(z)$ and $U_Y(z)$ have expectation zero. Note that

$$\begin{aligned}
&(1 + zR_\mu(z) + zR_\nu(z) - zX - zY)^{-1} \\
&\qquad = [(1 + U_X(z))^{-1} + (1 + U_Y(z))^{-1} - 1]^{-1} \\
&\qquad = (1 + U_X(z))[(1 + U_Y(z)) + (1 + U_X(z)) - (1 + U_Y(z))(1 + U_X(z))]^{-1}(1 + U_Y(z)) \\
&\qquad = (1 + U_X(z))[1 - U_Y(z)U_X(z)]^{-1}(1 + U_Y(z)).
\end{aligned}$$

Now because $U_X(0) = 0 = U_Y(0)$, we know that for sufficiently small z, we can expand $[1 - U_Y(z)U_X(z)]^{-1}$ as a geometric series, and thus for small z,

$$(1 - zR_\mu(z) - zR_\nu(z) - zX - zY)^{-1} = (1 + U_X(z))\left(\sum_{k=0}^{\infty}(U_Y(z)U_X(z))^k\right)(1 + U_Y(z)).$$

Next, we take the expectation. Because $U_X(z)$ and $U_Y(z)$ have expectation zero and because X and Y are free, all the terms on the right hand side have zero expectation except the term 1 which comes from multiplying together the 1 from $1 + U_X(z)$, the 1 from the geometric series, and the 1 from $1 + U_Y(z)$. Therefore, as desired,

$$E[(1 - zR_\mu(z) - zR_\nu(z) - zX - zY)^{-1}] = 1.$$

This shows that

$$R_{\mu\boxplus\nu}(z) = R_\mu(z) + R_\nu(z)$$

holds in a neighborhood of zero.

This implies that $\Phi_{\mu\boxplus\nu} = \Phi_\mu + \Phi_\nu$ if $\operatorname{Im} z$ is sufficiently large, and hence by Corollary 2.9.7, we have $\Phi_{\mu\boxplus\nu} = \Phi_\mu + \Phi_\nu$ on $\mathbb{H}_{+,2\delta}(\mathcal{A})$, provided that this lies inside the common domain of $\Phi_{\mu\boxplus\nu}$, Φ_μ, and Φ_ν. Since $\operatorname{Var}_{\mu\boxplus\nu}(1) = \operatorname{Var}_\mu(1) + \operatorname{Var}_\nu(1)$ and all these elements are positive, we have $\|\operatorname{Var}_{\mu\boxplus\nu}(1)\| \geq \max(\|\operatorname{Var}_\mu(1)\|, \|\operatorname{Var}_\nu(1)\|)$, and hence it is sufficient that $\delta > \|\operatorname{Var}_{\mu\boxplus\nu}(1)\|^{1/2}$. □

The Boolean Case

The results of this section can be found in [SW97] [Ber06, Theorem 2.2] for the scalar case and [PV13, §2 and §5.3] in the operator-valued case.

Definition 4.7.4. For an $\mathcal{A}$-valued law μ, we define the *B-transform* as

$$B_\mu(z) := z - F_\mu(z).$$

We caution that some authors use this notation for $\tilde{B}_\mu(z) = B_\mu(z^{-1})$ instead.

Remark 4.7.5. We showed in Theorem 3.5.3 that $B_\mu(z) = \mu(X)^{(n)} + G_\sigma(z)$ for some generalized law σ.

Theorem 4.7.6. *$B_{\mu \uplus \nu}(z) = B_\mu(z) + B_\nu(z)$ as fully matricial functions.*

Proof. Let X and Y be freely independent random variables in $(\mathcal{B}, E)$ which realize the laws μ and ν. For small z, define

$$U_X(z) = (1 - zX)^{-1} - 1 = \sum_{k=1}^{\infty} (zX)^k,$$

and note that

$$1 + E[U_X(z)] = E[(1 - zX)^{-1}] = \tilde{G}_\mu(z) z^{-1}$$

or in other words

$$(1 + E[U_X(z)])^{-1} = z\tilde{F}_\mu(z)$$

Note that $U_X(z)$ is in the closed span of $\mathcal{A}\langle X \rangle_0$. Define $U_Y(z)$ analogously. Then

$$1 - zX - zY = (1 + U_X(z))^{-1} + (1 + U_Y(z))^{-1} - 1$$

Therefore,

$$\begin{aligned} (1 - zX - zY)^{-1} &= [(1 + U_X(z))^{-1} + (1 + U_Y(z))^{-1} - 1]^{-1} \\ &= (1 + U_X(z))[1 - U_Y(z)U_X(z)]^{-1}(1 + U_Y(z)) \\ &= (1 + U_X(z)) \left(\sum_{k=0}^{\infty} (U_Y(z)U_X(z))^k \right) (1 + U_Y(z)). \end{aligned}$$

Next, we take the expectation. Because $U_X(z)$ and $U_Y(z)$ are in the closures of $M_n(\mathcal{A}\langle X \rangle_0)$ and $M_n(\mathcal{A}\langle Y \rangle_0)$ respectively and because X and Y are Boolean independent, we have

$$\begin{aligned} E[(1 - zX - zY)^{-1}] &= (1 + E[U_X(z)]) \left(\sum_{k=0}^{\infty} (E[U_Y(z)]E[U_X(z)])^k \right) (1 + E[U_Y(z)]) \\ &= [(1 + E[U_X(z)])^{-1} + (1 + E[U_Y(z)])^{-1} - 1]^{-1} \end{aligned}$$

Therefore,

$$\tilde{G}_{\mu \uplus \nu}(z) z^{-1} = [(1 + E[U_X(z)])^{-1} + (1 + E[U_Y(z)])^{-1} - 1]^{-1}$$

By taking reciprocals,

$$\begin{aligned} z\tilde{F}_{\mu \uplus \nu}(z) &= (1 + E[U_X(z)])^{-1} + (1 + E[U_Y(z)])^{-1} - 1 \\ &= z\tilde{F}_\mu(z) + z\tilde{F}_\nu(z) - 1, \end{aligned}$$

Because $z\tilde{F}_\mu(z) - 1 = z\tilde{B}_\mu(z)$ and the same holds for Y and $X + Y$, this means precisely that

$$z\tilde{B}_{\mu\uplus\nu}(z) = z\tilde{B}_\mu(z) + z\tilde{B}_\nu(z)$$

for z in a neighborhood of 0. By Corollary 2.9.7, we have $B_{\mu\uplus\nu} = B_\mu + B_\nu$ on the upper half plane. □

The (Anti-)Monotone Case

The following result is due to [Mur00, Theorem 3.1] in the scalar-valued case and [Pop08a, Theorems 3.2 and 3.7] in the operator-valued case, whose proof we follow here. Another proof in the scalar case is in [Ber05].

Theorem 4.7.7. *We have $F_{\mu\rhd\nu}(z) = F_\mu(F_\nu(z))$ and $F_{\mu\lhd\nu}(z) = F_\nu(F_\mu(z))$ as fully matricial functions.*

Proof. Let inv denote the fully matricial function $z \mapsto z^{-1}$ where defined. Since $F_\mu = \operatorname{inv} \circ \tilde{G}_\mu \circ \operatorname{inv}$ and inv is an involution, it suffices to show that $\tilde{G}_{\mu\rhd\nu} = \tilde{G}_\mu \circ \tilde{G}_\nu$.

Let X and Y be monotone independent random variables in $(\mathcal{B}, E)$ realizing the laws μ and ν. We know that

$$E[f_0(Y)g_1(X)f_1(Y)\dots g_n(X)f_n(Y)] = E[E[f_0(Y)]g_1(X)E[f_1(Y)]\dots g_n(X)E[f_n(Y)]]$$

whenever $f(Y) \in \mathcal{A}\langle Y\rangle_0$ and $f(X) \in \mathcal{A}\langle X\rangle_0$. However, this also holds trivially if $f_j(Y) \in \mathcal{A}$, and thus by linearity it holds when $f_j(Y) \in \mathcal{A}\langle Y\rangle$.

Now for small z we have

$$\begin{aligned}\tilde{G}_{\mu\rhd\nu}(z) &= E[(1 - zX - zY)^{-1}z] \\ &= E[(1 - (1 - zY)^{-1}zX)^{-1}(1 - zY)^{-1}z] \quad = E\left[\sum_{k=1}^{\infty}[(1 - zY)^{-1}zX]^k(1 - zY)^{-1}z\right].\end{aligned}$$

Note that $(1 - zY)^{-1}$ is in the closure of $M_n(\mathcal{A}\langle Y\rangle)$ and $zX \in M_n(\mathcal{A}\langle X\rangle_0)$ and hence by monotone independence

$$\begin{aligned}\tilde{G}_{\mu\rhd\nu}(z) &= E\left[\sum_{k=1}^{\infty}[E[(1 - zY)^{-1}z]X]^k E[(1 - zY)^{-1}z]\right] \\ &= E\left[\sum_{k=1}^{\infty}[\tilde{G}_\nu(z)X]^k\tilde{G}_\nu(z)\right] \\ &= \tilde{G}_\mu \circ \tilde{G}_\nu(z).\end{aligned}$$

This equality extends to all z by Corollary 2.9.7. The anti-monotone case follows from the monotone case since $\mu \lhd \nu = \nu \rhd \mu$. □

4.8 Problems and Further Reading

Problem 4.1. Complete the details of the proof of Lemma 4.5.2 in all four cases.

Problem 4.2. Let $(\mathcal{B}, E)$ be an $\mathcal{A}$-valued probability space. Let $\mathcal{B}_1, \dots, \mathcal{B}_n$ be $\mathcal{A}$-subalgebras of $\mathcal{B}$. Show that for each type of independence, the following are equivalent:

1. $\mathcal{B}_1, \dots, \mathcal{B}_n$ are independent.

2. For each j, the algebras $\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_{j-1}$ and $\mathcal{B}_j \vee \cdots \vee \mathcal{B}_n$ are independent.

Problem 4.3. For each type of independence, show the following: Let $(\mathcal{B}, E)$ be an $\mathcal{A}$-valued probability space and let $\mathcal{B}_1, \dots, \mathcal{B}_n$ be $\mathcal{A}$-subalgebras (unital in the free case). Then $\mathcal{B}_1, \dots, \mathcal{B}_n$ are independent over $\mathcal{A}$ if and only if $M_n(\mathcal{B}_1), \dots, M_n(\mathcal{B}_n)$ are independent over $M_n(\mathcal{A})$ in the probability space $(M_n(\mathcal{B}), E^{(n)})$.

Chapter 5

Combinatorial Theory of Independence

5.1 Introduction

While analytic transforms allow us to compute the convolution of two laws, one would like more generally to compute arbitrary mixed moments of independent random variables X_1, ..., X_N. The theory of cumulants provides a combinatorial tool to do this. The formula for converting between moments and cumulants is phrased in terms of non-crossing partitions of the set $[n] = \{1, \dots, n\}$. This makes the computations easy to visualize even if they are not numerically tractable in high degree.

Cumulants also provide a way to characterize independence in the free and Boolean cases, and the power series coefficients of the analytic transforms in the last chapter are given by cumulants. Although free cumulants were defined by Voiculescu [Voi86], the combinatorial approach is due to Speicher [Spe94], [Spe98], and the adaptation to other types of operator-valued independence is due to various authors, whom we will cite individually below. We begin with basic definitions concerning non-crossing partitions.

5.2 Non-crossing Partitions

Definition 5.2.1. A *partition* of $[n]$ is collection of nonempty subsets V_1, ..., V_k such that $[n] = \bigsqcup_{j=1}^k V_j$. We call the subsets V_j *blocks.* We denote by $|\pi|$ the number of blocks. We denote the collection of partitions by $\mathcal{P}(n)$.

Definition 5.2.2. Let π be a partition of $[n]$. A *crossing* is a set of indices $i_1 < j_1 < i_2 < j_2$ such that i_1 and i_2 are in the same block V and j_1 and j_2 are in the same block $W \neq V$. A partition is said to be *non-crossing* if it has no crossings. We denote the set of non-crossing partitions by $\mathcal{NC}(n)$.

A partition is non-crossing if and only if it can be drawn in the plane without crossings. See Figures 5.2 and Figure 5.2.

Definition 5.2.3. A partition $\pi \in \mathcal{P}(n)$ is an *interval partition* if every block V has the form $\{j, j+1, \dots, k\}$ for some $j \leq k$. We denote the set of interval partitions by $\mathcal{I}(n)$.

Note that every interval partition is non-crossing.

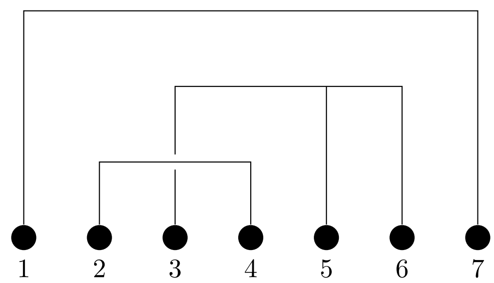

Figure 51: The partition $\{\{1,7\},\{2,4\},\{3,5,6\}\}$ has a crossing $2 < 3 < 4 < 5$.

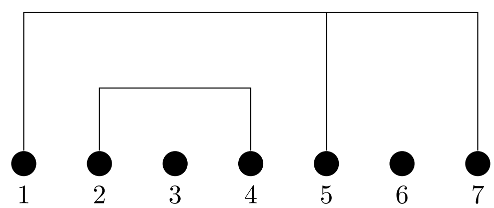

Figure 52: A non-crossing partition with blocks $V_1 = \{1,5,7\}$, $V_2 = \{2,4\}$, $V_3 = \{3\}$ and $V_4 = \{6\}$.

Lattice Properties

Definition 5.2.4. We say that a partition π *refines* a partition σ, or $\pi \leq \sigma$, if every block of π is contained in some block of σ.

Definition 5.2.5. Given a partitions $\pi_1, \dots, \pi_m$ of $[n]$, we define their *common refinement*

$$\bigwedge_{k=1}^{m} \pi_k = \pi_1 \wedge \cdots \wedge \pi_m = \{V = V_1 \cap \cdots \cap V_m : V_k \text{ a block of } \pi_k \text{ and } V \neq \varnothing\}.$$

The notation here makes sense because $\wedge$ is commutative and associative.

Lemma 5.2.6. *If $\pi_1, \dots, \pi_m$ are non-crossing partitions of $[n]$, then $\pi_1 \wedge \cdots \wedge \pi_m$ is non-crossing. If $\pi_1, \dots, \pi_m$ are interval partitions, then $\pi_1 \wedge \cdots \wedge \pi_m$ is an interval partition.*

Proof. First, consider the non-crossing case. Suppose for contradiction that $\pi = \pi_1 \wedge \cdots \wedge \pi_m$ has a crossing $i_1 < j_1 < i_2 < j_2$, where $i_1, i_2 \in V$ and $j_1, j_2 \in W$ for two distinct blocks V and W of π. By definition of $\wedge$, for each partition π_k, the indices i_1 and i_2 must be in the same block V_k, and the indices j_1 and j_2 must be in the same block W_k. Since π_k is non-crossing, V_k must equal W_k, so that i_1, i_2, j_1, and j_2 are all in the same block of π_k. But since this holds for every k, the four indices must have been in the same block of π, which contradicts our assumption that $i_1 < j_1 < i_2 < j_2$ is a crossing.

The interval case is immediate because the intersection of intervals is an interval. □

The common refinement $\pi_1 \wedge \cdots \wedge \pi_m$ can be thought of as the *minimum* or *greatest lower bound* of $\pi_1, \dots, \pi_m$ with respect to the refinement partial order $\leq$. Indeed, we have $\pi_1 \wedge \cdots \wedge \pi_m \leq \pi_k$ for each k; on the other hand, if $\pi \leq \pi_k$ for each k, then $\pi \leq \pi_1 \wedge \cdots \wedge \pi_m$.

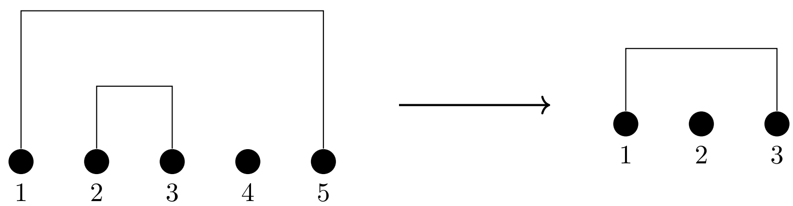

Figure 53: The partition $\pi = \{\{1,5\},\{2,3\},\{4\}\}$ and $\pi \setminus \{2,3\}$.

Moreover, each collection $\mathcal{P}(n)$, $\mathcal{NC}(n)$, and $\mathcal{I}(n)$ has a *maximum* operation given by

$$\pi_1 \vee_{\mathcal{P}} \cdots \vee_{\mathcal{P}} \pi_m := \bigwedge_{\substack{\pi \in \mathcal{P}(n) \\ \pi \geq \pi_k \forall k}} \pi$$
$$\pi_1 \vee_{\mathcal{NC}} \cdots \vee_{\mathcal{NC}} \pi_m := \bigwedge_{\substack{\pi \in \mathcal{NC}(n) \\ \pi \geq \pi_k \forall k}} \pi$$
$$\pi_1 \vee_{\mathcal{I}} \cdots \vee_{\mathcal{I}} \pi_m := \bigwedge_{\substack{\pi \in \mathcal{I}(n) \\ \pi \geq \pi_k \forall k}} \pi.$$

In each case, the maximum is the smallest partition that is $\geq$ each of the π_k's. This shows that each of the partially ordered sets $\mathcal{P}(n)$, $\mathcal{NC}(n)$, and $\mathcal{I}(n)$ is a *lattice*, that is, a partially ordered set with greatest lower bounds and least upper bounds.

Further Terminology

Definition 5.2.7. Let V and W be blocks in a non-crossing partition π. We say that $V \succ W$ if there exist $j, k \in W$ with $V \subseteq \{j+1, \dots, k-1\}$. In terms of the picture, this means that V is inside of W. Note that $\prec$ is a strict partial order on the blocks of π.

Example. In the partition in Figure 5.2, we have $V_3 \succ V_2 \succ V_1$ and $V_4 \succ V_1$, but V_4 is incomparable with V_2 and V_3.

Definition 5.2.8. Let $\pi \in \mathcal{NC}(n)$ and let V be a block of π. Then we denote by $\pi \setminus V$ the partition of $[n - |V|]$ given by deleting V from π and reindexing the terms $[n] \setminus V$ in order. For example, see Figure 5.2.

5.3 Partitions as Composition Diagrams

A non-crossing partition can be interpreted as a diagram for composition of certain multilinear forms.

Definition 5.3.1. Let $\mathcal{B}$ and $\mathcal{C}$ be $\mathcal{A}$-$\mathcal{A}$-bimodules. A multlinear form $\Lambda : \mathcal{B}^k \to \mathcal{C}$ will be called an $\mathcal{A}$-*quasi-multlinear* if we have

$$\Lambda[ab_1, b_2, \dots, b_k] = a\Lambda[b_1, \dots, b_k]$$
$$\Lambda[b_1, \dots, b_{k-1}, b_k a] = \Lambda[b_1, \dots, b_{k-1}, b_k]a$$
$$\Lambda[b_1, \dots, b_j a, b_{j+1}, \dots, b_k] = \Lambda[b_1, \dots, b_j, ab_{j+1}, \dots, b_k].$$

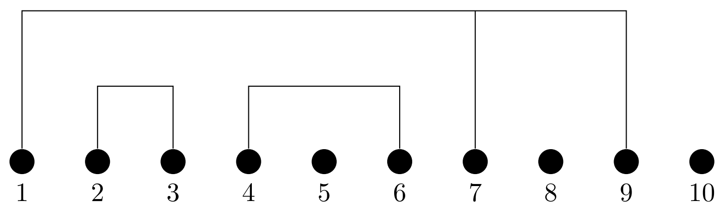

Figure 54: For the partition $\pi = \{\{1,7,9\},\{2,3\},\{4,6\},\{5\},\{8\},\{10\}\}$, we have $\Lambda_\pi[b_1,\dots,b_{10}] = \Lambda_3[b_1,\Lambda_2[b_2,b_3]\Lambda_2[b_4,\Lambda_1[b_5]b_6]b_7,\Lambda_1[b_8]b_9]\Lambda_1[b_{10}]$.

We remark that

- Λ is $\mathcal{A}$-quasi-multilinear if and only if it induces an $\mathcal{A}$-$\mathcal{A}$-bimodule map
$$\mathcal{B} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{B} \to \mathcal{C},$$
where $\otimes_{\mathcal{A}}$ denotes the algebraic tensor product over $\mathcal{A}$ of a right $\mathcal{A}$-module with a left $\mathcal{A}$-module.
- If $\mathcal{V}$ and $\mathcal{W}$ are vector spaces and $\Lambda : \mathcal{V}^k \to \mathcal{W}$ is a multilinear form, then the matrix ampflication $\Lambda^{(n,\dots,n)} : M_n(\mathcal{V})$ is $M_n(\mathbb{C})$-quasi-multilinear.
- If $(\mathcal{B}, E)$ is an $\mathcal{A}$-valued probability space, then $\Lambda[b_1,\dots,b_k] = E[b_1 \dots b_k]$ is $\mathcal{A}$-quasi-multilinear.

Definition 5.3.2. Let $\mathcal{B}$ be an $\mathcal{A}$-algebra. Let $\Lambda_n : \mathcal{B}^n \to \mathcal{A}$ be a sequence of $\mathcal{A}$-quasi-multilinear forms. For $\pi \in \mathcal{NC}(n)$, we define Λ_π be the following recursive relation. If $V = \{j+1,\dots,k\}$ is a block of π with $k < n$, then
$$\Lambda_\pi[b_1,\dots,b_n] = \Lambda_{\pi \setminus V}[b_1,\dots,b_j,\Lambda_{k-j}[b_{j+1},\dots,b_k]b_{k+1},\dots,b_n].$$
and if $k = n$, we have
$$\Lambda_\pi[b_1,\dots,b_n] = \Lambda_{\pi \setminus V}[b_1,\dots,b_j]\Lambda_{n-j}[b_{j+1},\dots,b_n]$$
For example, see Figure 5.3.

To show that this is well-defined, first note that every partition must have some interval block because a maximal block with respect to $\prec$ must be an interval. Moreover, by the associativity properties of composition and the fact that Λ_n is $\mathcal{A}$-quasi-multilinear, the resulting multilinear form Λ_π is independent of the sequence of recursive steps taken to evaluate it. Moreover, it is straighforward to check that Λ_π is $\mathcal{A}$-quasi-multilinear.

Remark 5.3.3. The $\mathcal{A}$-quasi-multilinear forms Λ_π respect disjoint unions of partitions in the following sense. Suppose that $\pi = \pi_1 \sqcup \pi_2$ where π_1 is a non-crossing partition of $\{1,\dots,k\}$ and π_2 is a non-crossing partition of $\{k+1,\dots,n\}$. Then
$$K_\pi[b_1,\dots,b_n] = K_{\pi_1}[b_1,\dots,b_k]K_{\pi_2}[b_{k+1},\dots,b_n].$$

Now given a sequence of $\mathcal{A}$-quasi-multilinear forms Λ_n and coefficients $\alpha_\pi \in \mathbb{C}$ for each non-crossing partition π, we can define a new sequence of $\mathcal{A}$-quasi-multilinear forms Γ_n by
$$\Gamma_n = \sum_{\pi \in \mathcal{NC}(n)} \alpha_\pi \Lambda_\pi. \tag{5.3.1}$$

In the moment-cumulant formulas to come, we will take $\Gamma_n[b_1, \dots, b_n] = E[b_1 \dots b_n]$ and the coefficients α_π will depend on the type of independence. In order to determine the cumulants from the moments $E[b_1 \dots b_n]$, we will need to apply Möbius inversion to the formula (5.3.1).

Lemma 5.3.4. *Let $\mathcal{B}$ be an $\mathcal{A}$-$\mathcal{A}$-module. Let $\Gamma_n : \mathcal{B}^n \to \mathcal{A}$ be a $\mathcal{A}$-quasi-multilinear form. For each non-crossing partition π, let $\alpha_\pi \in \mathbb{C}$, and assume that $\alpha_\pi \neq 0$ when π consists of a single block. Then there exist unique $\mathcal{A}$-quasi-multilinear forms $\Lambda_n : \mathcal{B}^n \to \mathcal{A}$ such that* (5.3.1) *holds.*

Proof. By separating out the partition $\{[n]\}$ with only one block on the right hand side of (5.3.1), we arrive at the formula

$$\Gamma_n = \frac{1}{\alpha_{\{[n]\}}} \left(\Lambda_n - \sum_{\substack{\pi \in \mathcal{NC}(n) \\ |\pi| > 1}} c_\pi \Gamma_\pi \right).$$

Here Γ_π can be computed from $\{\Gamma_k : k < n\}$, and thus we can solve for Γ_n inductively. One checks inductively that Γ_n is $\mathcal{A}$-quasi-multilinear. □

5.4 Cumulants and Independence

The Free Case

The free cumulants were defined by Voiculescu in [Voi85], [Voi86]. Speicher defined joint cumulants and discovered the relationship with non-crossing partitions [Spe94]. He also defined the operator-valued free cumulants in [Spe98]. We follow essentially the arguments of [Spe98, §3] which can also be found in [AGZ09, §5.3.2] in the scalar-valued case.

Definition 5.4.1. Let $(\mathcal{B}, E)$ be an $\mathcal{A}$-valued probability space. The *free cumulants* are the $\mathcal{A}$-quasi-multilinear forms $K_n : \mathcal{B}^n \to \mathcal{A}$ given by

$$E[b_1 \dots b_n] = \sum_{\pi \in \mathcal{NC}(n)} K_\pi[b_1, \dots, b_n]. \tag{5.4.1}$$

This is well-defined by Lemma 5.3.4.

The next result show that free independence is characterized by vanishing of mixed cumulants.

Theorem 5.4.2. *Let $\mathcal{B}_1, \dots, \mathcal{B}_N$ be unital $\mathcal{A}$-subalgebras of the $\mathcal{A}$-valued probability space $(\mathcal{B}, E)$ and let K_n be the nth free cumulant function for $\mathcal{B}$. The following are equivalent:*

1. *$\mathcal{B}_1, \dots, \mathcal{B}_N$ are freely independent.*
2. *If $b_1, \dots, b_n \in \mathcal{B}$ with $b_j \in \mathcal{B}_{i_j}$ and not all the i_j's are the same, then $K_n[b_1, \dots, b_n] = 0$.*

Proof of (2) $\implies$ (1). First, suppose that (2) holds. To demonstrate free independence, suppose that $b_1, \dots, b_n \in \mathcal{B}$ with $b_j \in \mathcal{B}_{i_j}$, $i_j \neq i_{j+1}$, and $E[b_j] = 0$. We have

$$E[b_1 \dots b_n] = \sum_{\pi \in \mathcal{NC}(n)} K_\pi[b_1, \dots, b_n].$$

Each partition π must have a maximal block with respect to $\prec$ which must be an interval block $I = \{j+1, \dots, k\}$. If $k > j+1$, then I contains elements from more than one $\mathcal{B}_i$ since $i_j \neq i_{j+1}$.

Hence, $K_{k-j}[b_{j+1},\dots,b_k]=0$ by condition (2). On the other hand, if $k=j+1$, then I is a singleton and $K_1[b_{j+1}]=E[b_{j+1}]=0$. Therefore, all the terms in the sum vanish, and hence $\mathcal{B}_1, \dots, \mathcal{B}_N$ are freely independent. □

For the other direction, we use the following lemma which describes how to evaluate the cumulants of products.

Lemma 5.4.3. *Let $m_1, \dots, m_n \geq 1$, and let $M_k=\sum_{j=1}^k m_j$. Let $w:[M_n]\to[n]$ be the function that maps $\{M_{j-1}+1,\dots,M_j\}$ onto j. For $\pi\in\mathcal{NC}(n)$, let $w^*\pi=\{w^{-1}(V):V\in\pi\}$. Let $\tau\in\mathcal{NC}(M_n)$ be the partition with blocks $\{M_{j-1}+1,\dots,M_j\}$. Let $\vee$ denote the maximum operation $\vee_{\mathcal{NC}}$. Then for $\pi\in\mathcal{NC}(n)$, we have*

$$K_\pi[(b_1\dots b_{M_1}),(b_{M_1+1}\dots b_{M_2}),\dots,(b_{M_{n-1}+1}\dots b_{M_n})]=\sum_{\substack{\sigma\in\mathcal{NC}(M_n)\\ \sigma\vee\tau=w^*\pi}} K_\sigma[b_1,\dots,b_{M_n}]. \tag{5.4.2}$$

Proof.

Step 1: Given $N\geq 1$, we will show that if the claim holds for the partitions $\{[n]\}$ for $n\leq N$, then it holds for every partition π with blocks of size $\leq N$. We proceed by induction on $|\pi|$. If $|\pi|=1$, then $\pi=\{[n]\}$, so there is nothing to prove. Otherwise, there exists an interval block $V=\{j+1,\dots,k\}$ of π. By the inductive hypothesis, the claim holds for the partition $\{[k-j]\}$ and it holds for $\pi\setminus V$. Therefore, using the recursive definition of the cumulants and splitting up all the partitions named into subpartitions of $w^{-1}(V)$ and $[M_n]\setminus w^{-1}(V)$, we have

$$\begin{aligned}&K_\pi[(b_1\dots b_{M_1}),(b_{M_1+1}\dots b_{M_2}),\dots,(b_{M_{n-1}+1}\dots b_{M_n})]\\ &\qquad=\sum_{\substack{\sigma_1\in\mathcal{NC}(M_n-(M_k-M_j))\\ \sigma_1\vee(\tau\setminus w^{-1}(V))=w^*\pi\setminus w^{-1}(V)}}\ \sum_{\substack{\sigma_2\in\mathcal{NC}(M_k-M_j)\\ \sigma_2\vee\tau|_V=[M_k-M_j]}} K_{\phi(\sigma_1,\sigma_2)}[b_1,\dots,b_{M_n}],\end{aligned}$$

where $\phi(\sigma_1,\sigma_2)$ is the partition of $[M_n]$ given by

$$\phi(\sigma_1,\sigma_2)|_{w^{-1}(V)}=\sigma_2 \qquad \phi(\sigma_1,\sigma_2)|_{[M_n]\setminus w^{-1}(V)}=\sigma_1.$$

Since $w^{-1}(V)$ is an interval, it is clear that $\phi(\sigma_1,\sigma_2)$ is non-crossing. Also, the two conditions $\sigma_1\vee(\tau\setminus w^{-1}(V))=w^*\pi\setminus w^{-1}(V)$ and $\sigma_2\vee\tau|_V=[M_k-M_j]$ are equivalent to

$$\phi(\sigma_1,\sigma_2)\vee\tau=w^*\pi.$$

Moreover, every partition σ with $\sigma\vee\tau=w^*\pi$ must restrict to subpartitions of $w^{-1}(V)$ and its complement since it is $\sigma\leq w^*\pi$. Thus, σ can be expressed as $\phi(\sigma_1,\sigma_2)$ where σ_1 and σ_2. Therefore, the claim holds for the partition π.

Step 2: It remains to show that the claim holds for the partitions $\{[n]\}$, which we will prove by induction on n. The base case $n=1$ follows immediately from the moment cumulant formula because in this case $\tau=\{[M_1]\}$. For the inductive step, choose $n>1$ and note that

$$\begin{aligned}E[b_1\dots b_{M_n}]&=\sum_{\sigma\in\mathcal{NC}([M_n])}K_\sigma[b_1,\dots,b_{M_n}]\\ &=\sum_{\pi\in\mathcal{NC}(n)}\ \sum_{\substack{\sigma\in\mathcal{NC}(M_n)\\ \sigma\vee\tau=w^*\pi}}K_\sigma[b_1,\dots,b_{M_n}].\end{aligned}$$

On the other hand,

$$E[(b_1 \dots b_{M_1})(b_{M_1+1} \dots b_{M_2}) \dots (b_{M_{n-1}+1} \dots b_{M_n})] = \sum_{\pi \in \mathcal{NC}(n)} K_\pi[(b_1 \dots b_{M_1}), (b_{M_1+1} \dots b_{M_2}), \dots, (b_{M_{n-1}+1} \dots b_{M_n})].$$

For every partition π other than $\{[n]\}$, we know that (5.4.2) holds by the inductive hypothesis and Step 1. But summing up the left hand side and the right hand side of (5.4.2) over all $\pi \in \mathcal{NC}(n)$ yields $E[b_1 \dots b_{M_n}]$. Thus, the terms for the partition $\{[n]\}$ must also be equal, so that (5.4.2) holds for $\{[n]\}$ also. $\square$

Lemma 5.4.4. *Let $(\mathcal{B}, E)$ be an $\mathcal{A}$-valued probability space. If $n \geq 1$, $a \in \mathcal{A}$ and $b_1, \dots, b_n \in \mathcal{B}$, then*

$$K_{n+1}[b_1, \dots, b_j, a, b_{j+1}, \dots, b_n] = 0.$$

Proof. We proceed by induction on n, but we include the argument for base case $n = 1$ in the induction proof. Consider the case where $j < n$. Then we may apply the product formula to evaluate $K_n(b_1, \dots, b_j, ab_{j+1}, \dots b_n)$. In this case, $\pi = \{[n]\}$, $w^*\pi = \{[n+1]\}$ and τ is the partition where every block is a singleton except $\{j+1, j+2\}$. Other than $\{[n+1]\}$, every partition σ with $\sigma \vee \tau = w^*\pi$ must have two blocks $V_1 \ni j+1$ and $V_2 \ni j+2$. If $|V_1| > 1$, then K_π is evaluated by applying $K_{|V_1|}$ to the indices in V_1, which include the index $j+1$ of the term a, and by induction hypothesis, this evaluates to zero. The only remaining partition is when $|V_1| = 1$ which means that $V_1 = j+1$. Thus,

$$K_n(b_1, \dots, b_j, ab_{j+1}, \dots b_n) = K_{n+1}[b_1, \dots, b_j, a, b_{j+1}, \dots, b_n] + K_n[b_1, \dots, b_j, K_1[a]b_{j+1}, \dots, b_n].$$

Since $K_1(a) = a$, the second term on the right is equal to the left hand side, so that the K_{n+1} term vanishes. In the case $j = n$, we apply the same reasoning to $K_n(b_1, \dots, b_{n-1}, b_n a)$. $\square$

Proof of Theorem 5.4.2 (1) $\implies$ (2).

Step 1: First, we show that given $b_1, \dots, b_n$ with $n \geq 2$, $b_j \in \mathcal{B}_{i_j}$ and $i_j \neq i_{j+1}$, we have $K_n(b_1, \dots, b_n) = 0$. We proceed by induction on n, the base case being included in the same proof as the induction step. In light of Lemma 5.4.4 and multilinearity, the value of K_n is unchanged if we replace b_j by $b_j - E[b_j]$, so we may assume without loss of generality that $E[b_j] = 0$. Then it follows from free independence that

$$0 = E[b_1 \dots b_n] = \sum_{\pi \in \mathcal{NC}(n)} K_\pi[b_1, \dots, b_n].$$

Now if $\pi \in \mathcal{NC}(n)$ has a singleton block $\{j\}$, then the term $K_\pi[b_1, \dots, b_n]$ vanishes because $K_1[b_j] = E[b_j] = 0$. If π has no singleton blocks and π is not the partition $\{[n]\}$, then π must have some interval block $V = \{j+1, \dots, k\}$ with $1 < k - j < n$. By the induction hypothesis, $K_{k-j}[b_{j+1}, \dots, b_k] = 0$ and therefore $K_\pi[b_1, \dots, b_n] = 0$. The only remaining term is $K_n[b_1, \dots, b_n]$ and since the terms add up to zero, this last term must be zero as well.

Step 2: Now we prove the general case of (2), again by induction. Let $n \geq 2$, and suppose $b_1, \dots, b_n$ with $b_j \in \mathcal{B}_{i_j}$ and not all the i_j's equal. If $i_j \neq i_{j+1}$ for each j, we are done by Step 1. So suppose that $i_j = i_{j+1}$ for some j. Applying the product formula, we have

$$K_{n-1}[b_1, \dots, b_{j-1}, b_j b_{j+1}, b_{j+2}, \dots, b_n] = \sum_{\substack{\sigma \in \mathcal{NC}(n+1) \\ \sigma \vee \tau = \{[n]\}}} K_\pi[b_1, \dots, b_n],$$

where τ is the partition in which every block is a singleton except $\{j, j+1\}$. The left hand side vanishes by induction hypothesis. On the right hand side, every partition other than $\{[n]\}$ has two blocks $V_1 \ni j$ and $V_2 \ni j+1$. Because not all the i_j's are equal, either V_1 or V_2 contains indices j with more than one value of i_j. Thus, $K_\pi[b_1, \dots, b_n]$ vanishes by induction hypothesis. So all the terms vanish other than $K_n[b_1, \dots, b_n]$, hence $K_n[b_1, \dots, b_n] = 0$ also. □

Theorem 5.4.2 has as a corollary the following rule for evaluating joint moments of freely independent random variables.

Lemma 5.4.5. *Let $\mathcal{B}_1, \dots, \mathcal{B}_N$ be freely independent subalgebras of $(\mathcal{B}, E)$ and let K_n be the nth free cumulant. Let $b_1, \dots, b_n \in \mathcal{B}$ with $b_j \in \mathcal{B}_{i_j}$. Let σ be the partition with blocks $V_i = \{j : b_j \in \mathcal{B}_i\}$. Then*

$$E[b_1 \dots b_n] = \sum_{\substack{\pi \in \mathcal{NC}(n) \\ \pi \leq \sigma}} K_\pi[b_1, \dots, b_n],$$

where the right hand side is expressed purely in terms of the free cumulants for the $\mathcal{B}_i$'s.

Proof. We express $E[b_1, \dots, b_n]$ as the sum over all partitions $\pi \in \mathcal{NC}(n)$. Then by Lemma 5.4.2, we $K_\pi[b_1, \dots, b_n] = 0$ unless each block of π only contains b_j's from the same algebra $\mathcal{B}_i$. But this is equivalent to $\pi \leq \sigma$. □

Lemma 5.4.6. *Let $(\mathcal{B}_0, E_0)$ be an $\mathcal{A}$-valued probability space. Let $(\mathcal{B}, E)$ be the free product of N copies of $(\mathcal{B}_0, E_0)$ with inclusions $\rho_j : \mathcal{B}_0 \to \mathcal{B}$ for $j = 1, \dots, N$. Then we have*

$$K_n\left[\sum_{j=1}^N \rho_j(b_1), \dots, \sum_{j=1}^N \rho_j(b_n)\right] = N K_n(b_1, \dots, b_n),$$

where the left hand side is the free cumulant of $(\mathcal{B}, E)$ and the right hand side is the free cumulant of $(\mathcal{B}_0, E_0)$.

Proof. This follows by expanding the left hand side by multilinearity and then applying Theorem 5.4.2. We also use the fact that since $E_0 = E \circ \rho_j$, we also have $K_n[\rho_j(b_1), \dots, \rho_j(b_n)] = K_n[b_1, \dots, b_n]$. □

Remark 5.4.7. Another consequence is that to check free independence of algebras $\mathcal{B}_1, \dots, \mathcal{B}_N$, it suffices to check the mixed cumulants vanish for a generating set of each algebra. More explicitly, suppose that $\mathcal{B}_i$ is contained in the closed $\mathcal{A}$-algebra generated by an $\mathcal{A}$-$\mathcal{A}$-bimodule $S_i \subseteq \mathcal{B}$ with $S_i = S_i^*$. Suppose also that $K_n[s_1, \dots, s_n] = 0$ whenever $s_j \in S_{i_j}$ and not all the i_j's are equal. Then for every string $s_1, \dots, s_n$ with $s_j \in S_{i_j}$, we have by the same argument as Lemma 5.4.5 that

$$E[s_1 \dots s_n] = \sum_{\pi \leq \sigma} K_\pi[s_1, \dots, s_n],$$

where σ is the partition with blocks $V_i = \{j : i_j = i\}$. This agrees with the expectation of the same string in the free product of the algebras $\mathcal{A}\langle S_j \rangle$ generated by S_j. Moreover, the span of such strings is dense in the $\mathcal{A}$-algebra generated by $S_1, \dots, S_N$. Therefore, the algebras generated by $S_1, \dots, S_N$ have the same expectation as if they were freely independent, which means that they *are* freely independent.

The Boolean Case

The Boolean cumulants were developed by [SW97] in the scalar case and [Pop09] in the operator-valued case; see also [PV13, §2].

Definition 5.4.8. A partition π of $[n]$ is called an *interval partition* if every block is an interval $\{j+1,\dots,k\}$. We denote the set of interval partitions by $\mathcal{I}(n) \subseteq \mathcal{NC}(n)$.

Definition 5.4.9. Let $(\mathcal{B}, E)$ be an $\mathcal{A}$-valued probability space. The *Boolean cumulants* are the $\mathcal{A}$-quasi-multilinear forms $K_n : \mathcal{B}^n \to \mathcal{A}$ given by

$$E[b_1 \dots b_n] = \sum_{\pi \in \mathcal{I}(n)} K_\pi[b_1, \dots, b_n]. \tag{5.4.3}$$

This is well-defined by Lemma 5.3.4 using $\alpha_\pi = 1_{\pi \in \mathcal{I}(n)}$ for $\pi \in \mathcal{NC}(n)$.

Theorem 5.4.10. *Let $\mathcal{B}_1$, ..., $\mathcal{B}_N$ be non-unital $\mathcal{A}$-subalgebras of the $\mathcal{A}$-valued probability space $(\mathcal{B}, E)$, and let K_n be the nth Boolean cumulant of $\mathcal{B}$. The following are equivalent:*

1. *$\mathcal{B}_1$,..., $\mathcal{B}_N$ are freely independent.*
2. *If b_1, ..., $b_n \in \mathcal{B}$ with $b_j \in \mathcal{B}_{i_j}$ and not all the i_j's are the same, then $K_n[b_1, \dots, b_n] = 0$.*

Proof of (2) $\implies$ (1). First, suppose that (2) holds. To demonstrate Boolean independence, suppose that b_1, ..., $b_n \in \mathcal{B}$ with $b_j \in \mathcal{B}_{i_j}$ and $i_j \neq i_{j+1}$. Then we have

$$E[b_1 \dots b_n] = \sum_{\pi \in \mathcal{I}(n)} K_\pi[b_1, \dots, b_n].$$

If π is an interval partition that has an interval I with $|I| > 1$, then I contains elements from more than one algebra $\mathcal{B}_i$ and hence the cumulant corresponding to that block vanishes. Therefore, the only partition that contributes to the sum is the partition of singletons, which implies that

$$E[b_1 \dots b_n] = K_1[b_1] \dots K_1[b_n] = E[b_1] \dots E[b_n].$$

□

For the other direction, as in the free case, we use the following product formula for cumulants. The proof is word-for-word the same as in the free case, except that non-crossing partitions are replaced by interval partitions.

Lemma 5.4.11. *Let m_1, ..., $m_n \geq 1$, and let $M_k = \sum_{j=1}^k m_j$. Let $w : [M_n] \to [n]$ be the function that maps $\{M_{j-1}+1, \dots, M_j\}$ onto j. For $\pi \in \mathcal{I}(n)$, let $w^*\pi = \{w^{-1}(V) : V \in \pi\}$. Let $\tau \in \mathcal{I}(M_n)$ be the partition with blocks $\{M_{j-1}+1, \dots, M_j\}$. Let $\vee$ denote the maximum operation $\vee_{\mathcal{I}}$. Then for $\pi \in \mathcal{I}(n)$, we have*

$$K_\pi[(b_1 \dots b_{M_1}), (b_{M_1+1} \dots b_{M_2}), \dots, (b_{M_{n-1}+1} \dots b_{M_n})] = \sum_{\substack{\sigma \in \mathcal{I}(M_n) \\ \sigma \vee \tau = w^*\pi}} K_\sigma[b_1, \dots, b_{M_n}]. \tag{5.4.4}$$

Proof of Theorem 5.4.10 (1) $\implies$ (2).

Step 1: First, we show that $K_n[b_1, \dots, b_n] = 0$ when $n \geq 2$, $b_j \in \mathcal{B}_{i_j}$, and $i_j \neq i_{j+1}$. We proceed by induction on n. By Boolean independence, we have

$$E[b_1] \dots E[b_n] = E[b_1 \dots b_n] = \sum_{\pi \in \mathcal{I}(n)} K_\pi[b_1, \dots, b_n].$$

Now if π is not equal to the singleton partition or the partition $\{[n]\}$, then π has some interval block $\{j+1,\dots,k\}$ with $k-j \geq 2$. By induction hypothesis $K_{k-j}[b_{j+1},\dots,b_k] = 0$, so that $K_\pi[b_1,\dots,b_n] = 0$. The singleton partition yields the term $K_1[b_1]\dots K_1[b_n] = E[b_1]\dots E[b_n]$ which is equal to the right hand side. The only term which is unaccounted for is the term $K_n[b_1,\dots,b_n]$ on the right hand side, so this must equal zero.

Step 2: Now we prove the general case of (2) by induction on n. Let $n \geq 2$ and let $b_j \in \mathcal{B}_{i_j}$ with not all the i_j's equal. If $i_j \neq i_{j+1}$, we are done by Step 1. So assume that $i_j = i_{j+1}$. Let τ be the partition of $[n]$ with all singleton blocks except for $\{j, j+1\}$ and note that the only two interval partitions σ with $\sigma \vee \tau = \{[n]\}$ are $\{[n]\}$ and $\{\{1,\dots,j\},\{j+1,\dots,n\}$. So by the product formula

$$K_{n-1}[b_1,\dots,b_{j-1},b_j b_{j+1},b_{j+2},\dots,b_n] = K_n[b_1,\dots,b_n] + K_j[b_1,\dots,b_j]K_{n-j}[b_{j+1},\dots,b_n].$$

By induction hypothesis, the left hand side and the second term on the right hand side vanish, and therefore, $K_n[b_1,\dots,b_n] = 0$. ☐

Lemma 5.4.12. *Let $\mathcal{B}_1$, ..., $\mathcal{B}_N$ be Boolean independent $\mathcal{A}$-subalgebras of $(\mathcal{B}, E)$, and let K_n be the nth Boolean cumulant. Let b_1, ..., $b_n \in \mathcal{B}$ with $b_j \in \mathcal{B}_{i_j}$. Let σ be the partition with blocks $V_i = \{j : b_j \in \mathcal{B}_i\}$. Then*

$$E[b_1 \dots b_n] = \sum_{\substack{\pi \in \mathcal{I}(n) \\ \pi \leq \sigma}} K_\pi[b_1,\dots,b_n],$$

where the right hand side is expressed purely in terms of the Boolean cumulants for the $\mathcal{B}_i$'s.

Proof. We express $E[b_1,\dots,b_n]$ as the sum over all partitions $\pi \in \mathcal{NC}(n)$. Then by Lemma 5.4.2, we $K_\pi[b_1,\dots,b_n] = 0$ unless each block of π only contains b_j's from the same algebra $\mathcal{B}_i$. But this is equivalent to $\pi \leq \sigma$. ☐

Lemma 5.4.13. *Let $(\mathcal{B}_0, E_0)$ be an $\mathcal{A}$-valued probability space. Let $(\mathcal{B}, E)$ be the Boolean product of N copies of $(\mathcal{B}_0, E_0)$ with non-unital inclusions $\rho_j : \mathcal{B}_0 \to \mathcal{B}$ for $j = 1,\dots,N$. Then we have*

$$K_n\left[\sum_{j=1}^N \rho_j(b_1),\dots,\sum_{j=1}^N \rho_j(b_n)\right] = N K_n(b_1,\dots,b_n),$$

where the left hand side is the Boolean cumulant of $(\mathcal{B}, E)$ and the right hand side is the Boolean cumulant of $(\mathcal{B}_0, E_0)$.

Proof. This follows by expanding the left hand side by multilinearity and then applying Theorem 5.4.10. We also use the fact that since $E_0 = E \circ \rho_j$, we also have $K_n[\rho_j(b_1),\dots,\rho_j(b_n)] = K_n[b_1,\dots,b_n]$. ☐

Remark 5.4.14. Similar to the free case, in order to check Boolean independence of algebras $\mathcal{B}_1$, ..., $\mathcal{B}_N$, it suffices to check the mixed cumulants vanish for a generating set of each algebra. More explicitly, suppose that $\mathcal{B}_i$ is contained in the closed $\mathcal{A}$-algebra generated by an $\mathcal{A}$-$\mathcal{A}$-bimodule $S_i \subseteq \mathcal{B}$ with $S_i = S_i^*$. Suppose also that $K_n[s_1,\dots,s_n] = 0$ whenever $s_j \in S_{i_j}$ and not all the i_j's are equal. Then $\mathcal{B}_1$, ..., $\mathcal{B}_N$ are freely independent.

The Monotone Case

The monotone cumulants were developed in the scalar case by [HS11b] [HS11a] and in the operator-valued case by [HS14].

Definition 5.4.15. Let $\pi \in \mathcal{NC}(n)$. Let $\mathbb{R}^\pi$ be the product of copies of $\mathbb{R}$ indexed by the blocks of the partition (or equivalently labelings of the blocks by real numbers). We say that $t \in \mathbb{R}^\pi$ is *compatible with* π or $t \models \pi$ if we have $V \prec W$ implies $t_V < t_W$ for $V, W \in \pi$.

Definition 5.4.16. Let $(\mathcal{B}, E)$ be an $\mathcal{A}$-valued probability space. The *monotone cumulants* are the $\mathcal{A}$-quasi-multilinear forms $K_n : \mathcal{B}^n \to \mathcal{A}$ given by

$$E[b_1 \dots b_n] = \sum_{\pi \in \mathcal{NC}(n)} \gamma_\pi K_\pi[b_1, \dots, b_n], \tag{5.4.5}$$

where

$$\gamma_\pi = |\{t \in [0,1]^\pi : t \models \pi\}|.$$

This is well-defined by Lemma 5.3.4 using $\alpha_\pi = \gamma_\pi$.

In the monotone (and anti-monotone) cases, we cannot have an analogue of Theorems 5.4.2 and 5.4.10 because monotone independence is not invariant under reordering the algebras. However, the analogues of Lemmas 5.4.5, 5.4.12 and Lemmas 5.4.6, 5.4.13 do hold.

Lemma 5.4.17. *Let $\mathcal{B}_1, \dots, \mathcal{B}_N$ be monotone independent subalgebras of $(\mathcal{B}, E)$ and let K_n be the nth monotone cumulant. Let $b_1, \dots, b_n \in \mathcal{B}$ with $b_j \in \mathcal{B}_{i_j}$. Let σ be the partition with blocks $V_i = \{j : b_j \in \mathcal{B}_i\}$. For each $\pi \leq \sigma$, denote*

$$\pi|_{\mathcal{V}_i} = \{W \in \pi : W \subseteq V_i\}.$$

Then we have

$$E[b_1 \dots b_n] = \sum_{\substack{\pi \in \mathcal{NC}(n) \\ \pi \leq \sigma}} \left|\{t \in [0,1)^{\pi|_{V_1}} \times \dots \times [N-1, N)^{\pi|_{V_N}} : t \models \pi\}\right| K_\pi[b_1, \dots, b_n].$$

Proof. We define a new partition $\tau \leq \sigma$ as follows. Let τ_j be the partition of V_j defined by taking the common refinement of the partitions $\{V_j \cap [1, k-1], V_j \cap [k+1, n]\}$ for each $k \in \bigcup_{i<j} V_i$. Then let τ be the partition with blocks $\bigcup_{j=1}^n \tau_j$. In other words, τ is chosen to be the maximal refinement of σ such that if two elements r and s are in the same block within V_j, then there are no elements of V_i, $i < j$, between r and s.

For example, suppose that $V_1 = \{1,3,8\}$, $V_2 = \{2,4,6\}$, and $V_3 = \{5,7,9\}$. Then V_3 is subdivided into blocks $\{5\}$ and $\{7\}$ and $\{9\}$, and V_2 is subdivided into blocks $\{2\}$ and $\{4,6\}$, and V_1 remains one block.

We can use monotone independence to evaluate $E[b_1 \dots b_n]$ in terms of $E|_{\mathcal{B}_i}$ in a way which mimics the construction of τ. Indeed, if $\{j+1, \dots, k\}$ is a block of τ_N, that means that $b_{j+1}, \dots, b_k$ are in $\mathcal{B}_N$, while b_j and b_{k+1} are not (when b_j and b_{k+1} exist). Therefore, by monotone independence,

$$E[b_1 \dots b_n] = E[b_1 \dots b_j E[b_{j+1} \dots b_k] b_{k+1} \dots b_n].$$

We can apply the same reasoning to each block of τ_N. Then we are left with a string of length $n - |V_N|$ after we group each term $E[b_{j+1} \dots b_k] \in \mathcal{A}$ for $\{j+1, \dots, k\} \in \tau_N$ together with the following element b_{k+1}. Next, by monotone independence, we may apply E to each block of τ_{N-1}, and so forth.

Altogether, $E[b_1 \dots b_n]$ is equivalent to the expression we get by applying the expectation to each block of τ_j (together with the intervening terms in $\bigcup_{i>j} V_j$). For example, in the case where $V_1 = \{1,3,8\}$, $V_2 = \{2,4,6\}$, and $V_3 = \{5,7,9\}$, we have

$$E[b_1 \dots b_9] = E[b_1 E[b_2] b_3 E[b_4 E[b_5] b_6 E[b_7]] b_8 E[b_9]].$$

Next, we apply the moment-cumulant formula (5.4.5) within each block of τ_j. We obtain

$$\sum_{\substack{\pi \leq \tau \\ \pi|_{W_j} \in \mathcal{NC}(W_j)}} \gamma_{\pi|_{W_1}} \dots \gamma_{\pi|_{W_m}} K_\pi(b_1, \dots, b_n).$$

where $W_1, \dots, W_m$ are the blocks of τ and $\mathcal{NC}(W_j)$ is the set of non-crossing partitions of the set W_j with the standard ordering of the elements.

We claim that for each term in the sum, $\pi \in \mathcal{NC}(n)$. Indeed, consider two blocks P and Q of π. If P and Q are in the same block of τ, then they are not allowed to cross. If P and Q are not in the same block of τ, but they are in the same V_j, then they also cannot cross because the blocks of τ_j in V_j are intervals in V_j and hence do not cross. Finally, suppose that P is within V_i and Q is within V_j with $i < j$. By construction of τ, the blocks of τ_j do not cross V_i, and therefore a fortiori P and Q cannot cross. Therefore, we can replace the condition "$\pi \leq \tau; \pi|_{W_j} \in \mathcal{NC}(W_j)$" in the sum by "$\pi \in \mathcal{NC}(n); \pi \leq \tau$."

Next, we rewrite the term $\gamma_{\pi|_{W_1}} \dots \gamma_{\pi|_{W_m}}$. Suppose that $V_j = \bigcup_{i \in I_j} W_i$. Then because the W_i's form an interval partition of V_j, the blocks in different W_j's are incomparable with respect to $\succ$. Therefore, $t \in [0,1]^{\pi|_{V_j}}$ is compatible with $\pi|_{V_j}$ if and only if $t|_{W_i}$ is compatible with $\pi|_{W_i}$ for each $i \in I_j$. This means that

$$\prod_{i \in I_j} \gamma_{\pi|_{W_i}} = \gamma_{\pi|_{V_j}}$$

and hence

$$\gamma_{\pi|_{W_1}} \dots \gamma_{\pi|_{W_m}} = \gamma_{\pi|_{V_1}} \dots \gamma_{\pi|_{V_N}}.$$

We write this as

$$\prod_{j=1}^N \left|\{t \in [0,1]^{\pi|_{V_j}} : t \models \pi|_{V_j}\}\right| = \prod_{j=1}^N \left|\{t \in [j-1,j)^{\pi|_{V_j}} : t \models \pi|_{V_j}\}\right|,$$

where the equality follows from translation-invariance of Lebesgue measure. Now note that if $P \subseteq V_i$ and $Q \subseteq V_j$ with $i < j$, then there cannot be any elements of P between any two elements of Q and therefore $P \not\succ Q$. This implies that $t \in [0,N)^\pi$ satisfies $t|_{V_j} \in [j-1,j)^{\pi|_{V_j}}$ and $t|_{V_j}$ is compatible with $\pi|_{V_j}$ for each j, then t is compatible with π. Thus,

$$\prod_{j=1}^N \left|\{t \in [j-1,j)^{\pi|_{V_j}} : t \models \pi|_{V_j}\}\right| = \left|\{t \in [0,1)^{\pi|_{V_1}} \times \dots \times [N-1,N)^{\pi|_{V_N}} : t \models \pi\}\right|.$$

Therefore,

$$E[b_1 \dots b_n] = \sum_{\substack{\pi \in \mathcal{NC}(n) \\ \pi \leq \tau}} \left|\{t \in [0,1)^{\pi|_{V_1}} \times \dots \times [N-1,N)^{\pi|_{V_N}} : t \models \pi\}\right| K_\pi[b_1, \dots, b_n].$$

It only remains to replace $\pi \leq \tau$ by $\pi \leq \sigma$ in the index set for the sum by showing that the terms for $\pi \not\leq \tau$ vanish. Suppose that $\pi \leq \sigma$ but $\pi \not\leq \tau$. Then π must have some blocks $P \subseteq V_j$ and $Q \subseteq V_i$ with $i < j$ such that there exist $r < k < s$ with $r, s \in P$ and $k \in Q$. But then $Q \succ P$. However, if

$$t \in [0,1)^{\pi|_{V_1}} \times \cdots \times [N-1, N)^{\pi|_{V_N}},$$

then $t_Q < t_P$ and hence t cannot be compatible with π. This implies

$$\{t \in [0,1)^{\pi|_{V_1}} \times \cdots \times [N-1, N)^{\pi|_{V_N}} : t \models \pi\} = \varnothing,$$

so that π does not contribute to the sum. □

Lemma 5.4.18. *Let $(\mathcal{B}_0, E_0)$ be an $\mathcal{A}$-valued probability space. Let $(\mathcal{B}, E)$ be the monotone product of N copies of $(\mathcal{B}_0, E_0)$ with non-unital inclusions $\rho_j : \mathcal{B}_0 \to \mathcal{B}$ for $j = 1, \dots, N$. Then we have*

$$K_n \left[\sum_{j=1}^N \rho_j(b_1), \dots, \sum_{j=1}^N \rho_j(b_n) \right] = N K_n[b_1, \dots, b_n],$$

where the left hand side is the monotone cumulant of $(\mathcal{B}, E)$ and the right hand side is the monotone cumulant of $(\mathcal{B}_0, E_0)$.

Proof. We can define two sequences of $\mathcal{A}$-quasi-multilinear forms $\mathcal{B}^n \to \mathcal{A}$ by

$$\Lambda_n[b_1, \dots, b_n] = N K_n[b_1, \dots, b_n]$$

and

$$\Gamma_n[b_1, \dots, b_n] = K_n \left[\sum_{j=1}^N \rho_j(b_1), \dots, \sum_{j=1}^N \rho_j(b_n) \right].$$

(To show that Γ_n is $\mathcal{A}$-quasi-multilinear, we use that fact that ρ_j is an $\mathcal{A}$-$\mathcal{A}$-bimodule map even though $\rho_j|_{\mathcal{A}}$ is not identity.) We want to prove that $\Gamma_n = \Lambda_n$ and to do this, it suffices by Lemma 5.3.4 to show that

$$\sum_{\pi \in \mathcal{NC}(n)} \gamma_\pi \Gamma_\pi[b_1, \dots, b_n] = \sum_{\pi \in \mathcal{NC}(n)} \gamma_\pi \Lambda_\pi[b_1, \dots, b_n].$$

Note that $\Lambda_\pi[b_1, \dots, b_n] = N^{|\pi|} K_\pi[b_1, \dots, b_n]$ by multilinearity of K_n. Meanwhile, on the right side, we simply have a joint moment of the variables $\sum_{j=1}^N \rho_j(b_i)$ in $(\mathcal{B}, E)$. In other words, we must show that

$$E\left[\sum_{j=1}^N \rho_j(b_1), \dots, \sum_{j=1}^N \rho_j(b_n) \right] = \sum_{\pi \in \mathcal{NC}(n)} N^{|\pi|} \gamma_\pi K_n(b_1, \dots, b_n).$$

Let $c_i = \sum_{j=1}^N \rho_j(b_i)$. We will first evaluate $E[c_1 \dots c_n]$ in terms of the joint cumulants of the b_j's. By multilinearity,

$$E[c_1, \dots, c_n] = \sum_{j_1, \dots, j_n \in [N]} E[\rho_{j_1}(b_1), \dots, \rho_{j_n}(b_n)].$$

Given $j_1, \ldots, j_n$, we define sets $V_j(j_1, \ldots, j_n) = \{i : j_i = j\}$ and let $\sigma_{j_1,\ldots,j_n}$ be the corresponding partition. Then we have by Lemma 5.4.17 that

$$E[c_1, \ldots, c_n] = \sum_{j_1,\ldots,j_n \in [N]} \sum_{\substack{\pi \in \mathcal{NC}(n) \\ \pi \leq \sigma_{j_1,\ldots,j_n}}} \left|\{t \in [0,1)^{\pi|_{V_1}} \times \cdots \times [N-1,N)^{\pi|_{V_N}} : t \models \pi\}\right| K_\pi[\rho_{j_1}(b_1), \ldots, \rho_{j_n}(b_n)].$$

In this sum, K_π is computed completely from the joint cumulants K_n applied to tuples of variables within the same algebra $\mathcal{B}_i$. Therefore, the sum is equal to

$$\sum_{j_1,\ldots,j_n \in [N]} \sum_{\substack{\pi \in \mathcal{NC}(n) \\ \pi \leq \sigma_{j_1,\ldots,j_n}}} \left|\{t \in [0,1)^{\pi|_{V_1}} \times \cdots \times [N-1,N)^{\pi|_{V_N}} : t \models \pi\}\right| K_\pi[b_1, \ldots, b_n].$$

We can view this as a sum over pairs π and $(V_1, \ldots, V_N)$ where $\pi \leq \{V_1, \ldots, V_N\}$ because the indices $j_1, \ldots, j_n$ are uniquely determined by the sets $V_1, \ldots, V_N$, and we thus obtain

$$\sum_{\pi \in \mathcal{NC}(n)} \sum_{\substack{(V_1,\ldots,V_N) \\ \{V_1,\ldots,V_N\} \geq \pi}} \left|\{t \in [0,1)^{\pi|_{V_1}} \times \cdots \times [N-1,N)^{\pi|_{V_N}} : t \models \pi\}\right| K_\pi[b_1, \ldots, b_n].$$

Next, note that

$$[0,N)^\pi = \bigsqcup_{(j_W)_{W \in \pi}} \prod_{W \in \pi} [j_W - 1, j_W) = \bigsqcup_{\substack{(V_1,\ldots,V_N) \\ \{V_1,\ldots,V_N\} \geq \pi}} [0,1)^{\pi|_{V_1}} \times \cdots \times [N-1,N)^{\pi|_{V_N}},$$

and therefore

$$\{t \in [0,N)^\pi : t \models \pi\} = \bigsqcup_{\substack{(V_1,\ldots,V_N) \\ \{V_1,\ldots,V_N\} \geq \pi}} \{t \in [0,1)^{\pi|_{V_1}} \times \cdots \times [N-1,N)^{\pi|_{V_N}} : t \models \pi\}.$$

Therefore,

$$\begin{aligned} E[c_1, \ldots, c_n] &= \sum_{\pi \in \mathcal{NC}(n)} \left|\{t \in [0,N)^\pi : t \models \pi\}\right| K_\pi[b_1, \ldots, b_n] \\ &= \sum_{\pi \in \mathcal{NC}(n)} N^{|\pi|} \gamma_\pi K_\pi[b_1, \ldots, b_n] \end{aligned}$$

as desired. □

5.5 Non-Commutative Generating Functions

In order to understand the relationship between cumulants and the analytic transforms discussed in the last chapter, we first establish some basic terminology for formal power series of multilinear forms. For further background, see [Dyk07], [Pop08b].

Definition 5.5.1. A *non-commutative generating function* is a formal power series $F(z) = \sum_{k=0}^\infty \Lambda_k[z, \ldots, z]$ where $\Lambda_k : \mathcal{A}^k \to \mathcal{A}$ is a multilinear form. (Here a multilinear form $\mathcal{A}^0 \to \mathcal{A}$ is interpreted as a map $\mathbb{C} \to \mathcal{A}$ or equivalently a constant in $\mathcal{A}$.)

Definition 5.5.2. If $F = \sum_k \Lambda_k$ and $G = \sum_k \Gamma_k$, then we define the composition

$$F \circ G(z) = \sum_{k=0}^{\infty} \left(\sum_{\substack{m_1 \dots m_\ell \geq 1 \\ m_1 + \dots + m_\ell = k}} \Lambda_\ell[\Gamma_{m_1}[z, \dots, z], \dots \Gamma_{m_\ell}[z, \dots, z]] \right).$$

or more succinctly

$$F \circ G = \sum_{k=0}^{\infty} \left(\sum_{\ell \geq 1} \sum_{\substack{m_1 \dots m_\ell \geq 1 \\ m_1 + \dots + m_\ell = k}} \Lambda_\ell[\Gamma_{m_1}, \dots \Gamma_{m_\ell}] \right).$$

Note that composition is associative.

Lemma 5.5.3. *Let $F = \sum_{k=0}^\infty \Lambda_k$ with $\Lambda_0 = 0$ and Λ_1 invertible as a linear map. Then there exists a non-commutative generating function G such that $F \circ G(z) = z$ and $G \circ F(z) = z$.*

Proof. First, let us show the existence of a unique right inverse for F. If $G = \sum_{k \geq 1} \Gamma_k$, then G being a right inverse for F means that

$$\sum_{\substack{m_1 \dots m_\ell \geq 1 \\ m_1 + \dots + m_\ell = k}} \Lambda_\ell[\Gamma_{m_1}, \dots \Gamma_{m_\ell}] = \begin{cases} \mathrm{id}, & k = 1 \\ 0, & k > 1. \end{cases}$$

From the $k = 1$ case, we see that $\Gamma_1 = \Lambda_1^{-1}$. For $k > 1$, can write

$$\Lambda_k[\Gamma_1, \dots, \Gamma_1] = -\sum_{\ell > 1} \sum_{\substack{m_1 \dots m_\ell \geq 1 \\ m_1 + \dots + m_\ell = k}} \Lambda_\ell[\Gamma_{m_1}, \dots \Gamma_{m_\ell}]$$

by separating out one term from the sum. Because Γ_1 is invertible, this allows us to solve for Λ_k inductively. This proves existence and uniqueness of a right inverse G. There is a similar argument for the existence and uniqueness of a left inverse H. But then we have $G = (H \circ F) \circ G = H \circ (F \circ G) = H$, so the right and left inverse agree. ☐

The following observations are elementary from the theory of fully matricial functions already developed. Every fully matricial function defined in a neighborhood of zero can be viewed as a non-commutative generating function $\sum_{k=0}^\infty \Delta^k F(0, \dots, 0)[z, \dots, z]$. The composition and inverse function operations for fully matricial functions agree with the more general definitions for generating functions.

Conversely, a non-commutative generating function defines a fully matricial function if the power series converges in $\|\cdot\|_\#$ on some ball $B(0, R)$. However, in general, the multilinear forms in a generating function need not be completely bounded, let alone have their sum converge absolute in $\|\cdot\|_\#$.

However, we *can* view a non-commutative generating function as a literal function defined on upper triangular nilpotent matrices. Let $\mathcal{N}_n(\mathcal{A})$ be the algebra of strictly upper triangular (hence nilpotent) matrices over $\mathcal{A}$. A non-commutative generating function $F(z) = \sum_{k=0}^\infty \Lambda_k[z, \dots, z]$ defines a map $F^{(n)} : \mathcal{N}_n(\mathcal{A}) \to \mathcal{N}_n(\mathcal{A})$ (where we evaluate $F(z)$ using $\Lambda_k^\#$) which respects direct sums and intertwinings. Formally, we have $\Lambda_k = \Delta^k F(0, \dots, 0)$. Moreover, the formal composition and inverse operations agree with the composition and inverse operations for functions $\mathcal{N}_n(\mathcal{A}) \to \mathcal{N}_n(\mathcal{A})$.

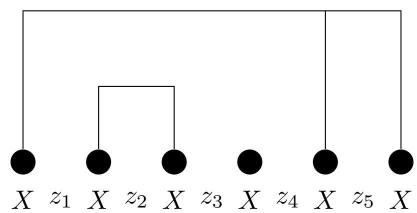

Figure 55: For $\pi = \{\{1,5,6\},\{2,3\},\{4\}\}$, we have $\operatorname{Cum}_\pi(\mu)[z_1,\dots,z_5] = \operatorname{Cum}_3(\mu)[z_1 \operatorname{Cum}_2(\mu)[z_2]z_3 \operatorname{Cum}_1(\mu)z_4, z_5]$.

5.6 Cumulants of a Law and Analytic Transforms

Cumulants of a Law

Definition 5.6.1. Let μ be an $\mathcal{A}$-valued law. We define the nth free / Boolean / monotone cumulant of μ as the multilinear form $\mathcal{A}^{n+1} \to \mathcal{A}$ given by

$$\operatorname{Cum}_n(\mu)[z_1,\dots,z_{n-1}] = K_n[Xz_1, Xz_2, \dots, Xz_{n-1}, X] = K_n[X, z_1 Z, \dots, z_{n-2}X, z_{n-1}X],$$

where X is a random variable in a probability space $(\mathcal{B}, E)$ which realizes the law μ, and K_n is the nth free / Boolean / monotone cumulant for $(\mathcal{B}, E)$.

Definition 5.6.2. More generally, for $\pi \in \mathcal{NC}(n)$, we define the π cumulant of μ by

$$\operatorname{Cum}_\pi(\mu)[z_1,\dots,z_{n-1}] = K_\pi[Xz_1, Xz_2, \dots, Xz_{n-1}, X] = K_\pi[X, z_1 Z, \dots, z_{n-2}X, z_{n-1}X].$$

These cumulants are equivalently given by the following recursive relations: First, if $I = \{j+1,\dots,k\}$ is a block of π, then

$$\operatorname{Cum}_\pi(\mu)[z_1,\dots,z_{n-1}] = \operatorname{Cum}_{\pi\setminus I}(\mu)[z_1,\dots,z_j \operatorname{Cum}_{k-j}(\mu)[z_{j+1},\dots,z_{k-1}]z_k,\dots,z_n].$$

Remark 5.6.3. Diagrammatically, we can view the elements of $\{1,\dots,n\}$ as copies of X, while the z_j's occupy the white space between the X's. See Figure 5.6.

Definition 5.6.4. The *moment generating function* of a law μ is the non-commutative generating function

$$\sum_{k=0}^\infty \operatorname{Mom}_k[\underbrace{z,\dots,z}_{k+1}].$$

Note that this is equal to $\tilde{G}_\mu(z)$ as a generating function.

Definition 5.6.5. The free / Boolean / monotone *cumulant generating function* of μ is the non-commutative generating function

$$\sum_{k=1}^\infty \operatorname{Cum}_k[\underbrace{z,\dots,z}_{k-1}].$$

Free Cumulants and the R-transform

Theorem 5.6.6. *For a non-commutative law μ, the free cumulant generating function is equal to R_μ.*

Proof. Recall that $z^{-1} + R_\mu(z)$ is the functional inverse of $G_\mu(z)$, which means that

$$(z^{-1} + R_\mu(z))^{-1} = \sum_{k=0}^{\infty} (-zR_\mu(z))^k z$$

is the functional inverse of $\tilde{G}_\mu(z)$.

Let $\Lambda_\pi = \mathrm{Cum}_\pi(\mu)$ and $K(z) = \sum_{k=1}^{\infty} \Lambda_k[z, \dots, z]$ be the cumulant generating function. Note that

$$H(z) = (z^{-1} + K(z))^{-1} = \sum_{k=0}^{\infty} (-zK(z))^k z$$

makes sense as a generating function and that $H(z)$ can be recovered from $K(z)$ by

$$H(z) = (K(z)^{-1} - z)^{-1} = \sum_{k=0}^{\infty} (K(z)z)^k K(z).$$

Therefore, if we can show that H is the inverse generating function to $\tilde{G}_\mu$, then it will follow that $K(z) = R_\mu(z)$ as desired.

We claim that $\tilde{G}_\mu(H(z)) = z$. Note that as a generating function, we have

$$\tilde{G}_\mu(z) = z + \sum_{n \geq 1} \sum_{\pi \in \mathcal{NC}(n)} z\Lambda_\pi[z, \dots, z]z$$

as a consequence of the free moment-cumulant formula. In order to show that $\tilde{G}_\mu(H(z)) = z$, it suffices to show that

$$\sum_{n \geq 1} \sum_{\pi \in \mathcal{NC}(n)} H(z)\Lambda_\pi[H(z), \dots, H(z)]H(z) = 0.$$

Substituting $H(z) = \sum_{k=0}^{\infty} (-zK(z))^m z$ yields

$$\sum_{n \geq 1} \sum_{\pi \in \mathcal{NC}(n)} \sum_{m_0, \dots, m_n \geq 1} (-1)^{m_1 + \dots + m_n} (zK(z))^{m_0} z\Lambda_\pi[(zK(z))^{m_1} z, \dots, (zK(z))^{m_{n-1}} z](zK(z))^{m_n} z.$$

Then we substitute $K(z) = \sum_{m \geq 1} \Lambda_m[z, \dots, z]$, so that

$$(zK(z))^{m_j} z = \sum_{k_1, \dots, k_{m_j}} z\Lambda_{k_1}[z, \dots, z]z \dots \Lambda_{k_{m_j}}[z, \dots, z]z.$$

Overall, we are summing over the following choices: We first choose a partition π in the sum. Then we make a choice of m_0, …, m_n. Then for each j, we choose a list of m_j terms from $K(z)$. The jth collection of m_j terms is then inserted into the jth position of the partition π (that is, in the white space between the indices $j - 1$ and j if $0 < j < n$, in the white space before 1 if $j = 0$, and in the white space after n if $j = n$).

Diagrammatically, we are taking the partition π and then creating a larger partition $\tilde{\pi}$ by inserting m_j chosen interval blocks between positions $j - 1$ and j. For such a choice, we include the term $(-1)^m \Lambda_\pi[z, \dots, z]$ where $m = m_0 + \dots + m_j$ is the total number of interval blocks inserted. In other words, the expression we want to evaluate is

$$\sum_{\pi} \sum_{\substack{\text{ways of inserting} \\ \text{interval blocks} \\ \text{to obtain } \tilde{\pi}}} (-1)^{\#\text{ of interval blocks inserted}} z\Lambda_{\tilde{\pi}}[z, \dots, z]z$$

The sum makes sense as a generating function because there are only finitely many terms of a given degree in z. Now we regroup the terms to sum over $\tilde{\pi}$ instead and obtain

$$\sum_{\tilde{\pi}} \left(\sum_{\substack{\text{ways of obtaining } \tilde{\pi} \\ \text{from some } \pi}} (-1)^{\#\text{ of interval blocks inserted}} \right) z\Lambda_{\tilde{\pi}}[z, \dots, z]z.$$

Given $\tilde{\pi}$, what possibilities are there to obtain it from another partition π by adding interval blocks? Letting S be the set of interval blocks of π, these possibilities can be enumerated by choosing $T \subseteq S$ and letting $\pi = \tilde{\pi} \setminus T$. This means that the coefficient of $z\Lambda_{\tilde{\pi}}[z, \dots, z]z$ is

$$\sum_{T \subseteq S} (-1)^{|T|} = 0.$$

Therefore, $\tilde{G}_\mu(H(z)) = z$ as desired. □

Boolean Cumulants and the B-Transform

The following result is due to [SW97] in the scalar case and [Pop09, Remark 4.2] in the operator-valued case.

Theorem 5.6.7. *For a non-commutative law μ, the Boolean cumulant generating function is equal to $\tilde{B}_\mu$.*

Proof. Recall that $B_\mu(z) = z - F_\mu(z)$ which means that $\tilde{G}_\mu(z) = (z^{-1} - \tilde{B}_\mu(z))^{-1}$. Let $\Lambda_\pi = \mathrm{Cum}_\pi(\mu)$ and let $K(z) = \sum_k \Lambda_k[z, \dots, z]$ be the Boolean cumulant generating function. To show that $K(z) = \tilde{B}_\mu(z)$, it suffices to show that $(z^{-1} - K(z))^{-1} = \tilde{G}_\mu(z)$, where we interpret

$$(z^{-1} - K(z))^{-1} = \sum_{m=0}^{\infty} (zK(z))^m z$$

as generating functions. By expanding $K(z)$ into the sum of cumulants, we obtain

$$\sum_{m=0}^{\infty} \sum_{k_1, \dots, k_m \geq 1} z\Lambda_{k_1}[z, \dots, z]z \dots \Lambda_{k_m}[z, \dots, z]z.$$

This is precisely

$$\sum_{\text{interval partitions } \pi} z\Lambda_\pi[z, \dots, z]z$$

which is equal to $\tilde{G}_\mu(z)$ by the Boolean moment-cumulant formula. □

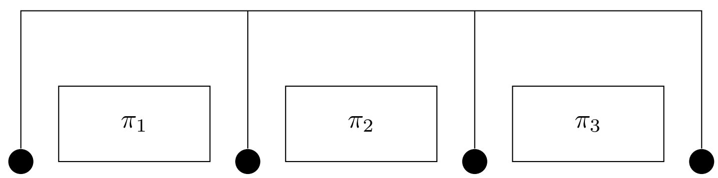

Figure 56: A partition with one outer block V and $|V| = 4$.

Monotone Cumulants and Composition Semigroups

The following result is due to Hasebe and Saigo [Has10b], [HS11b], [HS14].

Theorem 5.6.8. *Let μ be an $\mathcal{A}$-valued law and let $K(z)$ be the monotone cumulant generating function. For $t \geq 0$, let $M_t(z)$ be the generating function*

$$M_t(z) = z + \sum_{n \geq 1} \sum_{\pi \in \mathcal{NC}(n)} \gamma_\pi t^{|\pi|} z \operatorname{Cum}_\pi(\mu)[z, \dots, z] z.$$

Then $M_t(z)$ is the unique generating function which (formally) solves the initial value problem

$$\partial_t M_t(z) = M_t(z) K(M_t(z)) M_t(z), \qquad M_0(z) = z,$$

and $M_1(z) = \tilde{G}_\mu(z)$. Moreover, M_t forms a semigroup under composition.

Remark 5.6.9. The operation of monotone convolution (corresponding to composition of $\tilde{G}$ transforms) is not commutative and hence cannot be linearized like free or Boolean convolution. However, when we restrict to the monotone convolution semigroup $\mu_n = \mu^{\rhd n}$ generated by a single law μ, convolution is commutative, and hence we may expect that it can be linearized. The way to linearize it is to extend this semigroup formally to real values of t by setting $\tilde{G}_{\mu_t} = M_t$ (though μ_t is not necessarily a law). The function $zK(z)z$ is the infinitesimal generator of the composition semigroup M_t, and a straightforward rescaling argument shows that $\alpha z K(z) z$ is the infinitesimal generator of the semigroup $M_{\alpha t}$ for $\alpha > 0$. Hence, the map $\mu_\alpha \mapsto \alpha z K(z) z$ linearizes monotone convolution on $\{\mu_\alpha : \alpha \geq 0\}$.

Proof of Theorem 5.6.8. Note that $\partial_t M_t(z)$ makes sense as a generating function since the term for each degree of z is a polynomial in t, and we can compute the derivative term by term by considering each partition π individually.

Let us call a block V of π *outer* if it does not lie inside any other block of π (that is, V is minimal with respect to $\prec$). Consider first the case where π has only one outer block V and suppose that $|V| = m$. Then we have

$$\operatorname{Cum}_\pi[z, \dots, z] = \operatorname{Cum}_m[z \operatorname{Cum}_{\pi_1}[z, \dots, z] z, \dots, z \operatorname{Cum}_{\pi_{m-1}}[z, \dots, z]],$$

where $\pi_1, \dots, \pi_{m-1}$ are the subpartitions of π that are between the elements of V (see Figure 5.6). The partition π_j may be empty, in which we case we adopt the convention that $z \operatorname{Cum}_{\pi_j}[z, \dots, z] z = z$. Observe that

$$\gamma_\pi t^{|\pi|} = |\{s \in [1-t, 1]^\pi : s \models \pi\}|$$

Such tuples s can be enumerated by choosing s_V and then for each j, choosing $s_{\pi_j} \in (1-s_V, 1]^{\pi_j}$ which is compatible with π_j. Therefore, we have

$$\gamma_\pi t^{|\pi|} = \int_t^1 \prod_{j=1}^{m-1} \left|\{s_{\pi_j} \in (1-s_V, 1]^{\pi_j} : s_{\pi_j} \models \pi_j\}\right| \, ds_V.$$

Thus,

$$\frac{d}{dt}[\gamma_\pi t^{|\pi|}] = \prod_{j=1}^{m-1} \gamma_{\pi_j} t^{|\pi_j|}.$$

So

$$\frac{d}{dt}\left[\gamma_\pi t^{|\pi|} \operatorname{Cum}_\pi[z, \dots, z]\right] = \operatorname{Cum}_m[z\, \gamma_{\pi_1} t^{|\pi_1|} \operatorname{Cum}_{\pi_1}[z, \dots, z] z, \dots, z\, \gamma_{\pi_{m-1}} t^{|\pi_{m-1}|}[z, \dots, z] z].$$

Now consider the case of a general partition π. Let $V_1, \dots, V_k$ be the outer blocks of π from left to right. Let τ_j be the subpartition of π which lies inside V_j (that is, $\tau_j = \{W \in \pi : W \succeq V_j\}$. Then

$$z \operatorname{Cum}_\pi[z, \dots, z] z = z \operatorname{Cum}_{\tau_1}[z, \dots, z] z \operatorname{Cum}_{\tau_2}[z, \dots, z] \dots z \operatorname{Cum}_{\tau_k}[z, \dots, z] z$$

and

$$\begin{aligned}&\gamma_\pi t^{|\pi|} z \operatorname{Cum}_\pi[z, \dots, z] z \\ &= z \left(\gamma_{\tau_1} t^{|\tau_1|} \operatorname{Cum}_{\tau_1}[z, \dots, z]\right) z \left(\gamma_{\tau_2} t^{|\tau_2|} \operatorname{Cum}_{\tau_2}[z, \dots, z]\right) \dots z \left(\gamma_{\tau_k} t^{|\tau_k|} \operatorname{Cum}_{\tau_k}[z, \dots, z]\right) z.\end{aligned}$$

We differentiate this expression using the product rule, since each $\operatorname{Cum}_{\tau_j}$ term can be differentiated using the preceding computation. We obtain a sum of terms indexed by τ_j or equivalently indexed by the outer blocks of π, which can be written as

$$\begin{aligned}\frac{d}{dt}\left[\gamma_\pi t^{|\pi|} z \operatorname{Cum}_\pi[z, \dots, z] z\right] &= \sum_{\text{outer blocks } V} z \operatorname{Cum}_{\pi_0}[z, \dots, z] z \\ &\operatorname{Cum}_{|V|}[z \gamma_{\pi_1} t^{|\pi_1|} \operatorname{Cum}_{\pi_1}[z, \dots, z] z, \dots, z \operatorname{Cum}_{\pi_{m-1}}[z, \dots, z] z] z \operatorname{Cum}_{\pi_m}[z, \dots, z] z,\end{aligned}$$

where $\pi_0, \dots, \pi_m$ depend implicitly on V as follows; π_0 is the subpartition of π to the left of V, π_m is the partition of π to the right of V, and $\pi_1, \dots, \pi_{m-1}$ are the subpartitions of π in between the elements of V.

Now we sum this expression over all partitions π. It is convenient here to write $\mathcal{NC} = \bigcup_{n=1}^\infty \mathcal{NC}(n)$ and $\mathcal{NC}_0 = \{\varnothing\} \cup \mathcal{NC}$. Then

$$\begin{aligned}\sum_{\pi \in \mathcal{NC}} \frac{d}{dt}\left[\gamma_\pi t^{|\pi|} z \operatorname{Cum}_\pi[z, \dots, z] z\right] &= \sum_{m \geq 1} \sum_{\pi_0, \dots, \pi_m \in \mathcal{NC}_0} z \operatorname{Cum}_{\pi_0}[z, \dots, z] z \\ &\operatorname{Cum}_m[z \gamma_{\pi_1} t^{|\pi_1|} \operatorname{Cum}_{\pi_1}[z, \dots, z] z, \dots, z \operatorname{Cum}_{\pi_{m-1}}[z, \dots, z] z] z \operatorname{Cum}_{\pi_m}[z, \dots, z] z.\end{aligned}$$

This equation says precisely that

$$\frac{d}{dt}[M_t(z)] = M_t(z) K(M_t(z)) M_t(z).$$

Therefore, we have shown that $M_t(z)$ solves the initial value problem. Moreover, the uniqueness of a generating-function-valued solution to the equation is immediate because the derivative of the kth degree term of $M_t(z)$ is a function of the lower degree terms, and hence all the terms in the formal power series $M_t(z)$ can be computed inductively.

To see that $M_t(z)$ forms a composition semigroup, fix t_0 and consider the two functions

$$P_t(z) = M_{t+t_0}(z), \qquad Q_t(z) = M_t(M_{t_0}(z)).$$

Then we have $P_0(z) = M_{t_0}(z) = Q_0(z)$ and both these functions satisfy the equation

$$\frac{d}{dt} P_t(z) = P_t(z) K(P_t(z)) P_t(z).$$

Therefore, $P_t(z) = Q_t(z)$ for all t, so that $M_{t+t_0}(z) = M_t(M_{t_0}(z))$. □

5.7 Problems and Further Reading

Problem 5.1. Let $(\mathcal{B}, E)$ be an $\mathcal{A}$-valued probability space. Show that the free / Boolean / monotone cumulants K_n for $(M_m(\mathcal{B}), E^{(m)})$ are the matrix amplifications of the cumulants for $(\mathcal{B}, E)$.

Problem 5.2. Let $X_1, \dots, X_n$ be self-adjoint random variables in $(\mathcal{B}, E)$. Let $X = X_1 \oplus \dots \oplus X_n$ and let R_X be the $M_n(\mathcal{A})$-valued R-transform of X. Show that $X_1, \dots, X_n$ are freely independent if and only if $R_X(z)$ is diagonal for sufficiently small $z \in M_n(\mathcal{B})$. The analogous result also holds for Boolean independence and $\tilde{B}_X$.

Problem 5.3. For the free and Boolean cumulants, show that there are universal constants C and M such that

$$\|K_n[b_1, \dots, b_n]\| \leq C M^n \|b_1\| \dots \|b_n\|.$$

Problem 5.4. Let $\mathcal{NC}_{\mathrm{irr}}(n)$ be the set of non-crossing partitions with 1 and n in the same block (called *irreducible non-crossing partitions*). Show that

$$K_n^{\mathrm{bool}} = \sum_{\pi \in \mathcal{NC}_{\mathrm{irr}}(n)} K_\pi^{\mathrm{free}}.$$

and that

$$K_n^{\mathrm{free}} = \sum_{\pi \in \mathcal{NC}_{\mathrm{irr}}(n)} (-1)^{|\pi|-1} K_\pi^{\mathrm{bool}}.$$

Note: The second claim is harder to prove. These results were proved in [Leh02] [BN08]. Relations between all the different cumulants can be found in [Ari+15].

Further Reading

A unified approach to the theory of cumulants for the natural independences was given by Hasebe and Saigo in [HS11a]. They characterized cumulants by the axioms of multilinearity, polynomial dependence on moments which is universal for all probability spaces, and extensivity (the property described in Lemmas 5.4.6, 5.4.13, 5.4.18).

Chapter 6

The Central Limit Theorem

6.1 Introduction

The central limit theorem of classical probability states that $X_1, \dots, X_N$ are independent and identically distributed with mean zero and variance 1, and if $S_N = (X_1 + \cdots + X_N)/\sqrt{N}$, then the law of S_N approaches the standard normal distribution as $N \to \infty$.

There are analogous results for non-commutative independences. The limiting distributions are as follows

free	semicircle	$\frac{1}{2\pi}\sqrt{4-x^2}\,1_{\lvert x\rvert<2}\,dx$
Boolean	Bernoulli	$\frac{1}{2}(\delta_{-1}+\delta_1)$
(anti-)monotone	arcsine	$\frac{1}{\pi\sqrt{2-x^2}}1_{\lvert x\rvert<\sqrt{2}}\,dx$

For the free case, see [VDN92, §3.5]. For the Boolean case, see [SW97, Theorem 3.4]. For the monotone case, see [Mur01].

In the operator-valued setting, the variance is not just a scalar, but rather a completely positive map $\eta : \mathcal{A} \to \mathcal{A}$ given by

$$\eta[a] = \mu[(X - \mu(X))a(X - \mu(X))].$$

Therefore, we will define $\mathcal{A}$-valued semicircle, Bernoulli, and arcsine laws of variance η for each completely positive $\eta : \mathcal{A} \to \mathcal{A}$. Although there is no density in the operator-valued setting, the combinatorial formulas for the moments of these laws adapt without difficulty.

We will then show that if $X_1, \dots, X_N$ are $\mathcal{A}$-valued independent and identically distributed with mean zero and variance η, then the law of $(X_1 + \cdots + X_N)/\sqrt{N}$ approaches the semicircle/Bernoulli/arcsine law of variance η.

Because of the centrality of the central limit theorem, we will present three different approaches to the proof, first using analytic transforms, second using cumulants, and third using Lindeberg exchange. We will also comment on the case of variables which are independent but not identically distributed. The case of nonzero mean will be discussed in a later chapter.

The operator-valued central limit theorem can be found in the following references. For the free case, see [Voi95, Theorem 8.4], [Spe98, §4.2]. For the Boolean case, see [BPV13, §2.1]. For the monotone case, see [BPV13, §2.3], [HS14, Theorem 3.6].

6.2 Operator-valued Semicircle, Bernoulli, and Arcsine Laws

Semicircle Law

The operator-valued semicircle law was defined in [Voi95], [Spe98, §4.2 - 4.3].

Let $\mathcal{NC}_2(n)$ denote the set of non-crossing partitions on $[n]$ where every block has exactly two elements (which is empty if n is odd). Let $\eta : \mathcal{A} \to \mathcal{A}$ be completely positive. For $\pi \in \mathcal{NC}_2(n)$, denote by η_π the multilinear form $\mathcal{A}^{n-1} \to \mathcal{A}$ given recursively by the relation that if $\{j, j+1\} \in \pi$, then

$$\eta_\pi[z_1, \dots, z_{n-1}] = \begin{cases} \eta[z_1] z_2 \eta_{\pi \setminus \{1,2\}}[z_3, \dots, z_{n-1}], & j = 1 \\ \eta_{\pi \setminus \{1,2\}}[z_1, \dots, z_{j-2}, z_{j-1}\eta[z_j]z_{j+1}, z_{j+2}, \dots, z_{n-1}], & 1 < j < n-1 \\ \eta_{\pi \setminus \{n-1,n\}}[z_1, \dots, z_{n-3}] z_{n-2} \eta[z_{n-1}], & j = n-1. \end{cases}$$

Then we define the *$\mathcal{A}$-valued semicircle law with mean zero and variance* η as the map $\nu_\eta : \mathcal{A}\langle X \rangle \to \mathcal{A}$ given by

$$\nu_\eta[z_0 X z_1 \dots X z_n] = \sum_{\pi \in \mathcal{NC}_2(n)} z_0 \eta_\pi[z_1, \dots, z_{n-1}] z_n.$$

Equivalently, ν_η is given formally by the relation

$$\operatorname{Cum}_k(\nu_\eta) = \begin{cases} \eta, & k = 2 \\ 0, & \text{otherwise}, \end{cases}$$

where Cum_k is the kth free cumulant, or

$$R_{\nu_\eta} = \eta.$$

We have not yet shown that ν_η is an $\mathcal{A}$-valued law (completely positive and exponentially bounded). To do this, we will construct a self-adjoint operator S on a Hilbert bimodule which has the law ν_η. This is a special case of the construction in [Spe98, §4.7] and [PV13, Lemma 3.7].

Let $\mathcal{N}$ be the Hilbert $\mathcal{A}$-$\mathcal{A}$-bimodule $\mathcal{A} \otimes_\eta \mathcal{A}$. Then define

$$\mathcal{H} = \mathcal{A} \oplus \bigoplus_{n \geq 1} \underbrace{\mathcal{N} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{N}}_{n},$$

or more concisely

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{N}^{\otimes_{\mathcal{A}} n},$$

where $\mathcal{K}^{\otimes_{\mathcal{A}} 0} = \mathcal{A}$. Let ξ be the vector 1 in the first direct summand $\mathcal{A}$. Note that this is an $\mathcal{A}$-central unit vector. We call $(\mathcal{H}, \xi)$ the *free Fock space generated by* $\mathcal{N}$.

For $\zeta \in \mathcal{N}$, we define the *creation operator* $\ell(\zeta) : \mathcal{N}^{\otimes_{\mathcal{A}} n} \to \mathcal{N}^{\otimes_{\mathcal{A}} n+1}$ by

$$\ell(\zeta)[a\xi] = \zeta a$$
$$\ell(\zeta)[\zeta_1 \otimes \dots \otimes \zeta_n] = \zeta \otimes \zeta_1 \otimes \dots \otimes \zeta_n.$$

To show that this operator is bounded, observe that for h in the n-fold algebraic tensor product of $\mathcal{N}$ over $\mathcal{A}$, we have

$$\langle \ell(\zeta) h, \ell(\zeta) h \rangle = \langle h, \langle \zeta, \zeta \rangle h \rangle$$

Thus, $\ell(\zeta)$ defines a bounded right $\mathcal{A}$-linear operator $\ell(\zeta) : \mathcal{N}^{\otimes_{\mathcal{A}} n} \to \mathcal{N}^{\otimes_{\mathcal{A}} n+1}$ with $\|\ell(\zeta)\| \leq \|\zeta\|$. Because $\mathcal{H}$ is the orthogonal direct sum of the tensor powers of $\mathcal{N}$, we know $\ell(\zeta)$ defines a bounded map $\mathcal{H} \to \mathcal{H}$.

Moreover, we claim that $\ell(\zeta)$ has an adjoint given by the *annihilation operator* $\ell(\zeta)^*$, where

$$\ell(\zeta)^*[\xi] = 0$$
$$\ell(\zeta)^*[\zeta_1 \otimes \cdots \otimes \zeta_k] = \langle \zeta, \zeta_1 \rangle \zeta_2 \otimes \cdots \otimes \zeta_k.$$

Indeed, a straightforward computation shows that $\langle \ell(\zeta)h, h' \rangle = \langle h, \ell(\zeta)^* h' \rangle$ for h and h' in the algebraic direct sum of the algebraic tensor powers of $\mathcal{N}$. It follows that $\|\ell(\zeta)^* h\| \leq \|\zeta\| \|h\|$ as in Proposition 1.2.10 (4), and therefore, $\ell(\zeta)^*$ defines a bounded operator $\mathcal{H} \to \mathcal{H}$, and it is the adjoint of $\ell(\zeta)$.

Proposition 6.2.1. *Let $\eta : \mathcal{A} \to \mathcal{A}$ be completely positive, and let ν_η be the semicircular law of mean zero and variance η. Let $(\mathcal{H}, \xi)$ be the free Fock space generated by $\mathcal{N} = \mathcal{A} \otimes_\eta \mathcal{A}$.*

1. *The operator $S = \ell(1 \otimes 1) + \ell(1 \otimes 1)^*$ satisfies $\|S\| \leq 2\|\eta(1)\|^{1/2}$.*
2. *The law of S with respect to ξ is the operator-valued semicircle law ν_η.*
3. *In particular, ν_η is an $\mathcal{A}$-valued law.*
4. *The law ν_η has mean zero and variance η, that is, $\nu_\eta(X) = 0$ and $\operatorname{Var}_{\nu_\eta}(a) = \nu_\eta(XaX) = \eta(a)$.*

Proof. (1) Letting $\zeta = 1 \otimes 1 \in \mathcal{N}$, we have

$$\ell(\zeta)^* \ell(\zeta) = \langle \zeta, \zeta \rangle = \eta(1)$$

and hence $\|\ell(\zeta)\| = \|\eta(1)\|^{1/2}$ and $\|S\| \leq 2\|\eta(1)\|^{1/2}$.

(2) We want to compute the moment

$$\langle \xi, a_0 S a_1 \dots S a_n \xi \rangle$$

for every $a_0, \dots, a_n \in \mathcal{A}$. We write $S = \ell + \ell^*$ where $\ell = \ell(1 \otimes 1)$, and then expand

$$a_0(\ell + \ell^*)a_1 \dots (\ell + \ell^*)a_n$$

by the distributive property into a sum of terms

$$a_0 b_1 a_1 \dots b_n a_n,$$

where $b_j \in \{\ell, \ell^*\}$. Consider applying the operators a_n, b_n, a_{n-1}, ... to ξ in succession. Since ℓ maps $\mathcal{N}^{\otimes_{\mathcal{A}} j}$ to $\mathcal{N}^{\otimes_{\mathcal{A}} j+1}$ and ℓ^* does the opposite, each vector $a_j b_j \dots a_n b_n a_n \xi$ is in some $\mathcal{N}^{\otimes_{\mathcal{A}} j}$. Applying ℓ increases the index by 1 and applying ℓ^* decreases it (and $\ell^*(a\xi) = 0$ for $a \in \mathcal{A}$).

For each term in the sum, we define a sequence of ordered lists corresponding to the substrings $a_{j-1} b_j \dots a_n b_n a_n \xi$ as follows (by induction from n to 1). At time n, we have the empty list. If $b_j = \ell$, then we append j to the start of the list . If $b_j = \ell^*$ and the list at time $j+1$ is not empty, then we remove the first element from the list. If $b_j = \ell^*$ and the list at time $j+1$ is empty, then we terminate the process and do not define any more lists. In this case, $a_{j+1} b_j \dots a_{n-1} b_n a_n \xi$ is in $\mathcal{A}\xi$ and hence $b_j a_{j+1} b_j \dots a_{n-1} b_n a_n \xi = 0$, so this term does not contribute to the sum.

a_8	$\varnothing$
$a_7\ell a_8$	(8)
$a_6\ell a_7\ell a_8$	$(7,8)$
$a_5\ell a_6\ell a_7\ell a_8$	$(6,7,8)$
$a_4\ell^* a_5\ell a_6\ell a_7\ell a_8$	$(7,8)$
$a_3\ell^* a_4\ell^* a_5\ell a_6\ell a_7\ell a_8$	(8)
$a_2\ell a_3\ell^* a_4\ell^* a_5\ell a_6\ell a_7\ell a_8$	$(2,8)$
$a_1\ell^* a_2\ell a_3\ell^* a_4\ell^* a_5\ell a_6\ell a_7\ell a_8$	(8)
$a_0\ell a_1\ell^* a_2\ell a_3\ell^* a_4\ell^* a_5\ell a_6\ell a_7\ell a_8$	$\varnothing$

Figure 61: Construction of a sequence of ordered lists from a string of ℓ's and ℓ^*'s.

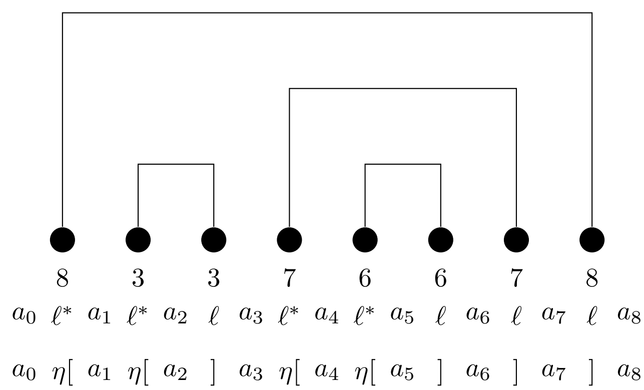

Figure 62: Constructing a planar partition π from a sequence of ℓ's and ℓ^*'s (top), and evaluation of η_π (bottom). This is the same sequence as in Figure 61.

Suppose that we never apply ℓ^* to a vector in $\mathcal{A}\xi$ and hence the lists are defined all the way to time 1. At time 1, if the list is not empty, then $a_0b_1a_1\dots b_na_n$ is not in $\mathcal{A}\xi$ and hence

$$\langle \xi, a_0b_1a_1\dots b_na_n\xi\rangle = 0.$$

Therefore, the term $a_0b_1a_1\dots b_na_n$ only contributes to the sum when the list at time 1 is empty. An example of such a string and a sequence of lists is shown in Figure 61.

Consider such a term. Let us label each occurrence of ℓ as b_j with its index j, and let us label each occurrence of ℓ^* as b_j with the list element which is removed from the list at time j. This produces a pairing between the occurrences of ℓ and the occurrences of ℓ^* because each list element was added to the list once and removed from the list once. See Figure 62 for an example. This pairing of ℓ's and ℓ^*'s represents a pair partition of $[n]$. This partition is non-crossing; indeed, if s is added to the list and then r is added to the list before s is removed, then r must be removed from the list before s.

Conversely, every planar partition of $[n]$ produces a term $a_0b_1a_1\dots b_na_n$ where for each $\{j,k\}\in\pi$ with $j<k$, we let $b_j=\ell^*$ and $b_k=\ell$. One can show by induction on the size of π that the corresponding string yields

$$\langle \xi, a_0b_1a_1\dots b_na_n\xi\rangle = a_0\eta_\pi[a_1,\dots,a_{n-1}a_n.$$

This follows from the identity $\ell^* a \ell = \eta(a)$ for $a \in \mathcal{A}$. Therefore,

$$\langle \xi, a_0 S a_1 \dots S a_n \xi \rangle = \sum_{\pi \in \mathcal{NC}_2(n)} a_0 \eta_\pi[a_1, \dots, a_{n-1}] a_n,$$

so S has the law ν_η as desired.

(3). Because ν_η is given by the moments of S with respect to ξ, we know that ν_η is completely positive and exponentially bounded. Also, ν_η is an $\mathcal{A}$-$\mathcal{A}$-bimodule map because ξ is an $\mathcal{A}$-central unit vector in $\mathcal{H}$. So ν_η is a law.

(4). This follows from direct computation. □

Lemma 6.2.2. *Let ν_η be the semicircle law of variance η. Then $\nu_\eta \boxplus \nu_{\eta'} = \nu_{\eta+\eta'}$.*

Proof. This follows from the definition of ν_η in terms of cumulants and the additivity of the R-transform under free convolution. □

Bernoulli Law

We define the *$\mathcal{A}$-valued Bernoulli law with mean zero and variance* η as the map $\nu_\eta : \mathcal{A}\langle X \rangle \to \mathcal{A}$ given by

$$\nu_\eta[z_0 X z_1 \dots X z_n] = \begin{cases} z_0 \eta(z_1) z_2 \dots \eta(z_{n-1}) z_n, & n \text{ is even} \\ 0, & n \text{ is odd.} \end{cases}$$

Equivalently, ν_η is given formally by the relation

$$\mathrm{Cum}_k(\nu_\eta) = \begin{cases} \eta, & k = 2 \\ 0, & \text{otherwise,} \end{cases}$$

where Cum_k is the kth Boolean cumulant, or

$$\tilde{B}_{\nu_\eta} = \eta.$$

Remark 6.2.3. The analogous definition to the free case would be take the sum of $z_0 \eta_\pi[z_1, \dots, z_{n-1}] z_n$ over all interval partitions into pairs. But there is only one such partition if n is even (namely $\{\{1,2\}, \{3,4\}, \dots, \{n-1, n\}\}$) and there are no such partitions if n is odd.

We have not yet shown that ν_η is an $\mathcal{A}$-valued law (completely positive and exponentially bounded). To do this, we will construct a self-adjoint operator S on a Hilbert bimodule which has the law ν_η. This is a special case of the construction in [PV13, Lemma 2.9].

Let $\mathcal{N}$ be the Hilbert $\mathcal{A}$-$\mathcal{A}$-bimodule $\mathcal{A} \otimes_\eta \mathcal{A}$. Let

$$\mathcal{H} = \mathcal{A} \oplus \mathcal{N}$$

Let ξ be the vector 1 in the first direct summand $\mathcal{A}$. Note that this is an $\mathcal{A}$-central unit vector. We call $(\mathcal{H}, \xi)$ the *Boolean Fock space generated by* $\mathcal{N}$.

For $\zeta \in \mathcal{N}$, we define the *creation operator* $\ell(\zeta) : \mathcal{H} \to \mathcal{H}$ by

$$\ell(\zeta)[a\xi] = \zeta a$$
$$\ell(\zeta)|_{\mathcal{N}} = 0.$$

The verification that $\ell(\zeta)$ is bounded is similar to the free case. Moreover, its adjoint is the *annihilation operator* $\ell(\zeta)^*$ which is given by

$$\ell(\zeta)^*[\xi a] = 0$$
$$\ell(\zeta)^*[\zeta'] = \langle \zeta, \zeta' \rangle \xi.$$

Proposition 6.2.4. *Let $\eta : \mathcal{A} \to \mathcal{A}$ be completely positive, and let ν_η be the Bernoulli law of mean zero and variance η. Let $(\mathcal{H}, \xi)$ be the Boolean Fock space generated by $\mathcal{N} = \mathcal{A} \otimes_\eta \mathcal{A}$.*

1. *The operator $S = \ell(1 \otimes 1) + \ell(1 \otimes 1)^*$ satisfies $\|S\| = \|\eta(1)\|^{1/2}$.*
2. *The law of S with respect to ξ is the operator-valued Bernoulli law ν_η.*
3. *In particular, ν_η is an $\mathcal{A}$-valued law.*
4. *The law ν_η has mean zero and variance η, that is, $\nu_\eta(X) = 0$ and $\mathrm{Var}_{\nu_\eta}(a) = \nu_\eta(XaX) = \eta(a)$.*

Proof. (1) Because $\ell(1 \otimes 1)$ maps $\mathcal{A}\xi$ to $\mathcal{N}$ and $\mathcal{N}$ to zero and $\ell(1 \otimes 1)^*$ does the reverse, we have

$$\begin{aligned}\|S\| \leq \max(\|\ell(1 \otimes 1)\|, \|\ell(1 \otimes 1)^*\|) &= \|\ell(1 \otimes 1)^*\ell(1 \otimes 1)\|^{1/2} \\ &= \|P_{\mathcal{A}\xi}\eta(1)P_{\mathcal{A}\xi}\|^{1/2} \\ &= \|\eta(1)\|^{1/2}.\end{aligned}$$

(2) To compute the moment $\langle \xi, a_0 S a_1 \dots S a_n \xi \rangle$ for every $a_0, \dots, a_n \in \mathcal{A}$, we write $S = \ell + \ell^*$ where $\ell = \ell(1 \otimes 1)$, and then expand

$$a_0(\ell + \ell^*)a_1 \dots (\ell + \ell^*)a_n$$

by the distributive property into a sum of terms

$$a_0 b_1 a_1 \dots b_n a_n,$$

where $b_j \in \{\ell, \ell^*\}$. Consider applying the operators a_n, b_n, a_{n-1}, …to ξ in succession. If we apply ℓ^* to $\mathcal{A}\xi$ or ℓ to $\mathcal{N}$, we get 0. Therefore, the only combination which could have a nonzero contribution is the term

$$a_0 \ell^* a_1 \ell a_2 \dots \ell^* a_{n-1} \ell a_n$$

when n is even. This yields the operator-valued Bernoulli law.

(3) and (4) are similar to the free case. □

Lemma 6.2.5. *Let ν_η be the Bernoulli law of mean zero and variance η. Then $\nu_\eta \uplus \nu_{\eta'} = \nu_{\eta+\eta'}$.*

Proof. This follows from the definition of ν_η in terms of cumulants and the additivity of the B-transform under Boolean convolution. □

Arcsine Law

We define the $\mathcal{A}$-*valued arcsine law with mean zero and variance* η as the map $\nu_\eta : \mathcal{A}\langle X \rangle \to \mathcal{A}$ given by

$$\nu_\eta[z_0 X z_1 \dots X z_n] = \sum_{\pi \in \mathcal{NC}_2(n)} \gamma_\pi z_0 \eta_\pi[z_1, \dots, z_{n-1}] z_n,$$

where

$$\gamma_\pi = |\{t \in [0,1]^\pi : t \models \pi\}|.$$

Equivalently, ν_η is given formally by the relation

$$\mathrm{Cum}_k(\nu_\eta) = \begin{cases} \eta, & k = 2 \\ 0, & \text{otherwise}, \end{cases}$$

where Cum_k is the kth monotone cumulant.

As in the free and Boolean cases, we will show that ν_η is a law by constructing a self-adjoint operator on a Hilbert $\mathcal{A}$-$\mathcal{A}$-bimodule which has the law ν_η. As we saw in §5.6, the monotone cumulants are naturally related to composition semigroups and differential equations. Thus, it is natural to include the time parameter t in the construction of our Hilbert bimodule. The following construction was done in the scalar case by [Lu97] and [Mur97].

We define $\mathcal{C} = C([0,1], \mathcal{A})$. We define a completely positive map $I : \mathcal{C} \to \mathcal{C}$ by

$$I[f](t) = \int_t^1 \eta[f(s)]\, ds$$

and a completely positive map $I' = \mathcal{C} \to \mathcal{A}$ by

$$I'[f] = \int_0^1 \eta[f(s)]\, ds.$$

Let $\mathcal{N} = \mathcal{C} \otimes_I \mathcal{C}$ and $\mathcal{N}' = \mathcal{C} \otimes_{I'} \mathcal{A}$. We claim that there is a well-defined embedding of $\mathcal{N}'$ into $\mathcal{N}$ given by

$$f(t) \otimes a \in \mathcal{C} \otimes_I \mathcal{C} \mapsto f(t) \otimes a \in \mathcal{C} \otimes_{I'} \mathcal{A}.$$

To see this, observe that by complete positivity of η,

$$\begin{aligned} \left\langle \sum_j f_j(t) \otimes a_j, \sum_j f_j(t) \otimes a_j \right\rangle_I (t) &= \int_t^1 \sum_{j,k} a_j^* \eta[f_j(s)^* f_k(s)] a_k\, ds \\ &\leq \int_0^1 \sum_{j,k} a_j^* \eta[f_j(s)^* f_k(s)] a_k\, ds \\ &= \left\langle \sum_j f_j(t) \otimes a_j, \sum_j f_j(t) \otimes a_j \right\rangle_{I'}, \end{aligned}$$

so that

$$\left\| \sum_j f_j(t) \otimes a_j \right\|_{\mathcal{C} \otimes_I \mathcal{C}} = \left\| \sum_j f_j(t) \otimes a_j \right\|_{\mathcal{C} \otimes_{I'} \mathcal{A}},$$

and hence $\mathcal{N}'$ embeds isometrically into $\mathcal{N}$ (even though the inner products take values in different algebras), and it is clear that this embedding is right $\mathcal{A}$-linear.

We define $\mathcal{H}$ as the Hilbert $\mathcal{A}$-$\mathcal{A}$-bimodule

$$\mathcal{H} = \mathcal{A} \oplus \bigoplus_{n \geq 1} \mathcal{N}^{\otimes_{\mathcal{C}} (n-1)} \mathcal{N}'.$$

Note that $\mathcal{H}$ can be regarded as a $\mathcal{C}$-$\mathcal{A}$-bimodule where the left $\mathcal{C}$ action on $\mathcal{A}$ is given by

$$f \cdot a = f(0) a.$$

Let ξ be the vector 1 in the first direct summand $\mathcal{A}$. We call $(\mathcal{H}, \xi)$ the *monotone Fock space generated by $\mathcal{N}'$*.

For $\zeta \in \mathcal{N}' \subseteq \mathcal{N}$, we define the *creation operator* $\ell(\zeta)$ by

$$\begin{aligned} \ell(\zeta)[\xi] &= \zeta, \\ \ell(\zeta)[\zeta_1 \otimes \dots \otimes \zeta_n] &= \zeta \otimes \zeta_1 \otimes \dots \otimes \zeta_n. \end{aligned}$$

Similar to the free case, one argues that this is well-defined and bounded with norm equal to $\|\zeta\|_I$. Moreover, the adjoint is given by the operator

$$\begin{aligned} \ell(\zeta)^*[\xi] &= 0 \\ \ell(\zeta)^*[\zeta'] &= \langle \zeta, \zeta' \rangle_{I'} \\ \ell(\zeta)^*[\zeta_1 \otimes \dots \otimes \zeta_n] &= \langle \zeta, \zeta_1 \rangle_I \zeta_2 \otimes \dots \otimes \zeta_n, \qquad n \geq 2. \end{aligned}$$

Proposition 6.2.6. *Let $\eta : \mathcal{A} \to \mathcal{A}$ be completely positive, and let ν_η be the arcsine law of mean zero and variance η. Let $\mathcal{C}$, I, and I' be as above. Let $(\mathcal{H}, \xi)$ be the monotone Fock space generated by $\mathcal{A} \otimes_\eta \mathcal{A}$.*

1. *The operator $S = \ell(1 \otimes 1) + \ell(1 \otimes 1)^*$ satisfies $\|S\| \leq 2\|\eta(1)\|^{1/2}$.*

2. *The law of S with respect to ξ is the operator-valued arcsine law ν_η.*

3. *In particular, ν_η is an $\mathcal{A}$-valued law.*

4. *The law ν_η has mean zero and variance η, that is, $\nu_\eta(X) = 0$ and $\operatorname{Var}_{\nu_\eta}(a) = \nu_\eta(XaX) = \eta(a)$.*

Proof. (1) We have $\|\ell(1 \otimes 1)\| = \|1 \otimes 1\|_{\mathcal{N}'} = \|\eta(1)\|^{1/2}$.

(2) Let $\ell = \ell(1 \otimes 1)$ and $\ell^* = \ell(1 \otimes 1)^*$. As in the free case, we expand

$$\langle \xi, a_0(\ell + \ell^*)a_1 \dots (\ell + \ell^*)a_n \xi \rangle$$

using the distributive property. After discarding some zero terms, this results in a sum of terms indexed by $\mathcal{NC}_2(n)$. For each partition, the first element of each block corresponds to a copy of ℓ^* and the second element of each block corresponds to a copy of ℓ. We also have the identity

$$\ell^* f \ell = I[f]$$

where f and $I[f] \in \mathcal{C}$ represent the left multiplication operators on $\mathcal{H}$ (which are multiplication by $f(0)$ and $I[f](0)$ on the direct summand $\mathcal{A}$). This implies that

$$\langle \xi, a_0(\ell + \ell^*)a_1 \dots (\ell + \ell^*)a_n \xi \rangle = \sum_{\pi \in \mathcal{NC}_2(n)} a_0 I_\pi[a_1, \dots, a_{n-1}] a_n \Big|_{t=0},$$

where $I_\pi : \mathcal{C}^{n-1} \to \mathcal{C}$ is defined recursively the same way as η_π. More explicitly, I_π is given by the recursive relation that if $\{k, k+1\}$ is an block of π, then

$$I_\pi[c_1, \dots, c_{n-1}] = \begin{cases} I[c_1] c_2 I_{\pi \setminus \{1,2\}}[c_3, \dots, c_{n-1}], & k = 1 \\ I_{\pi \setminus \{k,k+1\}}[c_1, \dots, c_{k-1} I[c_k] c_{k+1}, \dots, c_{n-1}], & 1 < k < n-1 \\ I_{\pi \setminus \{n-1,n\}}[c_1, \dots, c_{n-3}] c_{n-2} I[c_{n-1}], & k = n-1. \end{cases}$$

Now we claim that

$$\begin{aligned} I_\pi[a_1, \dots, a_{n-1}](t) &= \gamma_\pi (1-t)^{|\pi|} \eta_\pi[a_1, \dots, a_{n-1}] \\ &= |\{s \in [t,1]^\pi : s \models \pi\}| \, \eta_\pi[a_1, \dots, a_{n-1}], \end{aligned}$$

which we prove by induction on $|\pi|$. The base case where π has only one block is immediate. For inductive step, recall from the proof of Theorem 5.6.8 that a block of π is called *outer* if it minimal with respect to $\prec$. First, suppose that π has exactly one outer block V. This block must be $V = \{1, n\}$. So we see that

$$I_\pi[a_1, \dots, a_{n-1}] = I[a_1 I_{\pi \setminus V}[a_2, \dots, a_{n-2}] a_{n-1}].$$

Applying the inductive hypothesis, we get

$$\begin{aligned} I_\pi[a_1, \dots, a_{n-1}](t) &= \int_t^1 \eta[a_1 I_{\pi \setminus V}[a_2, \dots, a_{n-2}] a_{n-1}] \, ds_V \\ &= \int_t^1 \left| \left\{ s \in [s_V, 1]^{\pi \setminus V} : s \models \pi \setminus V \right\} \right| \, ds_V \, \eta_\pi[a_1, \dots, a_{n-1}] \\ &= |\{s \in [t,1]^\pi : s \models \pi\}| \, \eta_\pi[a_1, \dots, a_{n-1}]. \end{aligned}$$

On the other hand, suppose that π has more than one outer block. Then we can write $\pi = \pi_1 \sqcup \pi_2$ where π_1 and π_2 are non-crossing partitions of $\{1, \dots, k\}$ and $\{k+1, \dots, n\}$ for some $1 < k < n$. We can then apply the inductive hypothesis to π_1 and π_2 to prove the claim for π.

Altogether, we have shown that

$$\langle \xi, a_0 S a_1 \dots S a_n \xi \rangle = \sum_{\pi \in \mathcal{NC}(n)} \gamma_\pi a_0 \eta_\pi[a_1, \dots, a_{n-1}] a_n.$$

and hence the law of S is ν_η, proving claim (2). Claims (3) and (4) are then immediate. □

Example. As an example of the evaluation of I_π in the preceding argument, consider the partition $\{\{1,8\}, \{2,3\}, \{4,7\}, \{5,6\}\}$ from Figure 62. In this case,

$$\begin{aligned} I_\pi[a_1, \dots, a_7]|_{t=0} &= \int_0^1 \eta \left[a_1 \int_{s_8}^1 \eta[a_2] \, ds_3 \, a_3 \int_{s_8}^1 \eta \left[a_4 \int_{s_7}^1 \eta[a_5] \, ds_6 \, a_6 \right] \, ds_7 \, a_7 \right] \, ds_8 \\ &= |\{(s_3, s_6, s_7, s_8) : s_3 < s_8, s_6 < s_7 < s_8\}| \, \eta_\pi[a_1, \dots, a_7]. \end{aligned}$$

Lemma 6.2.7. *Let ν_η be the arcsine law of variance η. Then $\nu_\eta^{\rhd N} = \nu_{N\eta}$.*

Proof. This follows from the definition of ν_η in terms of cumulants together with Lemma 5.4.18. □

6.3 Central Limit Theorem via Cumulants

The proof of the central limit theorem using cumulants is straightforward given the results of the previous chapter. It also handles free, Boolean, and (anti-)monotone independence in one argument.

Definition 6.3.1. For an $\mathcal{A}$-valued law μ, we define the *dilation* $\operatorname{dil}_t(\mu)$ by

$$\operatorname{dil}_t(\mu)[a_0 X a_1 \dots X a_n] = t^n \mu[a_0 X a_1 \dots X a_n].$$

In other words, if μ is the law of X, then $\operatorname{dil}_t(\mu)$ is the law of tX.

Observation 6.3.2. *If μ is an $\mathcal{A}$-valued law, then* $\operatorname{Cum}_k(\operatorname{dil}_t(\mu)) = t^k \operatorname{Cum}_k(\mu)$ *for the free, Boolean, and monotone cumulants.*

Theorem 6.3.3. *Let μ be an $\mathcal{A}$-valued law with mean zero and variance η.*

1. *$\mu^{\boxplus N}$ converges to the semicircle law of variance η as $N \to \infty$.*
2. *$\mu^{\uplus N}$ converges to the Bernoulli law of variance η as $N \to \infty$.*
3. *$\mu^{\rhd N}$ converges to the arcsine law of variance η as $N \to \infty$.*

Proof. For each type of independence, let λ_N be the $N^{-1/2}$ dilation of the N-fold convolution of μ. Then

$$\operatorname{Cum}_k(\lambda_N) = N^{1-k/2} \operatorname{Cum}_k(\mu).$$

Therefore,

$$\lim_{N \to \infty} \operatorname{Cum}_k(\lambda_N) = \begin{cases} \eta, & k = 2 \\ 0, & \text{otherwise}, \end{cases}$$

where the limit occurs in $\|\cdot\|_{\#}$. Now by the moment-cumulant relations, the moments of λ_N also converge to the moments of μ in $\|\cdot\|_{\#}$. $\square$

6.4 Central Limit Theorem via Analytic Transforms

Next, we will give proofs of the central limit theorem using analytic transforms. We include explicit estimates on the rate of convergence of the analytic transforms to those of the central limit distribution.

The behavior of analytic transforms under dilation is as follows. The verification is a straightforward computation, which we leave as an exercise.

Observation 6.4.1.

$$\begin{aligned} G_{\operatorname{dil}_t(\mu)}(z) &= t^{-1} G_\mu(t^{-1} z) \\ F_{\operatorname{dil}_t(\mu)}(z) &= t F_\mu(t^{-1} z) \\ \Phi_{\operatorname{dil}_t(\mu)}(z) &= t \Phi_\mu(t^{-1} z) \\ R_{\operatorname{dil}_t(\mu)}(z) &= t R_\mu(t z) \\ B_{\operatorname{dil}_t(\mu)}(z) &= t B_\mu(t^{-1} z). \end{aligned}$$

Theorem 6.4.2 (Analytic Free CLT)**.** *Let μ be an $\mathcal{A}$-valued law with mean zero and variance η. Then $\operatorname{dil}_{1/\sqrt{N}}(\mu^{\boxplus N})$ converges in moments to the semicircle law of variance η. Moreover, there are universal constants C and C' such that*

$$\|z\| \le C N^{1/2} / \operatorname{rad}(\mu) \implies \|R_{\operatorname{dil}_{1/\sqrt{N}}(\mu^{\boxplus N})}(z) - \eta(z)\| \le C' N^{-1/2} \|\eta(1)\| \operatorname{rad}(\mu).$$

Proof. Let $\lambda_N = \operatorname{dil}_{1/\sqrt{N}}(\mu^{\boxplus N})$ and let ν_η be the semicircle law of variance η. By Theorem 4.3.1 (3),

$$\operatorname{rad}(\mu^{\boxplus N}) \le (2N^{1/2} + 1) \operatorname{rad}(\mu),$$

so that

$$\operatorname{rad}(\lambda_N) \le (2 + N^{-1/2}) \operatorname{rad}(\mu).$$

By the additivity and dilation properties of the R-transform,

$$R_{\lambda_N} = N^{-1/2}(NR_\mu)(N^{-1/2}z) = N^{1/2}R_\mu(N^{-1/2}z).$$

Now $R_\mu : B(0, C_1/\operatorname{rad}(\mu)) \to B(0, C_2\|\eta(1)\|\operatorname{rad}(\mu))$ for some universal constants C_1 and C_2, and hence

$$\|R_\mu(z) - \eta(z)\| = \|R_\mu(z) - \Delta R_\mu(0,0)[z]\| \leq \frac{C_2\|z\|^2\|\eta(1)\|\operatorname{rad}(\mu)}{C_1/\operatorname{rad}(\mu) - \|z\|} \leq C_3\|z\|^2\|\eta(1)\|\operatorname{rad}(\mu)$$

when $\|z\| \leq C_1/2\operatorname{rad}(\mu)$. Hence,

$$\|N^{1/2}R_\mu(N^{-1/2}z) - \eta(z)\| \leq C_3N^{-1/2}\|\eta(1)\|\operatorname{rad}(\mu)$$

for $\|z\| \leq C_1N^{1/2}/2\operatorname{rad}(\mu)$.

Recalling that $\tilde{G}_\mu^{-1}(z) = z(1 + R_\mu(z)z)^{-1}$, we see that for sufficiently large N, we have

$$\sup_{z \in B(0, C_4/\operatorname{rad}(\mu))} \left\| \tilde{G}_{\lambda_N}^{-1}(z) - \tilde{G}_{\nu_\eta}^{-1}(z) \right\| \leq C_5N^{-1/2}\|\eta(1)\|\operatorname{rad}(\mu).$$

Now as in the proof of Theorem 4.7.2, we have $\tilde{G}_{\lambda_N} : B(0, C_6/\operatorname{rad}(\lambda_N)) \to B(0, C_7/\operatorname{rad}(\lambda_N))$ (and of course $\operatorname{rad}(\lambda_N) \leq 3\operatorname{rad}(\mu)$). So by continuous dependence of the inverse function (Proposition 2.8.4, we have for sufficiently large N that

$$\sup_{z \in B(0, C_8/\operatorname{rad}(\mu))} \left\| \tilde{G}_{\lambda_N}(z) - \tilde{G}_{\nu_\eta}(z) \right\| \leq C_9N^{-1/2}\|\eta(1)\|\operatorname{rad}(\mu).$$

Therefore, by Proposition 3.6.6, we have $\lambda_N \to \nu_\eta$. □

Theorem 6.4.3 (Analytic Boolean CLT)**.** *Let μ be an $\mathcal{A}$-valued law with mean zero and variance η. Then $\operatorname{dil}_{1/\sqrt{N}}(\mu^{\uplus N})$ converges in moments to the Bernoulli law of variance η. Moreover, there are universal constants C and C' such that*

$$\|z\| \leq CN^{1/2}/\operatorname{rad}(\mu) \implies \|\tilde{B}_{\operatorname{dil}_{1/\sqrt{N}}(\mu^{\uplus N})}(z) - \eta(z)\| \leq C'N^{-1/2}\|\eta(1)\|\operatorname{rad}(\mu).$$

Proof. The argument is similar to and easier than the free case. Letting $\lambda_N = \operatorname{dil}_{1/\sqrt{N}}(\mu^{\uplus N})$ and ν_η be the Bernoulli law of variance η, we again have $\operatorname{rad}(\lambda_N) \leq 3\operatorname{rad}(\mu)$. Also, $\tilde{B}_{\lambda_N}(z) = N^{1/2}\tilde{B}_\mu(N^{-1/2}z)$. From here one argues that $\tilde{B}_{\lambda_N}(z) \to \eta(z)$ and $\tilde{G}_{\lambda_N}(z) \to \tilde{G}_{\nu_\eta}(z)$ in a neighborhood of zero, so that $\lambda_N \to \nu_\eta$ in moments. □

Theorem 6.4.4 (Analytic Monotone CLT)**.** *Let μ be an $\mathcal{A}$-valued law with mean zero and variance η. Then $\operatorname{dil}_{1/\sqrt{N}}(\mu^{\rhd N})$ converges in moments to the arcsine law of variance η. Moreover,*

$$\operatorname{Im} z \geq \epsilon \implies \left\| G_{\operatorname{dil}_{1/\sqrt{N}}(\mu^{\rhd N})}(z) - G_{\nu_\eta}(z) \right\| \leq \frac{4\|\eta(1)\|\operatorname{rad}(\mu)}{\epsilon^4N^{1/2}}.$$

Proof. Note that for a law μ of mean zero and variance η, we have

$$F_\mu(z) = z - G_\sigma(z),$$

where σ is a generalized law with $\operatorname{rad}(\sigma) \leq 2\operatorname{rad}(\mu)$ and $\sigma|_{\mathcal{A}} = \eta$. So

$$G_\sigma(z) - \eta^\#(z^{-1}) = \sigma[(z - X)^{-1} - z^{-1}] = -\sigma[(z - X)^{-1}Xz^{-1}],$$

so that

$$\operatorname{Im} z \geq \epsilon \implies \|G_\sigma(z) - \eta^\#(z^{-1})\| \leq \frac{\|\sigma(1)\| \operatorname{rad}(\sigma)}{\epsilon^2} \leq \frac{2\|\eta(1)\| \operatorname{rad}(\mu)}{\epsilon^2},$$

and hence

$$\|F_\mu(z) - z + \eta(z^{-1})\| \leq \frac{2\|\eta(1)\| \operatorname{rad}(\mu)}{\epsilon^2}$$

The same holds with μ replaced by the arcsine law ν_η, so that

$$\|F_\mu(z) - F_{\nu_\eta}(z)\| \leq \frac{2\|\eta(1)\|(\operatorname{rad}(\mu) + 2\|\eta(1)\|^{1/2})}{\epsilon^2} \leq \frac{4\|\eta(1)\| \operatorname{rad}(\mu)}{\epsilon^2}.$$

Because monotone convolution corresponds to composition of F-transforms, we have

$$G_{\mu^{\rhd N}} - G_{\nu_\eta^{\rhd N}} = \sum_{j=1}^N \left[G_{\mu^{\rhd(j-1)}} \circ F_\mu \circ F_{\nu_\eta^{\rhd(N-j)}} - G_{\mu^{\rhd(j-1)}} \circ F_{\nu_\eta} \circ F_{\nu_\eta^{\rhd(N-j)}} \right].$$

Recall that every F transform maps $\mathbb{H}_{+,\epsilon}(\mathcal{A})$ into itself, and $G_{\mu^{\rhd(j-1)}}$ is $1/\epsilon^2$ Lipschitz on $\mathbb{H}_{+,\epsilon}$. Therefore,

$$\begin{aligned}\left\| G_{\mu^{\rhd N}}(z) - G_{\nu_\eta^{\rhd N}}(z) \right\| &\leq \sum_{j=1}^N \frac{1}{\epsilon^2} \left\| (F_\mu - F_{\nu_\eta}) \circ F_{\nu_\eta^{\rhd(N-j)}}(z) \right\| \\ &\leq \frac{4N\|\eta(1)\| \operatorname{rad}(\mu)}{\epsilon^4}.\end{aligned}$$

Letting $\lambda_N = \operatorname{dil}_{N^{-1/2}}(\mu^{\rhd N})$ and applying rescaling, we have

$$\operatorname{Im} z \geq \epsilon \implies \left\| G_{\lambda_N}(z) - G_{\nu_\eta}(z) \right\| \leq \frac{4\|\eta(1)\| \operatorname{rad}(\mu)}{\epsilon^4 N^{1/2}}.$$

□

6.5 Central Limit Theorem via Lindeberg Exchange

Finally, we will give a proof of the central limit theorem using an exchange technique which Lindeberg invented for classical central limit theorem in 1922 [Lin22]. Considering a sum of independent random variables $X_1 + \cdots + X_N$ and a function f, we want to estimate

$$E[f((X_1 + \cdots + X_N)/\sqrt{N})] - E[f(Y)]$$

where Y has the normal distribution. We estimate the difference by replacing the X_j's one at a time by normal random variables Y_j, so that

$$\begin{aligned} &E[f(N^{-1/2}(X_1 + \cdots + X_N))] - E[f(N^{-1/2}(Y_1 + \cdots + Y_N))] \\ &= \sum_{j=1}^N \Big(E[f(N^{-1/2}(X_1 + \cdots + X_j + Y_{j+1} + \cdots + Y_N))] - E[f(N^{-1/2}(X_1 + \cdots + X_{j-1} + Y_j + \cdots + Y_N))] \Big) \end{aligned}$$

The individual error terms are then estimated using a Taylor expansion of f.

Non-Commutative Function Spaces

To adapt this to the non-commutative setting, we will take f to be a non-commutative polynomial in $\mathcal{A}\langle X \rangle$. To estimate error terms in the Taylor-Taylor expansion of f (Lemma 2.4.1) evaluated on self-adjoint operators, we want an non-commutative analogue of $\sup_{t \in [-R,R]} |(d/dt)^k f(t)|$ for a function f on the real line. Of course, once we have such an analogue, we can take the completion with respect to this norm and view it as a non-commutative analogue of $C^k([-R,R])$. Estimates for the rate of convergence in the central limit theorem will naturally extend to the completion as well.

Definition 6.5.1. Let $f \in \mathcal{A}\langle X \rangle$. We define

$$\|f\|_{k,R} = \sup\{\|E[\Delta^k f(x_0, \dots, x_k)[z_1, \dots, z_k]]]\| : x_j, z_k \in (\mathcal{B}, E), x_j = x_j^*, \|x_j\| \leq R, \|z_j\| \leq 1\},$$

where the supremum is taken over every tuple $x_0, \dots, x_k$ of self-adjoints with $\|x_j\| \leq R$ and every tuple $z_1, \dots, z_k$ with $\|z_j\| \leq 1$ in an $\mathcal{A}$-valued probability space $(\mathcal{B}, E)$.

Remark 6.5.2. The collection of probability spaces is not a set. However, we can rephrase the definition by taking the supremum over all possible joint laws of $x_0, \dots, x_k$ and real and imaginary parts of $z_1, \dots, z_k$. The space of joint laws is a set because it consists of functions from a formal polynomial algebra into $\mathcal{A}$.

Definition 6.5.3. We define $C^k_{nc}(\mathcal{A}, R)$ to the completion of $\mathcal{A}\langle X \rangle$ with respect to the norm

$$\|f\|_{C^k_{nc}(\mathcal{A},R)} = \sum_{j=0}^k \|f\|_{k,R}.$$

Note that $\|f\|_{k,R}$ extends continuously to the completion.

Remark 6.5.4. Although $\|f\|_{k,R}$ is difficult to compute, it is not hard to find upper bounds for it when f has an explicit form (e.g. a monomial). Moreover, if $\phi \in C_c^\infty(\mathbb{R})$, then $\phi(X)$ is in $C^k_{nc}(\mathcal{A}, R)$ for every k and R since $\|f\|_{k,R}$ can be estimated using results of Peller [Pel06]. However, we will leave such questions aside and return to the central limit theory.

Lindeberg-type Theorem

Theorem 6.5.5. *For free, Boolean, and monotone independence, the following holds. Suppose that $X_1, \dots, X_N$ are independent self-adjoint variables in $(\mathcal{B}_1, E)$ and $Y_1, \dots, Y_N$ are independent self-adjoint variables in $(\mathcal{B}_2, E)$. Suppose $\|X_j\| \leq M$, $\|Y_j\| \leq M$ and $\operatorname{Var}_{X_j} = \eta_j = \operatorname{Var}_{Y_j}$ and $E[X_j] = 0 = E[Y_j]$. Let $f \in C^3_{nc}(\mathcal{A}, R)$. Then*

$$\left\| E[f(N^{-1/2}(X_1 + \dots + X_N))] - E[f(N^{-1/2}(Y_1 + \dots + Y_N))] \right\| \leq 2N^{-1/2} M^3 \|f\|_{3,(2+N^{-1/2})M}.$$

Proof. Because $C^3_{nc}(\mathcal{A}, R)$ is the completion of polynomials, it suffices to prove the theorem when $f \in \mathcal{A}\langle X \rangle$. The expectations we want to compute depend only on the type of independence and the laws of $X_1, \dots, X_N$ and $Y_1, \dots, Y_N$. Thus, we can change the underlying probability space as long as these properties are preserved. Let $(\mathcal{B}, E)$ be the independent product of $\mathcal{A}\langle X_1 \rangle$, $\mathcal{A}\langle Y_1 \rangle$, $\dots$, $\mathcal{A}\langle X_N \rangle$, $\mathcal{A}\langle Y_N \rangle$. Let

$$S_j = N^{-1/2}(X_1 + \dots + X_j + Y_{j+1} + \dots + Y_N)$$
$$T_j = N^{-1/2}(X_1 + \dots + X_{j-1} + Y_{j+1} + \dots + Y_N).$$

Then we want to estimate

$$\|E[f(S_N)] - E[f(S_0)]\| \leq \sum_{j=1}^{N} \|E[f(S_j)] - E[f(S_{j-1})]\|.$$

Now

$$f(S_j) = f(T_j + N^{-1/2}Y_j)$$
$$= f(T_j) + N^{-1/2}\Delta F(T_j, T_j)[Y_j] + N^{-1}\Delta^2 F(T_j, T_j, T_j)[Y_j, Y_j] + N^{-3/2}\Delta^3 f(S_j, T_j, T_j, T_j)[Y_j, Y_j, Y_j]$$

and similarly

$$f(S_{j-1}) = f(T_j + N^{-1/2}X_j)$$
$$= f(T_j) + N^{-1/2}\Delta F(T_j, T_j)[X_j] + N^{-1}\Delta^2 F(T_j, T_j, T_j)[X_j, X_j] + N^{-3/2}\Delta^3 f(S_{j-1}, T_j, T_j, T_j)[X_j, X_j, X_j]$$

In the two Taylor expansions, the degree zero terms are equal.

We claim that the first degree terms have expectation zero, that is,

$$E[\Delta F(T_j, T_j)[Y_j]] = 0 = E[\Delta F(T_j, T_j)[X_j]].$$

To see this, note that $\Delta F(T_j, T_j)[Y_j]$ is a sum of monomials in the X_i's and Y_i's with only one occurrence of Y_j. Using the computation of joint moments for independent random variables (Lemmas 5.4.5, 5.4.12, 5.4.17), we see that the expectation is zero and the same holds for $\Delta F(T_j, T_j)[X_j]$.

We claim that the second degree terms in the Taylor expansion have the same expectation, that is,

$$E[\Delta^2 F(T_j, T_j, T_j)[Y_j, Y_j]] = E[\Delta^2 F(T_j, T_j, T_j)[X_j, X_j].$$

We can write out $\Delta^2 F(T_j, T_j, T_j)[Y_j, Y_j]$ as a sum of monomials which each have degree two in Y_j. Again invoking Lemmas 5.4.5, 5.4.12, 5.4.17, we see that a joint moment of X_1, …, X_{j-1}, Y_j, …, Y_{j+1} with degree two in Y_j only depends on the cumulants of X_1, …, X_{j-1}, Y_{j+1}, …, Y_N and the first and second cumulants of Y_j (that is, the mean and variance of Y_j). Moreover, if we replace Y_j with X_j, then the computation of moments is the same because X_1, …, X_j, Y_{j+1}, …, Y_N are also independent and because X_j has the same mean and variance as Y_j. This shows that the two quantities have the same expectation.

Next, we turn to the third degree terms in the Taylor expansion, which we treat as error terms. By Theorem 4.3.1 (3), we obtain

$$\|T_j\|, \|S_j\| \leq (2 + N^{-1/2})M.$$

and hence

$$\left\|E[\Delta^3 f(S_j, T_j, T_j, T_j)[Y_j, Y_j, Y_j]]\right\| \leq \|\Delta^3 f\|^*_{(2+N^{-1/2})M} \|Y_j\|^3 \leq \|\Delta^3 f\|^*_{(2+N^{-1/2})M} M^3.$$

The same holds for $\Delta^3 f(S_{j-1}, T_j, T_j, T_j)[X_j, X_j, X_j]$.

Putting together all the terms from the Taylor expansion, we have

$$\|f(S_j) - f(S_{j-1})\| \leq 2N^{-3/2} \|\Delta^3 f\|^*_{(2+N^{-1/2})M} M^3.$$

Then summing from $j = 1$ to N, we obtain

$$\|f(S_N) - f(S_0)\| \leq 2N^{-1/2} \|\Delta^3 f\|^*_{(2+N^{-1/2})M} M^3.$$ □

Estimates on the Cauchy-Stieltjes Transform

As a consequence, we now derive estimates for the rate of convergence of the Cauchy-Stieltjes transform, similar to the main theorem of [MS13].

Lemma 6.5.6. *Let $z \in \mathbb{H}_+^{(1)}(\mathcal{A})$. Then $f(z, X) = (z - X)^{-1}$, as a function of X, is in the space $C_{nc}^k(\mathcal{A}, R)$ for every R and k. Moreover, if $\operatorname{Im} z \geq \epsilon$, then*

$$\|f(z, \cdot)\|_{k,R} \leq \frac{1}{\epsilon^{k+1}}.$$

Proof. Choose a space $(\mathcal{B}, E)$. Let us write $f(z, X) = (z - X)^{-1}$ where defined for $z \in \mathcal{A}$ and $X \in \mathcal{B}$. Suppose that $\operatorname{Im} z \geq \epsilon$ and $\|\operatorname{Im}(X)\| \leq \epsilon/2$. Observe that

$$(z - X)^{-1} = ((z - 3iR - X) + 3iR)^{-1} = \sum_{k=0}^{\infty} [(z - 3iR - X)^{-1} 2iR]^k (z - 3iR - X)^{-1}.$$

where the series converges uniformly for such values of z and X because

$$\|(z - 2iR - X)^{-1}\| \leq \frac{1}{3R + \epsilon/2}.$$

On the other hand, if $\|X\| \leq 2R$, then

$$(z - 3iR - X)^{-1} = \sum_{m=0}^{\infty} [(z - 3iR)^{-1} X]^m (z - 3iR)^{-1},$$

where the series again converges uniformly. The upshot is that there exist polynomial functions $f_n(z, X)$ such that

$$f_n(z, X) \to (z - X)^{-1} \text{ uniformly when } \operatorname{Im} z \geq \epsilon, \|X\| \leq R, \|\operatorname{Im} X\| \leq \epsilon/2.$$

Because $\{X : \|X\| < R, \|\operatorname{Im}(X)\| < \epsilon/2\}$ is a matricial domain, we can apply the Cauchy-type estimates of Lemma 2.4.2 to conclude that

$$\Delta_X^k f_n(z, X_0, \dots, X_k) \to \Delta_X^k f(z, X_0, \dots, X_k)$$

uniformly for $\operatorname{Im} z \geq \epsilon$, $\|X_j\| \leq R$, $\|\operatorname{Im} X_j\| \leq \epsilon/4$. In particular, we have uniform convergence on self-adjoint variables X_j with $\|X_j\| \leq R$. This implies that $\{f_n(z, \cdot)\}$ is Cauchy in $C_{nc}^k(\mathcal{A}, R)$ for every k and that

$$\|f(z, \cdot)\|_{k,R} \leq \sup_{(\mathcal{B},E)} \sup_{\substack{\|X_j\| \leq R, \\ X_j^* = X_j}} \|\Delta_X^k f(z, X_0, \dots, X_k)\|.$$

Now

$$\Delta_X^k f(z, X_0, \dots, X_k)[Y_1, \dots, Y_k] = (z - X_0)^{-1} Y_1 (z - X_1)^{-1} \dots Y_k (z - X_k)^{-1},$$

and hence

$$\|f(z, \cdot)\|_{k,R} \leq \frac{1}{\epsilon^{k+1}}.$$

□

Proposition 6.5.7. *Let μ be an $\mathcal{A}$-valued law with mean zero and variance η. Let λ_N be the $N^{-1/2}$ dilation of the N-fold convolution of μ for free, Boolean, or monotone independence. Let ν_η be the semicircle / Bernoulli / arcsine law. Then*

$$\sup_{\operatorname{Im} z \geq \epsilon} \left\| G_{\lambda_N}(z) - G_{\nu_\eta}(z) \right\| \leq \frac{2\operatorname{rad}(\mu)^3}{N^{1/2}\epsilon^4}$$

Proof. As in Theorem 6.5.5, we can construct a probability space $(\mathcal{B}, E)$ with independent variables $X_1, \dots, X_N$ with law ν and independent variables $Y_1, \dots, Y_N$ with law ν_η.

Fix $z \in \mathbb{H}_+^{(n)}(\mathcal{A})$. Note that $X_1^{(n)}, \dots, X_N^{(n)}$ are independent over $M_n(\mathcal{A})$ and the same holds for $Y_1^{(n)}, \dots, Y_N^{(n)}$. Moreover, $f(X) = (z - X)^{-1}$ is in $C^3_{nc}(M_n(\mathcal{A}), R)$ with $\|f\|_{3,R} \leq 1/\epsilon^4$. Therefore, by Theorem 6.5.5,

$$\left\| E^{(n)}[(z - N^{-1/2}(X_1^{(n)} + \dots + X_N^{(n)}))^{-1} - E^{(n)}[(z - N^{-1/2}(Y_1^{(n)} + \dots + Y_N^{(n)}))^{-1} \right\| \leq \frac{2\operatorname{rad}(\mu)^3}{N^{1/2}\epsilon^4}.$$

□

6.6 Generalizations

Independent but Not Identically Distributed Variables

Since Theorem 6.5.5 applies even when the random variables do not have the same law, it is natural to generalize the central limit theorem as well. In the free case, the semicircle laws satisfy $\nu_\eta \boxplus \nu_{\eta'} = \nu_{\eta+\eta'}$. Thus, if μ_j has variance η_j, we can expect $\mu_1 \boxplus \dots \boxplus \mu_N$ to be well-approximated by $\nu_{\eta_1 + \dots + \eta_N}$ after an appropriate rescaling. The analogous statement holds in the Boolean case as well.

However, in the monotone case, we do not have $\nu_\eta \rhd \nu_{\eta'} = \nu_{\eta+\eta'}$ in general or even $\nu_\eta \rhd \nu_{\eta'} = \nu_{\eta'} \rhd \nu_\eta$. Nonetheless, $\mu_1 \rhd \dots \rhd \mu_N$ is still well-approximated by $\nu_{\eta_1} \rhd \dots \rhd \nu_{\eta_N}$, where η_j is the variance of μ_j. In light of this idea, we will call $\nu_{\eta_1} \rhd \dots \rhd \nu_{\eta_N}$ a *generalized arcsine law*, and we shall have more to say about such laws in the next chapter.

Altogether, we have the following generalization of Proposition 6.5.7.

Proposition 6.6.1. *For $j = 1, \dots, N$, let μ_j be an $\mathcal{A}$-valued law with mean zero, variance η_j, and $\operatorname{rad}(\mu_j) \leq M$. Let λ be the $N^{-1/2}$ dilation of the convolution of $\mu_1, \dots, \mu_N$. Let ν be the semicircle / Bernoulli law of variance $N^{-1}\sum_j \eta_j$ in the free / Boolean cases; and in the monotone case, let ν be the generalized arcsine law $\operatorname{dil}_{N^{-1/2}}(\nu_{\eta_1} \rhd \dots \rhd \nu_{\eta_N})$ where ν_{η_j} is the arcsine law of variance η. Then*

$$\sup_{\operatorname{Im} z \geq \epsilon} \|G_\lambda(z) - G_\nu(z)\| \leq \frac{2\operatorname{rad}(\mu)^3}{N^{1/2}\epsilon^4}$$

Mixing Types of Independence

The Lindeberg exchange approach actually works in far greater generality than we have presented it. As an example, suppose that $X_1, \dots, X_N$ are freely independent, $X_{N+1}, \dots, X_{2N}$ are Boolean independent, and that $\mathcal{A}\langle X_1, \dots, X_N \rangle$ and $\mathcal{A}\langle X_1, \dots, X_N \rangle$ are monotone independent. Suppose that $Y_1, \dots, Y_{2N}$ have the same independence properties as the X_j's. Suppose X_j and Y_j have mean zero and the same variance, and $\|X_j\|, \|Y_j\| \leq M$.

Then we have

$$\left\| E[f((2N)^{-1/2}(X_1 + \dots + X_{2N}))] - E[f((2N)^{-1/2}(Y_1 + \dots + Y_{2N}))] \right\| \leq 2(2N)^{-1/2} M^3 \|f\|^*_{3,(2N)^{1/2}M}.$$

The proof is exactly the same as for Theorem 6.5.5 with a few differences. First, we have substituted the crude bound $2NM$ for the norm of a sum of random variables rather than the more refined estimate $(2\sqrt{2N}+1)M$ which is available for $2N$ independent random variables with the same type of independence.

Second, when we compute the moments in the Taylor expansion, we must use all three Lemmas 5.4.5, 5.4.12, and 5.4.17 successively. But the end result is the same. The expectation of a non-commutative monomial in $X_1, \dots, X_{2N}$ which has degree n_j in X_j only depends on the first n_j moments of X_j. in fact, this property is sufficient to make the Lindeberg exchange method work, and hence we expect this method to generalize to other notions of independence.

6.7 Problems and Further Reading

Problem 6.1. Verify Observation 6.4.1 by computing the analytic transforms of $\mathrm{dil}_t(\mu)$.

Problem 6.2. Adapt the analytic proofs of the CLT for each type of independence to the case where the variables are independent but not identically distributed.

Chapter 7

Convolution Semigroups

7.1 Introduction

In classical probability theory, one considers convolution semigroups of measures μ_t on $\mathbb{R}$ satisfying $\mu_s * \mu_t = \mu_{s+t}$. Such semigroups are classified by the *L'evy-Hinčin formula*, which says, in the case of mean zero and finite moments, that the classical cumulants of μ_t are given by t times the moments of another measure σ. The measures which are part of a convolution semigroup are known as infinitely divisible.

Free, Boolean, and monotone analogues of the Lévy-Hinčin formula were considered by various authors; see for the free case [Voi86, Theorem 4.3], [BV92], [Bia98], [Spe98, §4.5 - 4.7], [PV13, §3]; for the Boolean case [SW97, Theorem 3.6], [PV13, §2]; for the monotone case [Has10a], [Has10b], [HS14], [AW16].

As a consequence, there are bijections in the scalar case between the infinitely divisible laws for classical, free, Boolean, and monotone independence, which we will refer to collectively as the *Bercovici-Pata correspondence*. The original paper of Bercovici and Pata studied the classical, free, and Boolean cases bijection [BP99]. Later authors established bijections in the monotone cases, then generalized the free-Boolean-monotone bijections to the multivariable and operator-valued setting; see [BN08], [BPV12], [AW14], [AW16]. Let us now summarize the main results that we will present in this chapter.

Definition 7.1.1. An $\mathcal{A}$*-valued free/Boolean/monotone convolution semigroup* is family of $\mathcal{A}$-valued laws $\{\mu_t\}_{t \in [0,+\infty)}$ such that the mean $\mu_t(X)$ depends continuously on t and

$$\begin{aligned} \mu_s \boxplus \mu_t &= \mu_{s+t} \quad \text{(free case)} \\ \mu_s \uplus \mu_t &= \mu_{s+t} \quad \text{(free case)} \\ \mu_s \rhd \mu_t &= \mu_{s+t} \quad \text{(monotone case)} \end{aligned}$$

Note that in the monotone case, we also have $\mu_s \lhd \mu_t = \mu_{s+t}$. Thus, an anti-monotone convolution semigroup is equivalent to a monotone convolution semigroup. In this chapter, we describe the analytic characterization of convolution semigroups in terms of their cumulant generating functions, which act as a kind of infinitesimal generator for the semigroup.

Theorem 7.1.2.

1. *If $\{\mu_t\}$ be a free/Boolean/monotone convolution semigroup, then there exists a generalized law σ and a self-adjoint $a \in \mathcal{A}$ such that*

$$K_{\mu_t}(z^{-1}) = t(a + G_\sigma(z)), \tag{7.1.1}$$

where K_{μ_t} is the free/Boolean/monotone cumulant generating function.

2. *Let us denote $G_{\sigma,a}(z) = a + G_\sigma(z)$ and denote $DF_{\mu_t}(z) = \Delta F_{\mu_t}(z,z)$. Then the F-transforms of the semigroup μ_t satisfy the differential equation*

$$\partial_t F_{\mu_t}(z) = \begin{cases} -DF_{\mu_t}(z)[G_{\sigma,a}(F_{\mu_t}(z))], & \textit{free case} \\ -G_{\sigma,a}(z), & \textit{Boolean case} \\ -DF_{\mu_t}(z)[G_{\sigma,a}(z)], & \textit{monotone case} \\ -G_{\sigma,a}(F_{\mu_t}(z)), & \textit{monotone case,} \end{cases} \tag{7.1.2}$$

where the differentiation occurs with respect to the operator norm on $M_n(\mathcal{A})$ and the equation holds for each $z \in \mathbb{H}_+^{(n)}(\mathcal{A})$.

3. *Conversely, given a generalized law σ and self-adjoint $a \in \mathcal{A}$, there exists a free/Boolean/monotone convolution semigroup satisfying (7.1.1) and (7.1.2).*

4. *We have the estimates*

$$\begin{aligned} t\|a\| &\leq \operatorname{rad}(\mu_t) \\ \operatorname{rad}(\sigma) &\leq C \operatorname{rad}(\mu_t) \\ \operatorname{rad}(\mu_t) &\leq \operatorname{rad}(\sigma) + 2\sqrt{t\|\sigma(1)\|} + t\|a\|, \end{aligned}$$

where C is a constant that can be taken to be $1/(3-2\sqrt{2})$ in the free case and 2 in the Boolean and monotone cases.

The proof of claims (1) and (2) will be handled in the next section §7.2, while (3) and (4) will be handled in the following section §7.3. Although claim (3) of theorem can be proved purely analytically by solving the differential equations (see [Jek17, §5.3] for the monotone case), we will instead approach the problem through Hilbert bimodules (as in [Spe98, §4.7]). We will construct operators $Y_{s,t}$ for $s < t$ such that

1. The law of $Y_{s,t}$ only depends on $t - s$.
2. $Y_{t_0,t_1}, \dots, Y_{t_{n-1},t_n}$ are independent.
3. $Y_{t_0,t_1} + \dots + Y_{t_{n-1},t_n} = Y_{t_0,t_n}$.

In other words, $Y_t = X_{0,t}$ is an operator-valued stochastic process with independent and stationary increments (a Lévy process). It follows that the law μ_t of $Y_{0,t}$ forms a convolution semigroup, and we will verify that it satisfies the given equations.

The Fock space construction also yield an alternative proof of the central limit theorem in the case of infinitely divisible laws, with sharper estimates, which we discuss in §7.4.

In §7.5, we present a combinatorial point of view on the Fock space construction, giving an alternative proof of some of the claims about the semigroup.

In §7.6, we relate the notions of semigroups and infinitely divisible laws. As a consequence of Theorem 7.1.2, such laws exist in bijection with pairs (σ, a) giving the cumulant generating function. We describe the resulting Bercovici-Pata correspondence between the infinitely divisible laws for $\mathcal{A}$-valued free, Boolean, and monotone independence.

7.2 Infinitesimal Generators and Differential Equation

In this section, we prove claims (1) and (2) of Theorem 7.1.2. Since the proofs are different for each type of independence, we will handle each in its own subsection.

The Free Case

For background on the scalar case, see [Voi86, §4], [BV92], [BP99]. For the operator-valued case, see [Spe98, §4.5], [PV13, §3], [Wil17, §4].

Proposition 7.2.1. *Let μ_t be a free convolution semigroup. Then there exists a generalized law σ such that $\Phi_{\mu_t}(z) = tG_{\sigma,a}(z)$, where $a = \mu(X)$.*

Proof. Recall that $K_{\mu_t}(z) = R_{\mu_t}(z) = \Phi_{\mu_t}(z^{-1})$. By Theorem 4.7.2, we know that $\Phi_{\mu_t}(z)$ is defined for $\operatorname{Im} z \geq 2\|\operatorname{Var}_{\mu_t}(1)\|^{1/2}$. By writing

$$\Phi_{\mu_t}(z) = n\Phi_{\mu_{t/n}}(z),$$

we see that Φ_{μ_t} extends to be fully matricial for $\operatorname{Im} z \geq 2n^{-1/2}\|\operatorname{Var}_{\mu_t}(1)\|^{1/2}$. These extensions must agree for different values of n by the uniqueness theorem. Taking $n \to \infty$, we see that Φ_{μ_t} extends to be fully matricial on $\mathbb{H}_+(\mathcal{A})$.

Moreover, by Theorem 4.7.2, Φ_{μ_t} maps $\mathbb{H}_+(\mathcal{A})$ into $\overline{\mathbb{H}}_-(\mathcal{A})$ and $\tilde{\Phi}_{\mu_t}$ is fully matricial in a neighborhood of 0 and preserves adjoints. If we let $a_t = \mu_t(X)$, then $\tilde{\Phi}_{\mu_t}(0) - a = 0$. Therefore, by Theorem 3.4.1, we have

$$\Phi_{\mu_t}(z) = a_t + G_{\sigma_t}(z)$$

for some generalized law σ_t.

Because we assumed that a_t is continuous in t and $a_{s+t} = a_s + a_t$, we have $a_t = ta$ where $a = a_1$. Letting $\sigma = \sigma_1$, we have $G_{\sigma_t} = tG_\sigma$ for rational values of t. Moreover, $\sigma_t(1)$ is an increasing function of t, which forces $\sigma_t(1) = t\sigma(1)$ for all real $t \geq 0$. Since $\operatorname{Im} z \geq \epsilon$ implies that $\|G_{\sigma_t}(z)\| \leq t\|\sigma(1)\|/\epsilon$, we see that G_{σ_t} depends continuously on t and hence $G_{\sigma_t} = tG_\sigma$ for all real $t \geq 0$. This completes the proof that $tG_{\sigma,a}(z)$ is the free cumulant generating function of μ_t. □

Lemma 7.2.2. *Let μ_t and $G_{\sigma,a}$ be as in the previous proposition. Then for $\delta > 0$,*

$$F_{t+\delta}(z) = F_t(z - \delta G_{\sigma,a}(F_{t+\delta}(z))).$$

Proof. Recall that for $\operatorname{Im} z$ sufficiently large, F_{μ_t} is invertible with inverse function given by $z + tG_{\sigma,a}(z)$. From this it follows that for $\delta > 0$,

$$\begin{aligned} F_{t+\delta}(z) &= F_t \circ (\operatorname{id} + tG_{\sigma,a}) \circ F_{t+\delta}(z) \\ &= F_t(F_{t+\delta}(z) + tG_{\sigma,a}(F_{t+\delta}(z))). \end{aligned}$$

But we also have

$$z = F_{t+\delta}(z) + (t+\delta)G_{\sigma,a}(F_{t+\delta}(z)),$$

and hence

$$F_{t+\delta}(z) = F_t(z - \delta G_{\sigma,a}(F_{t+\delta}(z))).$$

□

Lemma 7.2.3. *For a free convolution semigroup μ_t, $\epsilon > 0$ and $T > 0$, the function $F_t(z)$ is uniformly Lipschitz in t for $\operatorname{Im} z \geq \epsilon$ and $t \leq T$, where the Lipschitz constant only depends on $\|\operatorname{Var}_{\mu_1}(1)\|$, T, and ϵ.*

Proof. Recall that $F_t(z) - z + ta$ is the Cauchy transform of a generalized law ρ with $\rho(1) = \mathrm{Var}_{\mu_t}(1) = t\sigma(1)$. Hence for z, z' in the same size matrix algebra with imaginary part $\geq \epsilon$, we have

$$\|F_t(z) - F_t(z')\| \leq \left(1 + \frac{t\|\sigma(1)\|}{\epsilon^2}\right)\|z - z'\|.$$

Now examining Lemma 7.2.2, we see that for $\operatorname{Im} z \geq \epsilon$,

$$\begin{aligned}\|F_{t+\delta}(z) - F_t(z)\| &\leq \left(1 + \frac{t\|\sigma(1)\|}{\epsilon^2}\right)\delta\|G_{\sigma,a}(F_{t+\delta}(z))\| \\ &\leq \left(1 + \frac{t\|\sigma(1)\|}{\epsilon^2}\right)\frac{\|\sigma(1)\|}{\epsilon^2}\delta.\end{aligned}$$

Thus, F_t is depends in a Lipschitz way on t for t in a compact time interval $[0, T]$. □

Proposition 7.2.4. *Let μ_t be a free convolution semigroup and $\Phi_{\mu_t} = tG_{\sigma,a}$. Then*

$$\partial_t F_t(z) = DF_t(z)[-G_{\sigma,a}(F_t(z))].$$

Proof. Using Lemma 7.2.2, we see that $t \leq T$,

$$\begin{aligned}F_{t+\delta}(z) - F_t(z) &= F_t(z - \delta G_{\sigma,a}(F_{T+\delta}(z))) - F_t(z) \\ &= -\delta \cdot DF_t(z)[G_{\sigma,a}(F_{t+\delta}(z))] + O_{T,\epsilon,\sigma}(\delta^2) \\ &= -\delta \cdot DF_t(z)[G_{\sigma,a}(F_t(z))] + O_{T,\epsilon,\sigma}(\delta^2).\end{aligned}$$

This proves that the derivative of $F_t(z)$ from the right is $-DF_t(z)[G_{\sigma,a}(F_t(z))]$ as desired. The derivative from the left is handled similarly. To wit, for $\delta > 0$,

$$\begin{aligned}F_t(z) - F_{t-\delta}(z) &= F_{t-\delta}(z - \delta G_{\sigma,a}(F_t(z))) - F_{t-\delta}(z) \\ &= -\delta \cdot DF_{t-\delta}(z)[G_{\sigma,a}(F_t(z))] + O_{T,\epsilon,\sigma}(\delta^2) \\ &= -\delta \cdot DF_t(z)[G_{\sigma,a}(F_t(z))] + O_{T,\epsilon,\sigma}(\delta^2),\end{aligned}$$

where the last equality follows by applying the Cauchy estimates Lemma 2.4.2 to bound the first derivative of $F_t(z) - F_{t-\delta}(z)$. □

The Boolean Case

For background, see [SW97] and [PV13, §2].

Proposition 7.2.5. *Let $\{\mu_t\}$ be a Boolean convolution semigroup. Then $B_{\mu_t}(z) = tG_{\sigma,a}$ for some generalized law σ, with $a = \mu(X)$. We also have*

$$\partial_t F_t(z) = -G_{\sigma,a}(z).$$

Proof. We already know from Theorem 3.5.3 that $B_{\mu_t}(z) = G_{\sigma_t,a_t}(z)$ for some generalized law σ_t and $a_t = \mu_t(X)$. The same argument as in the free case shows that G_{σ_t,a_t} has the form $tG_{\sigma,a}$ where $\sigma = \sigma_1$ and $a = a_1$.

Thus, we have $B_{\mu_t}(z) = tG_{\sigma,a}(z)$ and $F_{\mu_t}(z) = z - tG_{\sigma,a}(z)$, which implies

$$\partial_t F_{\mu_t}(z) = -G_{\sigma,a}(z).$$ □

The Monotone Case

For background on the scalar case, see [Has10a], [Has10b], [HS11b]. For the operator-valued case, see [HS14],[AW16], [Jek17].

Suppose that μ_t is a monotone convolution semigroup. As in the previous cases, we have $\mu_t(X) = ta$ for some self-adjoint $a \in \mathcal{A}$. By Theorem 3.5.3, there exists a generalized law σ_t

$$F_{\mu_t}(z) = z - ta - G_{\sigma_t}(z) = z - G_{\sigma_t,ta}(z).$$

We will show that $t^{-1}G_{\sigma_t,ta}(z)$ converges to some function $G_{\sigma,a}(z)$ as $t \to 0$ along the sequence 2^{-k}, and that this limit satisfies the differential equations given in Theorem 7.1.2. Then using Theorem 5.6.8, we identify $\tilde{G}_{\sigma,a}(z)$ as the monotone cumulant generating function. We begin with some basic estimates which show that $G_{\sigma_t,ta}$ and F_{μ_t} depend continuously on t.

Lemma 7.2.6. *Let $s \leq t$. Then*

1. $\operatorname{Im} G_{\sigma_s}(z) \geq \operatorname{Im} G_{\sigma_t}(z)$ *for* $z \in \mathbb{H}_+(\mathcal{A})$.
2. $\operatorname{rad}(\sigma_s) \leq \operatorname{rad}(\sigma_t)$.
3. $\|G_{\sigma_t}(z) - G_{\sigma_s}(z)\| \leq (t-s)\|\operatorname{Var}_{\mu_1}(1)\|/\epsilon$ *for* $\operatorname{Im} z \geq \epsilon$.

Proof. To prove (1), observe that for $s \leq t$,

$$\operatorname{Im} F_{\mu_t}(z) = \operatorname{Im} F_{\mu_{t-s}} \circ F_{\mu_s}(z) \geq \operatorname{Im} F_{\mu_s}(z),$$

and hence

$$\operatorname{Im} G_{\sigma_t}(z) \leq \operatorname{Im} G_{\sigma_s}(z).$$

Now Corollary 3.4.8 verifies the claim (2) as well as the fact that for $\operatorname{Im} z \geq \epsilon$,

$$\|G_{\sigma_t}(z) - G_{\sigma_s}(z)\| \leq \|\sigma_t(1) - \sigma_s(1)\|/\epsilon = (t-s)\|\operatorname{Var}_{\mu_1}(1)\|/\epsilon.$$

□

As preparation to evaluating the limit of $t^{-1}G_{\sigma_{2t},2ta}$ as $t \to 0$, we show that $(2t)^{-1}G_{\sigma_{2t},2ta}$ is close to $t^{-1}G_{\sigma_t,ta}$.

Lemma 7.2.7. *We have for $\operatorname{Im} z \geq \epsilon$ that*

$$\left\| \frac{1}{2t} G_{\sigma_{2t}}(z) - \frac{1}{t} G_{\sigma_t}(z) \right\| \leq C_{\mu_1,\epsilon} t.$$

Proof. The identity $F_{\mu_{2t}} = F_{\mu_t} \circ F_{\mu_t}$ yields

$$z + 2ta + G_{\sigma_{2t}}(z) = (z + ta + G_{\sigma_t}(z)) + ta + G_{\sigma_t}(z + ta + G_{\sigma_t}(z)).$$

and hence

$$G_{\sigma_{2t}}(z) = G_{\sigma_t}(z)) + G_{\sigma_t}(z + ta + G_{\sigma_t}(z))$$

or

$$G_{\sigma_{2t}}(z) - 2G_{\sigma_t}(z) = G_{\sigma_t}(z + ta + G_{\sigma_t}(z)) - G_{\sigma_t}(z).$$

If we assume that $\operatorname{Im} z \geq \epsilon$, then

$$\begin{aligned}\|G_{\sigma_t}(z + ta + G_{\sigma_t}(z)) - G_{\sigma_t}(z)\| &\leq \frac{\|\sigma_t(1)\|}{\epsilon^2} \|ta + G_{\sigma_t}(z)\| \\ &\leq \frac{\|\sigma_t(1)\|}{\epsilon^2} \left(t\|a\| + \frac{\|\sigma_t(1)\|}{\epsilon^2} \right) \\ &\leq t^2 \frac{\|\operatorname{Var}_{\mu_1}(1)\|}{\epsilon^2} \left(\|a\| + \frac{\|\operatorname{Var}_{\mu_1}(1)\|}{\epsilon^2} \right).\end{aligned}$$

Therefore, dividing by $2t$, we obtain

$$\left\| \frac{1}{2t} G_{\sigma_{2t}}(z) - \frac{1}{t} G_{\sigma_t}(z) \right\| \leq C_{\mu_1,\epsilon} t. \qquad \square$$

Lemma 7.2.8. *Let μ_t be a monotone convolution semigroup and $a = \mu_1(X)$. Then there exists a generalized law σ such that*

$$\partial_t F_{\mu_t}(z) = -G_{\sigma,a}(F_{\mu_t}(z)).$$

Proof. From the previous lemma, we have

$$\left\| 2^{k+1} G_{\sigma_{2^{-k-1}}}(z) - 2^k G_{\sigma_{2^{-k}}}(z) \right\| \leq 2^{-k} C_{\mu_1,\epsilon}.$$

This implies that $\lim_{k\to\infty} 2^k G_{\sigma_{2^{-k}}}(z)$ exists. By Lemma 3.6.4 and Proposition 3.6.5, the limit function has the form $G_\sigma(z)$ for some generalized law σ. We also have $\sigma(1) = \operatorname{Var}_{\mu_1}(1)$. Moreover, we have the explicit rate of convergence

$$\|2^k G_{\sigma_{2^{-k}}}(z) - G_\sigma(z)\| \leq 2^{-k+1} C_{\mu_1,\epsilon}.$$

Now, to prove the differential equation, it suffices to show that

$$F_{\mu_t}(z) - z = \int_0^t -G_{\sigma,a}(F_{\mu_s}(z))\, ds,$$

where the latter is well-defined as a Riemann integral since G_{σ_s} and hence F_{μ_s} and $G_{\sigma,a}(F_{\mu_s})$ are continuous functions of s (uniformly for $\operatorname{Im} z \geq \epsilon$).

In fact, by continuity of F_{μ_t}, it suffices to prove the integral equality when t is a dyadic rational, that is, $t = 2^{-m} n$ for some $m \in \mathbb{Z}$ and $n > 0$. Fix such a value of t and let $k \geq m$. Then

$$\begin{aligned}F_{\mu_t}(z) - z &= \sum_{j=0}^{2^{k-m}n-1} [F_{\mu_{2^{-k}(j+1)}}(z) - F_{\mu_{2^{-k}j}}(z)] \\ &= \sum_{j=1}^{2^{k-m}n} [F_{\mu_{2^{-k}}} \circ F_{\mu_{2^{-k}j}}(z) - F_{\mu_{2^{-k}j}}(z)] \\ &= \sum_{j=1}^{2^{k-m}n} -G_{\sigma_{2^{-k}}, 2^{-k}a} \circ F_{\mu_{2^{-k}j}}(z).\end{aligned}$$

We can replace $G_{\sigma_{2^{-k}}, 2^{-k}a}$ by $2^{-k} G_{\sigma,a}$ at the cost of a small error since for $\operatorname{Im} z \geq \epsilon$, we have

$$\|G_{\sigma_{2^{-k}}}(z) - 2^{-k} G_{\sigma,a}(z)\| \leq 2^{-2k+1} C_{\mu_1,\epsilon}.$$

We can also replace $2^{-k}G_{\sigma,a}(F_{\mu_{2^{-k}j}}(z))$ by $\int_{2^{-k}j}^{2^{-k}(j+1)} G_{\sigma,a}(F_{\mu_s}(z))\,ds$ with an error bounded by

$$\begin{aligned}
&\int_{2^{-k}j}^{2^{-k}(j+1)} \left\| G_{\sigma,a}(F_{\mu_s}(z)) - G_{\sigma,a}(F_{\mu_{2^{-k}j}}(z)) \right\| ds \\
\leq &\frac{\|\sigma(1)\|}{\epsilon^2} \int_{2^{-k}j}^{2^{-k}(j+1)} \left\| F_{\mu_s}(z) - F_{\mu_{2^{-k}j}}(z) \right\| ds \\
= &\frac{\|\sigma(1)\|}{\epsilon^2} \int_{2^{-k}j}^{2^{-k}(j+1)} \left\| G_{\sigma_s}(z) - G_{\sigma_{2^{-k}j}}(z) \right\| ds \\
\leq &\frac{\|\sigma(1)\|^2}{\epsilon^3} 2^{-2k}
\end{aligned}$$

Overall, this yields

$$\begin{aligned}
\left\| F_{\mu_t}(z) - z - \int_0^t G_{\sigma,a}(F_s(z))\,ds \right\| &\leq \sum_{j=0}^{2^{k-m}n} \left\| G_{\sigma_{2^{-k}},2^{-k}a} \circ F_{\mu_{2^{-k}j}}(z) - \int_{2^{-k}j}^{2^{-k}(j+1)} G_{\sigma,a} \circ F_{\mu_s}(z)\,ds \right\| \\
&\leq 2^{(k-m)}n \cdot (2C_{\mu_1,\epsilon} + \|\sigma(1)\|\epsilon^{-3}) \cdot 2^{-2k} \\
&= 2^{-k}t(2C_{\mu_1,\epsilon} + \|\sigma(1)\|\epsilon^{-3}).
\end{aligned}$$

By taking $k \to +\infty$, we obtain equality and hence we have established the differential equation. □

Lemma 7.2.9. *Let μ_t be a monotone convolution semigroup and $a = \mu_1(X)$. Let σ be the generalized law given in the previous lemma. Let $DF_{\mu_t}(z)$ denote $\Delta F_{\mu_t}(z,z)$. Then F_{μ_t} also satisfies the differential equation*

$$\partial_t F_{\mu_t}(z) = -DF_{\mu_t}(z)[G_{\sigma,a}(z)].$$

Proof. Let $t, \delta > 0$. By the preceding arguments, for $\operatorname{Im} z \geq \epsilon$,

$$F_{\mu_\delta}(z) - z = -\int_0^\delta G_{\sigma,a}(F_{\mu_\delta}(z))\,ds = -\delta G_{\sigma,a}(z) + O_{\mu_1,\epsilon}(\delta^2).$$

This implies that for $\operatorname{Im} z \geq \epsilon$ that

$$\begin{aligned}
F_{\mu_{t+\delta}}(z) &= F_{\mu_t} \circ F_{\mu_\delta}(z) - F_{\mu_t}(z) \\
&= DF_{\mu_t}(z)[F_{\mu_\delta}(z) - z] + O_{\mu_1,\epsilon}(t\delta^2) \\
&= -DF_{\mu_t}(z)[G_{\sigma,a}(z)] + O_{\mu_1,\epsilon}((1+t)\delta^2),
\end{aligned}$$

where the error bounds again follow from Lemma 7.2.6 and from applying a priori estimates on the Cauchy-Stieltjes transform to estimate $DF_{\mu_t}(z) = \operatorname{id} - DG_{\sigma_t,at}(z)$. This proves that the derivative from the right of $F_{\mu_t}(z)$ is $-DF_{\mu_t}(z)[G_{\sigma,a}(z)]$. But because $DF_{\mu_t}(z)[G_{\sigma,a}(z)]$ is continuous in t and the error estimates are uniform on a compact time interval, the derivative from the left is also $-DF_{\mu_t}(z)[G_{\sigma,a}(z)]$. □

Lemma 7.2.10. *With the setup above, $t\tilde{G}_{\sigma,a}(z)$ is the monotone cumulant generating function for the law μ_t.*

Proof. Let inv be the fully matricial function $z \mapsto z^{-1}$. Recalling that $D\,\mathrm{inv}(z)[w] = -z^{-1}wz^{-1}$, we have

$$\begin{aligned}\partial_t[\tilde{G}_{\mu_t}(z)] &= \partial_t[\mathrm{inv} \circ F_{\mu_t} \circ \mathrm{inv}(z)] \\ &= -F_{\mu_t}(z^{-1})^{-1} \cdot \partial_t F_{\mu_t}(z^{-1}) \cdot F_{\mu_t}(z^{-1})^{-1} \\ &= \tilde{G}_{\mu_t}(z) \cdot G_{\sigma,a} \circ F_{\mu_t}(z^{-1}) \cdot \tilde{G}_{\mu_t}(z) \\ &= \tilde{G}_{\mu_t}(z) \cdot \tilde{G}_{\sigma,a} \circ \tilde{G}_{\mu_t}(z) \cdot \tilde{G}_{\mu_t}(z).\end{aligned}$$

Let $K_{\mu_t}(z)$ be the monotone cumulant generating function for μ_t. Since $K_{\mu_{s+t}}(z) = K_{\mu_s}(z) + K_{\mu_t}(z)$ and because the moments of μ_t depend continuously on t, we have $K_{\mu_t}(z) = tK_{\mu_1}(z)$. By Theorem 5.6.8, we have

$$\partial_t[\tilde{G}_{\mu_t}] = \tilde{G}_{\mu_t} \cdot K_{\mu_1} \circ \tilde{G}_{\mu_t} \cdot \tilde{G}_{\mu_t}.$$

It follows that as generating functions, we have

$$\tilde{G}_{\sigma,a} = \frac{d}{dt}\Big|_{t=0} \tilde{G}_{\mu_t} = K_{\mu_1}.$$

Thus, $\tilde{G}_{\sigma,a} = K_{\mu_1}$ and hence $t\tilde{G}_{\sigma,a} = K_{\mu_t}$ as desired. □

7.3 Fock Space Realization

We now turn to the converse direction (3) of Theorem 7.1.2, in which we must construct a semigroup μ_t from a constant a and a generalized law σ. We will proceed by constructing a process Y_t with independent increments which realizes the law μ_t, consisting of operators on a Fock space. We also establish the estimates (4) at the end of this section.

The Fock space realization of the law μ_t is due to Glockner, Schürmann, and Speicher [GSS92] in the scalar case and Speicher [Spe98, §4.7] in the operator-valued case. Popa and Vinnikov adapted this construction to the $\mathcal{A}$-valued Boolean case [PV13, Lemma 2.9]. The author earlier studied the monotone case (in fact, without assuming stationary increments) in [Jek17, §6]. Moreover, the construction is well-known in the case of Brownian motion (where the law μ_t is semicircle/Bernoulli/arcsine).

Before dividing into cases, we establish notation for some spaces and Hilbert bimodules.

Definition 7.3.1. Let (Ω, λ) be a measure space and $\mathcal{H}$ a right Hilbert $\mathcal{A}$-module. A *simple function* is a function $f : \Omega \to \mathcal{H}$ of the form $f(\omega) = \sum_{j=1}^n \xi_j \chi_{\Omega_j}(\omega)$, where the Ω_j's are disjoint and measurable and $\xi_j \in \mathcal{H}$. The *$\mathcal{A}$-valued Bochner L^2 space* $L^2(\Omega, \mathcal{H})$ is the completion of the simple functions with respect to the $\mathcal{A}$-valued inner product

$$\langle f, g \rangle = \int_\Omega \langle f(\omega), g(\omega) \rangle_{\mathcal{H}} \, d\lambda.$$

Definition 7.3.2. If (Ω, λ) is a measure space and $\mathcal{A}$ is a C^*-algebra, then a *countably-valued simple function* $f : \Omega \to \mathcal{A}$ is a function of the form $\sum_{j=1}^\infty a_j \chi_{\Omega_j}(\omega)$ where the Ω_j's are disjoint and measurable. We say that such a function f is *essentially bounded* if

$$\|f\|_\infty = \operatorname{esssup} \|f(\omega)\| = \sup_{\lambda(\Omega_j) > 0} \|a_j\|$$

is finite. We define the *Bochner L^∞ space* $L^\infty(\Omega, \mathcal{A})$ as the completion of essentially bounded, countably-valued simple functions with respect to this norm.

Observation 7.3.3. *Let $\mathcal{H}$ be a Hilbert $\mathcal{B}$-$\mathcal{A}$-bimodule and (Ω, λ) a measure space. Then $L^2(\Omega, \mathcal{H})$ is a Hilbert $L^\infty(\Omega, \mathcal{B})$-$\mathcal{A}$-bimodule with the multiplication given by*

$$f(\omega) \cdot g(\omega) = (fg)(\omega)$$

for a countably-valued simple function $f \in L^\infty(\Omega, \mathcal{B})$ and a simple function $g \in L^2(\Omega, \mathcal{H})$.

The verification is straightforward and left to the reader.

For each type of independence, the base for the Fock space will be $\mathcal{N} := L^2(\mathbb{R}_+, \mathcal{M})$, where $\mathbb{R}_+ = (0, +\infty)$ with Lebesgue measure and $\mathcal{M} = \mathcal{A}\langle X \rangle \otimes_\sigma \mathcal{A}$. We remark that if $\mathcal{B}$ is the C^*-algebra generated by $\mathcal{A}\langle X \rangle$ acting on $\mathcal{M}$, then $\mathcal{N}$ is a Hilbert $L^\infty(\mathbb{R}_+, \mathcal{B})$-$\mathcal{A}$-bimodule.

Furthermore, for $0 \leq s < t \leq +\infty$, we denote by $\mathcal{N}_{s,t} := L^2((s,t), \mathcal{M})$. This is a Hilbert $L^\infty((s,t), \mathcal{B})$-$\mathcal{A}$-bimodule. Moreover, there are natural inclusion $\iota_{s,t} : \mathcal{N}_{s,t} \to \mathcal{N}$ and $\iota'_{s,t} : L^\infty((s,t), \mathcal{B}) \to L^\infty(\mathbb{R}_+, \mathcal{B})$ given by extending a function by zero.

The Free Case

Let $\mathcal{N}$ be as above and let $\mathcal{H}$ be the free Fock space generated by $\mathcal{N}$, that is,

$$\mathcal{H} = \mathcal{A}\xi \oplus \bigoplus_{n \geq 1} \underbrace{\mathcal{N} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{N}}_{n}.$$

As in §6.2, we define for $\zeta \in \mathcal{N}$ the *creation operator* $\ell(\zeta)$ by

$$\ell(\zeta)\xi = \zeta\ell(\zeta)[\zeta_1 \otimes \cdots \otimes \zeta_n] \qquad = \zeta \otimes \zeta_1 \otimes \cdots \otimes \zeta_n,$$

for $\zeta_j \in \mathcal{N}$. Then $\ell(\zeta) \in B(\mathcal{H})$ with $\|\ell(\zeta)\| \leq \|\zeta\|_{\mathcal{N}}$ and $\ell(\zeta)^*$ given by the *annihilation operator*

$$\ell(\zeta)^*\xi = 0$$
$$\ell(\zeta)^*[\zeta_1 \otimes \cdots \otimes \zeta_n] = \langle \zeta, \zeta_1 \rangle \zeta_2 \otimes \cdots \otimes \zeta_n.$$

Furthermore, given the left action of $L^\infty(\mathbb{R}_+, \mathcal{B})$ on $\mathcal{N}$, we define a left multiplication operator $\mathfrak{m}(f)$ for $f \in L^\infty(\mathbb{R}_+, \mathcal{B})$ by

$$\mathfrak{m}(f)\xi = 0$$
$$\mathfrak{m}(f)[\zeta_1 \otimes \cdots \otimes \zeta_n] = f\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n.$$

This action is bounded with $\|\mathfrak{m}(f)\| \leq \|f\|$.

Proposition 7.3.4. *Let $a \in \mathcal{A}$ be self-adjoint, let σ be an $\mathcal{A}$-valued generalized law, and let $\mathcal{H}$ be the free Fock space over $\mathcal{N} = L^2(\mathbb{R}_+) \otimes (\mathcal{A}\langle X \rangle \otimes_\sigma \mathcal{A})$ described above. For $t \geq 0$, define*

$$Y_t = \ell((1 \otimes 1)\chi_{(0,t)}) + \ell((1 \otimes 1)\chi_{(0,t)})^* + \mathfrak{m}(X\chi_{(0,t)}) + ta.$$

Then Y_t is a process with freely independent and stationary increments, and the laws μ_t of Y_t form a free convolution semigroup with infinitesimal generator $G_{\sigma,a}$.

For the proof, it will be convenient for $s < t$ to denote

$$Y_{s,t} = Y_t - Y_s = \ell((1 \otimes 1)\chi_{(s,t)}) + \ell((1 \otimes 1)\chi_{(s,t)})^* + \mathfrak{m}(X\chi_{(s,t)}) + (t-s)a.$$

Thus, showing that Y_t has freely independent increments means showing that for $t_0 < \cdots < t_k$, the variables $Y_{0,t_1}, \ldots, Y_{t_{N-1},t_N}$ are freely independent. Rather than computing joint moments, we will prove this by expressing the Hilbert module $\mathcal{H}$ as a free product space.

Let

$$\mathcal{H}_{s,t} = \mathcal{A}\xi \oplus \bigoplus_{n\geq 1} \underbrace{\mathcal{N}_{s,t} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{N}_{s,t}}_{n}.$$

Clearly, for $\mathcal{H}_{s,t}$ has its own creation operators $\ell(\zeta)$ for $\zeta \in \mathcal{N}_{s,t}$ and multiplication operators $\mathfrak{m}(f)$ for $f \in L^\infty((s,t),\mathcal{B})$.

Lemma 7.3.5. *Let $0 < t_1 < \cdots < t_N$. Then $(\mathcal{H},\xi)$ is the free product Hilbert bimodule of $(\mathcal{H}_{0,t_1},\xi)$, …, $(\mathcal{H}_{t_{N-1},t_N},\xi)$, $(\mathcal{H}_{t_N,\infty},\xi)$. Moreover, the inclusion $B(\mathcal{H}_{t_{j-1},t_j}) \to B(\mathcal{H})$ is given by a map ρ_{t_{j-1},t_j} which depends only upon the pair (t_{j-1},t_j), not upon N or the other t_i's.*

Proof. Let us write $\mathcal{H}_{s,t} = \mathcal{A}\xi \oplus \mathcal{K}_{s,t}$, where

$$\mathcal{K}_{s,t} = \bigoplus_{n\geq 1} \underbrace{\mathcal{N}_{s,t} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{N}_{s,t}}_{n}.$$

If we denote $t_0 = 0$ and $t_{N+1} = \infty$, then the free product bimodule is

$$\mathcal{A}\xi \oplus \bigoplus_{n\geq 1} \bigoplus_{\substack{j_1,j_2,\dots,j_n\in[N+1] \\ j_i\neq j_{i+1}}} \mathcal{K}_{t_{j_1-1},t_{j_1}} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{K}_{t_{j_n-1},t_{j_n}}.$$

Substituting in the definition of $\mathcal{K}_{s,t}$, we obtain

$$\mathcal{A}\xi \oplus \bigoplus_{n\geq 1} \bigoplus_{\substack{j_1,j_2,\dots,j_n\in[N+1] \\ j_i\neq j_{i+1}}} \bigoplus_{k_1,\dots k_n\geq 1} \mathcal{N}_{t_{j_1-1},t_{j_1}}^{\otimes_{\mathcal{A}} k_1} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{N}_{t_{j_n-1},t_{j_n}}^{\otimes_{\mathcal{A}} k_n}.$$

Now regrouping the terms by $k = k_1 + \cdots + k_n$, we have

$$\mathcal{A}\xi \oplus \bigoplus_{k\geq 1} \bigoplus_{i_1,\dots,i_k\in[N+1]} \mathcal{N}_{t_{i_1-1},t_i} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{N}_{t_{i_k-1},t_{i_k}}.$$

By the distributive property of tensor products and the fact that $\mathcal{N} = \bigoplus_{i=1}^{N+1} \mathcal{N}_{t_{i-1},t_i}$, this is equal to

$$\mathcal{A}\xi \oplus \bigoplus_{k\geq 1} \underbrace{\mathcal{N} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{N}}_{k} = \mathcal{H}.$$

Thus, $(\mathcal{H},\xi)$ is the free product of $(\mathcal{H}_{0,t_1},\xi)$, …, $(\mathcal{H}_{t_{N-1},t_N},\xi)$, $(\mathcal{H}_{t_N,\infty},\xi)$.

Now consider the inclusion maps of bounded operators. For $s < t$, define $\rho_{s,t} : B(\mathcal{H}_{s,t}) \to B(\mathcal{H})$ by viewing $(\mathcal{H},\xi)$ as the free product of $\mathcal{H}_{0,s}$, $\mathcal{H}_{s,t}$, and $\mathcal{H}_{t,\infty}$. Given any $t_1 < \cdots < t_N$, we claim that the inclusion $B(\mathcal{H}_{t_{j-1},t_j}) \to B(\mathcal{H})$ given by expressing $(\mathcal{H},\xi)$ as the free product of $(\mathcal{H}_{0,t_1},\xi)$, …, $(\mathcal{H}_{t_{N-1},t_N},\xi)$, $(\mathcal{H}_{t_N,\infty},\xi)$ is the same as ρ_{t_{j-1},t_j}. To prove this, we note that by the above argument, $\mathcal{H}_{0,t_{j-1}}$ can be identified with the free product of $\mathcal{H}_{t_{i-1},t_i}$ for $i < j$ and similarly $\mathcal{H}_{t_j,\infty}$ can be identified with the free product of $\mathcal{H}_{t_{i-1},t_i}$ for $i > j$. Then we invoke the associativity properties of the free product. □

Lemma 7.3.6. *Let $0 \leq s < t \leq +\infty$. Let $\rho_{s,t} : B(\mathcal{H}_{s,t}) \to B(\mathcal{H})$ be the inclusion given by the free product construction. Then the following diagrams commute:*

$$\begin{array}{ccc} L^2((s,t),\mathcal{M}) & \xrightarrow{\iota_{s,t}} & L^2(\mathbb{R}_+,\mathcal{M}) \\ \downarrow \ell & & \downarrow \ell \\ B(\mathcal{H}_{s,t}) & \xrightarrow{\rho_{s,t}} & B(\mathcal{H}). \end{array}$$

$$\begin{array}{ccc} L^\infty((s,t),\mathcal{B}) & \xrightarrow{\iota'_{s,t}} & L^\infty(\mathbb{R}_+,\mathcal{B}) \\ \downarrow \mathfrak{m} & & \downarrow \mathfrak{m} \\ B(\mathcal{H}_{s,t}) & \xrightarrow{\rho_{s,t}} & B(\mathcal{H}). \end{array}$$

In other words, for $\zeta \in \mathcal{N}_{s,t}$, we have $\rho_{s,t}(\ell(\zeta)) = \ell(\iota_{s,t}(\zeta))$ and for $f \in L^\infty((s,t),\mathcal{B})$, we have $\rho_{s,t}(\mathfrak{m}(f)) = \mathfrak{m}(\iota'_{s,t}(f))$.

Proof. Let us fix s and t, and denote $\mathcal{N}_1 = \mathcal{N}_{0,s}$, $\mathcal{N}_2 = \mathcal{N}_{s,t}$, and $\mathcal{N}_3 = \mathcal{N}_{t,\infty}$. Define the notations $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ and $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ similarly.

For the first property, choose $\zeta \in \mathcal{N}_{s,t}$ and we will show that $\rho_{s,t}(\ell(\zeta))$ and $\ell(\iota_{s,t}(\zeta))$ act the same on simple tensors $\zeta_1 \otimes \cdots \otimes \zeta_n$ (by convention, when $n = 0$, we interpret this expression as ξ). By linearity, it suffices to consider the case where each ζ_j is either in $\mathcal{N}_{0,s}$, $\mathcal{N}_{s,t}$, or $\mathcal{N}_{t,\infty}$. Clearly,

$$\ell(\iota_{s,t}(\zeta))[\zeta_1 \otimes \cdots \otimes \zeta_n] = \zeta \otimes \zeta_1 \otimes \cdots \otimes \zeta_n,$$

where $\iota_{s,t}(\zeta)$ is identified with ζ on the right hand side. On the other hand, to understand the action of $\rho_{s,t}(\ell(\zeta))$, we write the simple tensor $\zeta_1 \otimes \cdots \otimes \zeta_n$ in terms of the free product decomposition as

$$(\zeta_1 \otimes \cdots \otimes \zeta_{n_1}) \otimes \cdots \otimes (\zeta_{n_{k-1}+1} \otimes \cdots \otimes \zeta_{n_k}),$$

where $0 = n_0 < n_1 < \cdots < n_k = n$, the terms $\zeta_{n_{j-1}+1}, \dots, \zeta_{n_j}$ all come from the same subspace $\mathcal{N}_{i_j}$ with $i_{j+1} \neq i_j$. Note that $\zeta_{n_{j-1}+1} \otimes \cdots \otimes \zeta_{n_j} \in \mathcal{K}_{i_j}$. Now if $i_1 \neq 2$, then we have

$$\begin{aligned} &\rho_{s,t}(\ell(\zeta))[(\zeta_1 \otimes \cdots \otimes \zeta_{n_1}) \otimes \cdots \otimes (\zeta_{n_{k-1}+1} \otimes \cdots \otimes \zeta_{n_k})] \\ =&\ell(\zeta)\xi \otimes (\zeta_1 \otimes \cdots \otimes \zeta_{n_1}) \otimes \cdots \otimes (\zeta_{n_{k-1}+1} \otimes \cdots \otimes \zeta_{n_k}) \\ =&\zeta \otimes (\zeta_1 \otimes \cdots \otimes \zeta_{n_1}) \otimes \cdots \otimes (\zeta_{n_{k-1}+1} \otimes \cdots \otimes \zeta_{n_k}) \end{aligned}$$

as desired. On the other hand, if $i_1 = 2$, then we have

$$\begin{aligned} &\rho_{s,t}(\ell(\zeta))[(\zeta_1 \otimes \cdots \otimes \zeta_{n_1}) \otimes \cdots \otimes (\zeta_{n_{k-1}+1} \otimes \cdots \otimes \zeta_{n_k})] \\ =&\ell(\zeta)[\zeta_1 \otimes \cdots \otimes \zeta_{n_1}] \otimes \cdots \otimes (\zeta_{n_{k-1}+1} \otimes \cdots \otimes \zeta_{n_k}) \\ =&(\zeta \otimes \zeta_1 \otimes \cdots \otimes \zeta_{n_1}) \otimes \cdots \otimes (\zeta_{n_{k-1}+1} \otimes \cdots \otimes \zeta_{n_k}) \end{aligned}$$

as desired.

For the second property, let us again compare the action of $\rho_{s,t}(\mathfrak{m}(f))$ and $\mathfrak{m}(\iota'_{s,t}(f))$ on a simple tensor $\zeta_1, \dots, \zeta_n$ and let n_j and i_j be as above. In the case where $i_1 \neq 2$, then we have $f\zeta_1 = 0$ and hence

$$\mathfrak{m}(\iota'_{s,t}(f))[\zeta_1 \otimes \cdots \otimes \zeta_n] = 0$$

while

$$\begin{aligned} &\rho_{s,t}(\mathfrak{m}(f))[(\zeta_1 \otimes \cdots \otimes \zeta_{n_1}) \otimes \cdots \otimes (\zeta_{n_{k-1}+1} \otimes \cdots \otimes \zeta_{n_k})] \\ =&\mathfrak{m}(f)\xi \otimes (\zeta_1 \otimes \cdots \otimes \zeta_{n_1}) \otimes \cdots \otimes (\zeta_{n_{k-1}+1} \otimes \cdots \otimes \zeta_{n_k}) = 0. \end{aligned}$$

On the other hand, if $i_1 = 2$, then we have

$$\mathfrak{m}(\iota'_{s,t}(f))[\zeta_1 \otimes \cdots \otimes \zeta_n] = b\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n,$$

while

$$\begin{aligned} &\rho_{s,t}(\mathfrak{m}(f))[(\zeta_1 \otimes \cdots \otimes \zeta_{n_1}) \otimes \cdots \otimes (\zeta_{n_{k-1}+1} \otimes \cdots \otimes \zeta_{n_k})] \\ =&\mathfrak{m}(f)[\zeta_1 \otimes \cdots \otimes \zeta_{n_1}] \otimes \cdots \otimes (\zeta_{n_{k-1}+1} \otimes \cdots \otimes \zeta_{n_k}) \\ =&(f\zeta_1 \otimes \cdots \otimes \zeta_{n_1}) \otimes \cdots \otimes (\zeta_{n_{k-1}+1} \otimes \cdots \otimes \zeta_{n_k}). \end{aligned}$$

□

Proof of Proposition 7.3.4. First, we claim that the process Y_t has freely independent increments. To see this, note that by the previous lemma, we have

$$Y_{s,t} = \rho_{s,t}\big(\ell(1 \otimes 1) + \ell(1 \otimes 1)^* + \mathfrak{m}(X) + (t-s)a\big)$$

and hence for $0 < t_1 < \cdots < t_N$, the variables Y_{0,t_1}, Y_{t_1,t_2}, ..., Y_{t_{N-1},t_N} are freely independent.

Second, we claim that the increments are stationary, meaning that the law of $Y_{s,t}$ only depends on $t-s$. To see this, note that $L^2((s,t))$ is isomorphic to $L^2((0,t-s))$ by a translation. This leads to an isomorphism of $\mathcal{H}_{s,t} \cong \mathcal{H}_{0,t-s}$ which respects the creation, annihilation, and multiplication operators. Hence, $Y_{s,t}$ has the same law as $Y_{0,t-s}$.

Third, we claim that the law μ_t of Y_t forms a free convolution semigroup. Indeed, $\mu_{s+t} = \mu_s \boxplus \mu_t$ because $Y_{s+t} = Y_s + Y_{s,s+t}$, where $Y_{s,s+t} \sim Y_t$. Also, the mean $\mu_t(X) = ta$ which is clearly continuous.

Finally, we claim that $G_{\sigma,a}$ is the infinitesimal generator of this semigroup. We proved in the last section that an infinitesimal generator $G_{\sigma',a'}$ exists and that

$$-G_{\sigma',a'} = \frac{d}{dt}\Big|_{t=0} F_{\mu_t}.$$

Thus, to show that $G_{\sigma,a} = G_{\sigma',a'}$, it suffices to show that the time-derivative of F_{μ_t} at $t=0$ is $-G_{\sigma,a}$. Let us use the abbreviated notation $\ell_t = \ell(\chi_{(0,t)} \otimes 1)$ and $\mathfrak{m}_t = \mathfrak{m}(\chi_{(0,t)} \otimes X)$, and note that $\|\ell_t\| \leq t^{1/2}\|\sigma(1)\|$ and $\|\mathfrak{m}_t\| \leq \operatorname{rad}(\sigma)$. Now for $\operatorname{Im} z \geq \epsilon$, we have from the Taylor-Taylor expansion of inv that

$$\begin{aligned}(z - \mathfrak{m}_t - \ell_t - \ell_t^* - ta)^{-1} &= (z-\mathfrak{m}_t)^{-1} + (z-\mathfrak{m}_t)^{-1}(\ell_t + \ell_t^* + ta)(z-\mathfrak{m}_t)^{-1} \\ &\quad + (z-\mathfrak{m}_t)^{-1}(\ell_t+\ell_t^*)(z-\mathfrak{m}_t)^{-1}(\ell_t+\ell_t^*)(z-\mathfrak{m}_t)^{-1} + O_\epsilon(t^{3/2}),\end{aligned}$$

where the equality holds for sufficiently small t since $\|\ell_t + \ell_t^*\| = O(t^{1/2})$ and $\|ta\| = O(t)$. In order to compute $\langle \xi, (z - Y_t)^{-1}\xi\rangle$, we first observe that because $\mathfrak{m}_t|_{\mathcal{A}\xi} = 0$,

$$(z-\mathfrak{m}_t)^{-1}\xi = z^{-1}\xi = \xi z^{-1},$$

hence

$$\langle \xi, (z-\mathfrak{m}_t)^{-1}\xi\rangle = z^{-1}.$$

For the next term, we have

$$\langle \xi, (z-\mathfrak{m}_t)^{-1}(\ell_t+\ell_t^*+ta)(z-\mathfrak{m}_t)^{-1}\xi\rangle = \langle \xi, z^{-1}(\ell_t+\ell_t^*+ta)z^{-1}\xi\rangle = t \cdot z^{-1}az^{-1}.$$

Finally,

$$\begin{aligned}&\langle \xi, (z-\mathfrak{m}_t)^{-1}(\ell_t+\ell_t^*)(z-\mathfrak{m}_t)^{-1}(\ell_t+\ell_t^*)(z-\mathfrak{m}_t)^{-1}\xi\rangle \\ &\quad = \langle (\ell_t+\ell_t^*)(z^*-\mathfrak{m}_t)^{-1}\xi, (z-\mathfrak{m}_t)^{-1}(\ell_t+\ell_t^*)z^{-1}\xi\rangle \\ &\quad = \langle (\ell_t+\ell_t^*)\xi(z^*)^{-1}, (z-\mathfrak{m}_t)^{-1}(\ell_t+\ell_t^*)\xi z^{-1}\rangle \\ &\quad = z^{-1}\langle \ell_t\xi, (z-\mathfrak{m}_t)^{-1}\ell_t\xi\rangle z^{-1} \\ &\quad = z^{-1}\langle (1\otimes 1)\chi_{(0,t)}, (z-\chi_{(0,t)}\otimes X)^{-1}(1\otimes 1)\chi_{(0,t)}\rangle_{L^2(\mathbb{R}_+,\mathcal{M})} z^{-1} \\ &\quad = t \cdot z^{-1}\sigma((z-X)^{-1})z^{-1}.\end{aligned}$$

Altogether,

$$\begin{aligned}G_{\mu_t}(z) = \langle \xi, (z-Y_t)^{-1}\xi\rangle &= z^{-1} + t\cdot z^{-1}az^{-1} + t\cdot z^{-1}\sigma((z-X)^{-1})z^{-1} + O_\epsilon(t^{3/2}) \\ &= z^{-1} + t\cdot z^{-1}G_{\sigma,a}(z)z^{-1} + O_\epsilon(t^{3/2}).\end{aligned}$$

Hence, again using the Taylor-Taylor expansion of inv,

$$F_{\mu_t}(z) = z - tG_{\sigma,a}(z) + O_\epsilon(t^{3/2}),$$

so that the derivative of F_{μ_t} at $t = 0$ is $-G_{\sigma,a}$ as desired. □

The Boolean Case

Let $\mathcal{N} = L^2(\mathbb{R}_+, \mathcal{M})$ as above, and let $\mathcal{H}$ be the Boolean Fock space generated by $\mathcal{N}$, that is,

$$\mathcal{H} = \mathcal{A}\xi \oplus \mathcal{N}.$$

As in §6.2, we define for $\zeta \in \mathcal{N}$ the *creation operator* $\ell(\zeta)$ by

$$\ell(\zeta)\xi = \zeta \ell(\zeta)|_{\mathcal{N}} = 0.$$

Then $\ell(\zeta) \in B(\mathcal{H})$ with $\|\ell(\zeta)\| \leq \|\zeta\|_{\mathcal{N}}$ and $\ell(\zeta)^*$ given by the *annihilation operator*

$$\begin{aligned} \ell(\zeta)^*\xi &= 0 \\ \ell(\zeta)^*\zeta' &= \langle \zeta, \zeta' \rangle \xi. \end{aligned}$$

Furthermore, given the left action of $L^\infty(\mathbb{R}_+, \mathcal{B})$ on $\mathcal{N}$, we can define a left multiplication operator $\mathfrak{m}(f)$ for $f \in L^\infty(\mathbb{R}_+, \mathcal{B})$ by

$$\begin{aligned} \mathfrak{m}(f)\xi &= 0 \\ \mathfrak{m}(f)\zeta &= f\zeta \text{ for } \zeta \in \mathcal{N}. \end{aligned}$$

This action is bounded with $\|\mathfrak{m}(f)\| \leq \|f\|$. We aim to prove the following.

Proposition 7.3.7. *Let $a \in \mathcal{A}$ be self-adjoint, let σ be an $\mathcal{A}$-valued generalized law, and let $\mathcal{H}$ be the Boolean Fock space over $\mathcal{N} = L^2(\mathbb{R}_+) \otimes (\mathcal{A}\langle X \rangle \otimes_\sigma \mathcal{A})$ described above. Let $P \in B(\mathcal{H})$ denote the projection onto $\mathcal{A}\xi$. For $t \geq 0$, define*

$$Y_t = \ell((1 \otimes 1)\chi_{(0,t)}) + \ell((1 \otimes 1)\chi_{(0,t)})^* + \mathfrak{m}(X\chi_{(0,t)}) + taP.$$

Then Y_t is a process with Boolean independent and stationary increments, and the laws μ_t of Y_t form a free convolution semigroup with infinitesimal generator $G_{\sigma,a}$.

The outline of the proof is the same as in the free case, and the individual steps are easier, so we will leave some details to the reader. For $s < t$, we denote

$$Y_{s,t} = Y_t - Y_s = \ell((1 \otimes 1)\chi_{(s,t)}) + \ell((1 \otimes 1)\chi_{(s,t)})^* + \mathfrak{m}(X\chi_{(s,t)}) + (t-s)aP.$$

We next express $\mathcal{H}$ as a Boolean product space in order to show that $Y_{0,t_1}, \dots, Y_{t_{N-1},t_N}$ are Boolean independent for $t_0 < \dots < t_k$.

For $0 \leq s < t \leq +\infty$, we will denote by $\mathcal{N}_{s,t}$ the space $L^2((s,t)) \otimes \mathcal{M}$. This space is a left module over $L^\infty((s,t)) \otimes \mathcal{A}\langle X \rangle$. We also denote $\mathcal{H}_{s,t} = \mathcal{A}\xi \oplus \mathcal{N}_{s,t}$, where ξ is an $\mathcal{A}$-central unit vector.

Lemma 7.3.8. *Let $0 < t_1 < \dots < t_N$. Then $(\mathcal{H}, \xi)$ is the Boolean product Hilbert bimodule of $(\mathcal{H}_{0,t_1}, \xi)$, ..., $(\mathcal{H}_{t_{N-1},t_N}, \xi)$, $(\mathcal{H}_{t_N,\infty}, \xi)$. Moreover, the inclusion $B(\mathcal{H}_{t_{j-1},t_j}) \to B(\mathcal{H})$ is given by a map ρ_{t_{j-1},t_j} which depends only upon the pair (t_{j-1}, t_j), not upon N or the other t_i's.*

Proof. Given $0 < t_1 < \cdots < t_N$, we have

$$\mathcal{N} = \mathcal{N}_{0,t_1} \oplus \mathcal{N}_{t_1,t_2} \oplus \cdots \oplus \mathcal{N}_{t_{N-1},t_N} \oplus \mathcal{N}_{t_N,\infty}.$$

From this, it is is immediate that $(\mathcal{H}, \xi)$ is the Boolean product space of $(\mathcal{H}_{t_{j-1},t_j}, \xi)$ for $j = 1$ to $N + 1$, where we denote $t_0 = 0$ and $t_{N+1} = +\infty$.

Let $\rho_{s,t} : B(\mathcal{H}_{s,t}) \to B(\mathcal{H})$ be the inclusion given by viewing $\mathcal{H}$ as the Boolean product of $(\mathcal{H}_{0,s}, \xi)$, $(\mathcal{H}_{s,t}, \xi)$, and $(\mathcal{H}_{t,+\infty}, \xi)$. Explicitly, $\rho_{s,t}(x)$ is given by viewing $\mathcal{H} = \mathcal{H}_{s,t} \oplus \mathcal{N}_{0,s} \oplus \mathcal{N}_{t,+\infty}$ and setting

$$\rho_{s,t}(x)|_{\mathcal{H}_{s,t}} = x, \qquad \rho_{s,t}(x)|_{\mathcal{H}_{s,t}^\perp} = 0.$$

It straightforward to check that inclusion map $B(\mathcal{H}_{t_{j-1},t_j}) \to B(\mathcal{H})$ for a given choice of $0 < t_1 < \cdots < t_N$ agrees with the map ρ_{t_{j-1},t_j} defined above. □

Lemma 7.3.9. *Let $0 \leq s < t \leq +\infty$. Let $\rho_{s,t} : B(\mathcal{H}_{s,t}) \to B(\mathcal{H})$ be the inclusion given by the Boolean product construction. Then for $\zeta \in \mathcal{N}_{s,t}$, we have $\rho_{s,t}(\ell(\zeta)) = \ell(\iota_{s,t}(\zeta))$ and for $f \in L^\infty((s,t), \mathcal{B})$, we have $\rho_{s,t}(\mathfrak{m}(f)) = \mathfrak{m}(\iota'_{s,t}(f))$. Also, if $P_{s,t} \in B(\mathcal{H}_{s,t})$ is the projection onto $\mathcal{A}\xi$, then we have $\rho_{s,t}(P_{s,t}) = P$.*

The verification is straightforward, especially after knowing the free case, so we leave it to the reader.

Proof of Proposition 7.3.7. Ro show that Y_t has Boolean independent increments, observe that

$$Y_{s,t} = \rho_{s,t}\big(\ell(1 \otimes 1) + \ell(1 \otimes 1)^* + \mathfrak{m}(X) + (t-s)P_{s,t}a\big)$$

and hence for $0 < t_1 < \cdots < t_N$, the variables Y_{0,t_1}, Y_{t_1,t_2}, …, Y_{t_{N-1},t_N} are Boolean independent.

To show that the increments are stationary, one proceeds as in the free case by arguing that time-translation produces an isomorphism $\mathcal{H}_{s,t} \cong \mathcal{H}_{0,t-s}$ which respects the creation, annihilation, and multiplication operators. From this, one concludes that the law μ_t of Y_t produces a Boolean convolution semigroup.

To show that the infinitesimal generator of this semigroup equals $G_{\sigma,a}$, it suffices to show that $\frac{d}{dt}|_{t=0} F_{\mu_t} = -G_{\sigma,a}$. We have for $\operatorname{Im} z \geq \epsilon$ that

$$\begin{aligned}(z - \mathfrak{m}_t - \ell_t - \ell_t^* - ta)^{-1} &= (z - \mathfrak{m}_t)^{-1} + (z - \mathfrak{m}_t)^{-1}(\ell_t + \ell_t^* + taP)(z - \mathfrak{m}_t)^{-1} \\ &\quad + (z - \mathfrak{m}_t)^{-1}(\ell_t + \ell_t^*)(z - \mathfrak{m}_t)^{-1}(\ell_t + \ell_t^*)(z - \mathfrak{m}_t)^{-1} + O_\epsilon(t^{3/2}).\end{aligned}$$

Arguing just as in the free case, we obtain

$$\langle \xi, (z - Y_t)^{-1}\xi \rangle = \langle \xi, (z - \mathfrak{m}_t - \ell_t - \ell_t^* - ta)^{-1}\xi \rangle = z^{-1} + t \cdot z^{-1} G_{\sigma,a}(z) z^{-1} + O_\epsilon(t^{3/2}),$$

and hence

$$F_{\mu_t}(z) = z - tG_{\sigma,a}(z) + O_\epsilon(t^{3/2}),$$

which completes the proof. □

The Monotone Case

Let $\mathcal{C} := C([0,+\infty],\mathcal{A}) \subseteq L^\infty(\mathbb{R}_+,\mathcal{A})$. We define a sesquilinear form $I : \mathcal{N} \times \mathcal{N} \to \mathcal{C}$ by

$$I(f,g)(t) = \int_t^{+\infty} \langle f(s), g(s)\rangle_{\mathcal{M}}\, ds = \langle f, \chi_{(t,+\infty)} g\rangle_{\mathcal{N}}.$$

The verification that $I(f,g)$ is in $C([0,+\infty],\mathcal{A})$ is straightforward, and we also remark that

$$\|I(f,f)\|_{L^\infty(\mathbb{R}_+,\mathcal{A})} = \|f\|_{\mathcal{N}}^2$$

Now we define

$$\widehat{\mathcal{N}} = \mathcal{N} \otimes_I \mathcal{C}$$

as the completion of the algebraic tensor product with respect to the pre-inner product given on simple tensors by

$$\langle f \otimes \phi, g \otimes \psi\rangle = \phi^* I(f,g)\psi.$$

Positivity of this pre-inner product follows from a similar argument as we used for tensor products of Hilbert bimodules in §1.3 and for the operator-valued GNS construction in §1.4. Moreover, $\widehat{\mathcal{N}}$ is a Hilbert $L^\infty(\mathbb{R}_+,\mathcal{B})$-$\mathcal{C}$-bimodule.

We can define a map $\mathcal{N} \to \widehat{\mathcal{N}}$ by $f \mapsto f \otimes 1$. Although the two spaces are right Hilbert modules over different algebras, this map is isometric (and hence an embedding of Banach spaces) because $\|I(f,f)\|_{L^\infty(\mathbb{R}_+,\mathcal{A})} = \|f\|_{\mathcal{N}}^2$.

We define the *monotone Fock space generated by* $\mathcal{N}$ as

$$\mathcal{H} = \mathcal{A} \oplus \bigoplus_{n\geq 1} \underbrace{\widehat{\mathcal{N}} \otimes_{L^\infty(\mathbb{R}_+,\mathcal{A})} \cdots \otimes_{\mathcal{C}} \widehat{\mathcal{N}}}_{n-1} \otimes_{\mathcal{C}} \mathcal{N}.$$

For $\zeta \in \mathcal{N}$, we define the *creation operator* $\ell(\zeta)$ by

$$\ell(\zeta)\xi = \zeta$$
$$\ell(\zeta)[\widehat{\zeta}_1 \otimes \cdots \otimes \widehat{\zeta}_{n-1} \otimes \zeta_n] = (\zeta \otimes 1) \otimes \widehat{\zeta}_1 \otimes \cdots \otimes \widehat{\zeta}_{n-1} \otimes \zeta_n],$$

where $\widehat{\zeta}_j \in \widehat{\mathcal{N}}$. One can check that $\|\ell(\zeta)\| \leq \|\zeta\|_{\mathcal{N}}$ and that the adjoint is given by the *annihilation operator*

$$\ell(\zeta)^*\xi = 0$$
$$\ell(\zeta)^*\zeta_1 = \langle \zeta, \zeta_1\rangle_{\mathcal{N}}\xi$$
$$\ell(\zeta)^*[\widehat{\zeta}_1 \otimes \cdots \otimes \widehat{\zeta}_{n-1} \otimes \zeta_n] = \langle \zeta \otimes 1, \widehat{\zeta}_1\rangle_{\widehat{\mathcal{N}}} \widehat{\zeta}_2 \otimes \cdots \otimes \widehat{\zeta}_{n-1} \otimes \zeta_n.$$

Moreover, for $f \in L^\infty(\mathbb{R}_+,\mathcal{B})$, we define the multiplication operator $\mathfrak{m}(f)$ to act by zero on $\mathcal{A}\xi$ and by left multiplication of the first coordinate by f on each of the tensor powers which make up $\mathcal{H}$. One can check that $\|\mathfrak{m}(f)\| \leq \|f\|$. We aim to prove the following.

Proposition 7.3.10. *Let σ be an $\mathcal{A}$-valued generalized law, let a be a self-adjoint element of $\mathcal{A}$. Let $\mathcal{H}$ be as above. Let $\chi_{(s,t)}$ be the indicator function of (s,t), let $\phi_{s,t}(x) = \int_x^\infty \chi_{s,t}$, let $P \in B(\mathcal{H})$ be the projection onto $\mathcal{A}\xi$, and let*

$$Q_{s,t} = (t-s)P + \mathfrak{m}(\phi_{s,t}).$$

Let $Y_t \in B(\mathcal{H})$ be given by

$$Y_t = \ell((1\otimes 1)\chi_{(0,t)}) + \ell((1\otimes 1)\chi_{(0,t)}) + \mathfrak{m}(X\chi_{(0,t)}) + aT_{0,t}$$

Then Y_t is a process with monotone independent and stationary increments. The laws μ_t of Y_t form a monotone convolution semigroup with infinitesimal generator $G_{\sigma,a}$.

Observe that since $\chi_{(0,s)}+\chi_{(s,t)} = \chi_{(0,t)}$ a.e., we have $\phi_{0,s}+\phi_{s,t} = \phi_{0,t}$ and hence $T_{0,s}+Q_{s,t} = T_{0,t}$. This implies that

$$Y_{s,t} := Y_t - Y_s = \ell((1\otimes 1)\chi_{(0,t)}) + \ell((1\otimes 1)\chi_{(0,t)}) + \mathfrak{m}(X\chi_{(0,t)}) + aT_{0,t}.$$

As before, we show that Y_{0,t_1}, $\dots$, Y_{0,t_N} are monotone independent by expressing $\mathcal{H}$ as a monotone product of subspaces $\mathcal{H}_{s,t}$. There is more to do than in the free and Boolean cases because of the more complicated structure of $\mathcal{H}$.

We define $\mathcal{H}_{s,t}$ in the same way as $\mathcal{H}$ except using the interval $[s,t]$ rather than $[0,+\infty]$. Explicitly, let $\mathcal{C}_{s,t} = C([s,t],\mathcal{A})$ and define $I_{s,t} : \mathcal{N}_{s,t}\times\mathcal{N}_{s,t} \to \mathcal{C}_{s,t}$ by $I_{s,t}(f,g)(x) = \int_x^t \langle f(y), g(y)\rangle_{\mathcal{M}}\, dy$ and let

$$\widehat{\mathcal{N}}_{s,t} := \mathcal{N}_{s,t} \otimes_{I_{s,t}} \mathcal{C}_{s,t}$$

Then define $\mathcal{H}_{s,t}$ similarly to $\mathcal{H}$ except using $\mathcal{C}_{s,t}$ rather than $\mathcal{C}$. The space $\mathcal{H}_{s,t}$ has similarly defined creation, annihilation, and multiplication operators.

Note that there is an inclusion $\varepsilon_{s,t} : \mathcal{C}_{s,t} \to \mathcal{C}$ given by extending a function to be constant on $[0,s]$ and on $[t,+\infty]$. Together with the inclusion $\iota_{s,t} : \mathcal{N}_{s,t} \to \mathcal{N}$, this defines an isometric injection

$$\widehat{\iota}_{s,t} : \widehat{\mathcal{N}}_{s,t} \to \widehat{\mathcal{N}}.$$

The image $\widehat{\iota}_{s,t}(\widehat{\mathcal{N}}_{s,t})$ is a left $L^\infty(\mathbb{R}_+,\mathcal{B})$ module. Moreover, $\widehat{\iota}_{s,t}(\widehat{\mathcal{N}}_{s,t})$ is a right $\varepsilon_{s,t}(\mathcal{C}_{s,t})$ submodule of $\widehat{\mathcal{N}}$ and the inclusion $\widehat{\iota}_{s,t}$ is an $L^\infty((s,t),\mathcal{B})$-$\mathcal{C}_{s,t}$-bimodule map, where $L^\infty((s,t),\mathcal{B})$ is viewed as a subspace of $L^\infty(\mathbb{R}_+,\mathcal{B})$ using the map $\iota_{s,t}'$.

Henceforth, we will identify $\widehat{\mathcal{N}}_{s,t}$ as a subspace of $\widehat{\mathcal{N}}$. We also denote by $\widehat{\mathcal{N}}_{s,t}\mathcal{C}$ the closed right $\mathcal{C}$-submodule which it generates. Now given $0 = t_0 < t_1 < \cdots < t_N < t_{N+1} = +\infty$, the subspaces $\widehat{\mathcal{N}}_{t_{j-1},t_j}$ are orthogonal in $\widehat{\mathcal{N}}$ because $I(f,g) = 0$ if f and g in $L^2(\mathbb{R}_+,\mathcal{M})$ have disjoint support. Moreover, because the spaces $\mathcal{N}_{t_{j-1},t_j}$ span $\mathcal{N}$, we have

$$\widehat{\mathcal{N}} = \bigoplus_{j=1}^{N+1} \widehat{\mathcal{N}}_{t_{j-1},t_j}\mathcal{C}.$$

Proceeding as in the free case, we will use this direct sum decomposition and the distributive property of tensor products to expand $\mathcal{H}$ into a monotone product space. This computation relies on the following lemma.

Lemma 7.3.11. *Let $s < t \leq s' < t'$. Then*

$$\begin{aligned}
\widehat{\mathcal{N}}_{s,t}\mathcal{C} \otimes_{\mathcal{C}} \widehat{\mathcal{N}}_{s',t'}\mathcal{C} &= 0 \\
\widehat{\mathcal{N}}_{s,t}\mathcal{C} \otimes_{\mathcal{C}} \mathcal{N}_{s',t'} &= 0, \\
\widehat{\mathcal{N}}_{s',t'}\mathcal{C} \otimes_{\mathcal{C}} \widehat{\mathcal{N}}_{s,t}\mathcal{C} &\cong \mathcal{N}_{s',t'} \otimes_{\mathcal{A}} \widehat{\mathcal{N}}_{s,t}\mathcal{C} \\
\widehat{\mathcal{N}}_{s',t'}\mathcal{C} \otimes_{\mathcal{C}} \mathcal{N}_{s',t'} &\cong \mathcal{N}_{s',t'} \otimes_{\mathcal{A}} \mathcal{N}_{s,t} \\
\widehat{\mathcal{N}}_{s,t}\mathcal{C} \otimes_{\mathcal{C}} \widehat{\mathcal{N}}_{s,t}\mathcal{C} &\cong \widehat{\mathcal{N}}_{s,t} \otimes_{\mathcal{C}_{s,t}} \widehat{\mathcal{N}}_{s,t}\mathcal{C} \\
\widehat{\mathcal{N}}_{s,t}\mathcal{C} \otimes_{\mathcal{C}} \mathcal{N}_{s,t} &\cong \widehat{\mathcal{N}}_{s,t} \otimes_{\mathcal{C}_{s,t}} \mathcal{N}_{s,t},
\end{aligned}$$

where the isomorphisms in the last four lines are given by the natural map defined explicitly below. In the fifth and sixth lines, the left action of $\mathcal{C}_{s,t}$ on $\widehat{\mathcal{N}}_{s,t}$ is defined by viewing $\mathcal{C}_{s,t} \subseteq L^\infty((s,t),\mathcal{B})$ or equivalently $\mathcal{C}_{s,t} \subseteq L^\infty(\mathbb{R}_+,\mathcal{B})$ by extension by zero.

Proof. Consider the first claim. Recalling the definition of $\widehat{\mathcal{N}}_{s,t}$, we rewrite the tensor product as

$$(\mathcal{N}_{s,t} \otimes_{I_{s,t}} \mathcal{C}_{s,t})\mathcal{C} \otimes_{\mathcal{C}} (\mathcal{N}_{s',t'} \otimes_{I_{s',t'}} \mathcal{C}_{s',t'})\mathcal{C}.$$

Consider a simple tensor $(f_1 \otimes c_1)c_1' \otimes (f_2 \otimes c_2)c_2'$. The inner product of this element with itself is given by

$$(c_2')^* c_2^* I_{s',t'}\Big(f_2, (c_1')^* c_1^* I_{s,t}(f_1, f_1) c_1 c_1' f_2\Big) c_2 c_2' = (c_2 c_2')^* I_{s',t'}\Big((c_1 c_1')f_2, I_{s,t}(f_1, f_1)(c_1 c_1')f_2\Big)(c_2 c_2').$$

Now $I_{s,t}(f_1, f_1)$ is identically zero on $[t, +\infty)$, while f_2 is supported in $[s', t'] \subseteq [t, +\infty)$. Therefore, $I_{s,t}(f_1, f_1)(c_1 c_1')f_2 = 0$. This shows that the simple tensors are zero and hence the whole space $\widehat{\mathcal{N}}_{s,t}\mathcal{C} \otimes_{\mathcal{C}} \widehat{\mathcal{N}}_{s,t'}\mathcal{C}$ is zero. The proof of the second claim is similar.

For the third claim, we want to define a map

$$w : \mathcal{N}_{s',t'} \otimes_{\mathcal{A}} \widehat{\mathcal{N}}_{s,t}\mathcal{C} \to \widehat{\mathcal{N}}_{s',t'}\mathcal{C} \otimes_{\mathcal{C}} \widehat{\mathcal{N}}_{s,t}\mathcal{C}$$

by

$$f \otimes_{\mathcal{A}} gc \mapsto (f \otimes 1) \otimes_{\mathcal{C}} gc,$$

where $f \in \mathcal{N}_{s,t}$, $g \in \widehat{\mathcal{N}}_{s',t'}$, and $c \in \mathcal{C}$. Consider two such elements $f_1 \otimes g_1 c_1$ and $f_2 \otimes g_2 c_2$. Then

$$\langle f_1 \otimes_{\mathcal{A}} g_1 c_1, f_2 \otimes_{\mathcal{A}} g_2 c_2 \rangle = c_1^* \langle g_1, \langle f_1, f_2 \rangle_{\mathcal{N}_{s,t}} g_2 \rangle_{\widehat{\mathcal{N}}_{s',t'}} c_2^*,$$

while on the other hand the inner product between their desired images under w is given by

$$\langle (f_1 \otimes 1) \otimes_{\mathcal{C}} g_1 c_1, (f_2 \otimes 1) \otimes_{\mathcal{C}} g_2 c_2 \rangle = c_1^* \langle g_1, I(f_1, f_2) g_2 \rangle_{\widehat{\mathcal{N}}_{s',t'}} c_2^*.$$

But since f_1 and f_2 are support in $[s, t]$, the function $I(f_1, f_2)$ is constant on $(0, s)$ and equal to $\langle f_1, f_2 \rangle_{\mathcal{N}_{s,t}} \in \mathcal{A}$. The functions g_1 and g_2 are supported in $[0, s]$ and hence we can replace $I(f_1, f_2)$ by $\langle f_1, f_2 \rangle_{\mathcal{N}_{s,t}}$. Thus, the two inner products are the same. This shows that the map w is well-defined and isometric.

It is also clearly an $L^\infty(\mathbb{R}_+, \mathcal{B})$-$\mathcal{C}$-bimodule map. It only remains to show that w is surjective. For this purpose, it suffices to show that the image contains simple tensors $(f \otimes c) \otimes_{\mathcal{C}} gc'$ for $f \in \mathcal{N}_{s',t'}$, $c, c' \in \mathcal{C}$ and $g \in \widehat{\mathcal{N}}_{s,t}$. But

$$(f \otimes c) \otimes_{\mathcal{C}} gc' = (f \otimes 1) \otimes_{\mathcal{C}} cgc' = w(f \otimes_{\mathcal{A}} (cg)c').$$

The proof of the fourth claim is similar.

For the fifth claim, we want to define a map

$$w : \widehat{\mathcal{N}}_{s,t} \otimes_{\mathcal{C}_{s,t}} \widehat{\mathcal{N}}_{s,t}\mathcal{C} \to \widehat{\mathcal{N}}_{s,t}\mathcal{C} \otimes_{\mathcal{C}} \widehat{\mathcal{N}}_{s,t}\mathcal{C}$$

by

$$f \otimes_{\mathcal{C}_{s,t}} gc \mapsto (f \otimes 1) \otimes_{\mathcal{C}} gc.$$

To show that this map is well-defined and isometric, we verify that it preserves the inner product for pairs of simple tensors $f_1 \otimes_{\mathcal{C}_{s,t}} g_1 c_1$ and $f_2 \otimes_{\mathcal{C}_{s,t}} g_2 c_2$. The key point is that when we compute the inner product $\langle f_1, f_2 \rangle_{\widehat{\mathcal{N}}_{s,t}} \in \mathcal{C}$ is multiplied by g_2, which is supported in $[s, t]$. Thus, the values of $\langle f_1, f_2 \rangle_{\widehat{\mathcal{N}}_{s,t}}$ outside $[s, t]$ do not affect the inner product, so the answer is the same whether we view $\langle f_1, f_2 \rangle_{\widehat{\mathcal{N}}_{s,t}}$ as an element of $\mathcal{C}$ which is constant on $[0, s]$ and $[t, +\infty]$ or as an element of $\mathcal{C}_{s,t} \subseteq L^\infty((s, t), \mathcal{A})$ which is then extended to be zero on $[0, s)$ and $(t, +\infty]$. The proof of surjectivity of w is the same as in the previous case, and the proof of the sixth claim is similar. $\square$

Lemma 7.3.12. *Let $0 = t_0 < t_1 < \cdots < t_N < t_{N+1} = +\infty$. The space $(\mathcal{H}, \xi)$ is the monotone product Hilbert bimodule of the spaces $(\mathcal{H}_{t_{j-1},t_j}, \xi)$ for $j = 1, \dots, N+1$. Moreover, the inclusion $B(\mathcal{H}_{t_{j-1},t_j}) \to B(\mathcal{H})$ only depends on t_{j-1} and t_j.*

Proof. As a short-hand, let us denote $\mathcal{H}_j = \mathcal{H}_{t_{j-1},t_j}$ and define similarly the notations $\mathcal{N}_j$ and $\widehat{\mathcal{N}}_j$ and $\mathcal{C}_j$. We substitute the direct sum decompositions

$$\widehat{\mathcal{N}} = \bigoplus_{j=1}^{N+1} \mathcal{N}_j \mathcal{C} \qquad \mathcal{N} = \bigoplus_{j=1}^{N+1} \mathcal{N}_j$$

into the definition of $\mathcal{H}$ and obtain

$$\mathcal{H} = \mathcal{A}\xi \oplus \bigoplus_{n \geq 1} \bigoplus_{j_1, \dots, j_n \in [N+1]} \widehat{\mathcal{N}}_{j_1} \mathcal{C} \otimes_{\mathcal{C}} \cdots \otimes_{\mathcal{C}} \widehat{\mathcal{N}}_{j_{n-1}} \otimes_{\mathcal{C}} \mathcal{N}_{j_n}.$$

Now we apply Lemma 7.3.11 to simplify the tensor product. If any of the above terms has indices $j_k < j_{k+1}$, then it becomes zero. The remaining terms have weakly decreasing sequences of indices. If $j_k > j_{k+1}$, then we can replace $\widehat{\mathcal{N}}_{j_k} \otimes_{\mathcal{C}}$ with $\mathcal{N}_{j_k} \otimes_{\mathcal{A}}$. If $j_k = j_{k+1}$, then we can replace $\widehat{\mathcal{N}}_{j_k} \otimes_{\mathcal{C}}$ with $\widehat{\mathcal{N}}_{j_k} \otimes_{\mathcal{C}_j}$. Therefore, we have

$$\mathcal{H} \cong \mathcal{A}\xi \oplus \bigoplus_{n \geq 1} \bigoplus_{\substack{j_1 > \cdots > j_m \\ \ell_1 + \cdots + \ell_m = n}} \left(\widehat{\mathcal{N}}_{j_1}^{\otimes_{\mathcal{C}_{j_1}} (\ell_1 - 1)} \otimes_{\mathcal{C}_{j_1}} \mathcal{N}_{j_1} \right) \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \left(\widehat{\mathcal{N}}_{j_m}^{\otimes_{\mathcal{C}_{j_m}} (\ell_m - 1)} \otimes_{\mathcal{C}_{j_m}} \mathcal{N}_{j_m} \right),$$

where we have grouped the terms with the same indices together. Now we denote

$$\mathcal{K}_j = \bigoplus_{\ell \geq 1} \widehat{\mathcal{N}}_j^{\otimes_{\mathcal{C}_j} (\ell - 1)} \otimes_{\mathcal{C}_j} \mathcal{N}_j,$$

so that $\mathcal{H}_j = \mathcal{A}\xi \oplus \mathcal{K}_j$. After rearrangement, we have

$$\mathcal{H} = \mathcal{A}\xi \oplus \bigoplus_{m \geq 1} \bigoplus_{j_1 > \cdots > j_m} \mathcal{K}_{j_1} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{K}_{j_m},$$

which is the monotone product space of the $\mathcal{H}_j$'s.

For the second claim, let $\rho_{s,t}$ be the inclusion $B(\mathcal{H}_{s,t}) \to B(\mathcal{H})$ given by the decomposition of $\mathcal{H}$ as the monotone product of $\mathcal{H}_{0,s}$, $\mathcal{H}_{s,t}$, and $\mathcal{H}_{t,+\infty}$. Then for $t_0, \dots, t_{N+1}$ as above the inclusion $B(\mathcal{H}_{t_{j-1},t_j}) \to B(\mathcal{H})$ is given by ρ_{t_{j-1},t_j}. This is verified by a straightforward associativity argument similar to the free case. □

Lemma 7.3.13. *Let $0 \leq s < t \leq +\infty$. Let $\rho_{s,t} : B(\mathcal{H}_{s,t}) \to B(\mathcal{H})$ be the inclusion given by the monotone product construction. Then for $\zeta \in \mathcal{N}_{s,t}$, we have $\rho_{s,t}(\ell(\zeta)) = \ell(\iota_{s,t}(\zeta))$ and for $f \in L^\infty((s,t), \mathcal{B})$, we have $\rho_{s,t}(\mathfrak{m}(f)) = \mathfrak{m}(\iota'_{s,t}(f))$.*

Proof. The verification is straightforward casework which combines the type of reasoning used in the free case with Lemma 7.3.11. For instance, suppose that we apply $\rho_{s,t}(\ell(\zeta))$ or $\ell(\iota_{s,t}(\zeta))$ to a simple tensor $\widehat{\zeta}_1 \otimes \cdots \otimes \widehat{\zeta}_{n-1} \otimes \zeta_n$ where $\widehat{\zeta}_j \in \widehat{\mathcal{N}}$ and $\zeta_n \in \mathcal{N}$. Using the decomposition of $\mathcal{N}$ into $\mathcal{N}_1 = \mathcal{N}_{0,s}$ and $\mathcal{N}_2 = \mathcal{N}_{s,t}$ and $\mathcal{N}_3 = \mathcal{N}_{t,+\infty}$, we may assume that each $\widehat{\zeta}_k$ belongs to $\widehat{\mathcal{N}}_{j_k}$. By Lemma 7.3.11, the indices j_k must be decreasing, and the position of $\widehat{\zeta}_1 \otimes \cdots \otimes \widehat{\zeta}_{n-1} \otimes \zeta_n$ in the monotone product decomposition can be seen by grouping the terms with the same index together.

If $\widehat{\zeta}_1$ is from $\widehat{\mathcal{N}}_3$, then $\ell(\iota_{s,t}(\zeta))$ acts by zero because $(\zeta \otimes 1) \otimes \widehat{\zeta}_1 = 0$ by Lemma 7.3.11. Meanwhile, $\rho_{s,t}(\ell(\zeta))$ also acts by zero because $\widehat{\zeta}_1 \otimes \cdots \otimes \widehat{\zeta}_{n-1} \otimes \zeta_n$ lies in a term of the form $\mathcal{K}_3 \otimes_{\mathcal{A}} \dots$ in the monotone product space and the index 3 is greater than the index 2 of the space $\mathcal{H}_2 = \mathcal{H}_{s,t}$.

On the other hand, if $\widehat{\zeta}_1$ is from $\widehat{\mathcal{N}}_2$ or $\widehat{\mathcal{N}}_1$, then we apply similar reasoning as in the free case to show that $\rho_{s,t}(\ell(\zeta))$ and $\ell(\iota_{s,t}(\zeta))$ agree on $\widehat{\zeta}_1 \otimes \cdots \otimes \widehat{\zeta}_{n-1} \otimes \zeta_n$.

The details of the remaining cases are left to the reader. □

Lemma 7.3.14. *Let $P_{s,t} \in B(\mathcal{H}_{s,t})$ be the projection on $\mathcal{A}\xi$. We have $Q_{s,t} = \rho_{s,t}((t-s)P_{s,t} + \mathfrak{m}(\phi_{s,t}|_{(s,t)}))$.*

Proof. We claim that $\rho_{s,t}(P_{s,t}) = P + \mathfrak{m}(\chi_{(0,s)})$. It is straighforward to check that these operators agree on $\mathcal{A}\xi$. Next, using the notation $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{N}_3$ as in the previous proof, consider the action of these operators on a simple tensor $z = \widehat{\zeta}_1 \otimes \cdots \otimes \widehat{\zeta}_{n-1} \otimes \zeta_n$ where $\widehat{\zeta}_k \in \widehat{\mathcal{N}}_{j_k}$ and $\zeta_n \in \mathcal{N}_{j_n}$. If $n_1 = 1$ or $\widehat{\zeta}_1 \in \mathcal{N}_1 = \mathcal{N}_{0,s}$, then z is in the direct summand $\mathcal{K}_{0,s}$ of $\mathcal{H}$ in the product decomposition, which is contained in $(\mathcal{A}\xi \oplus \mathcal{K}_{s,t}) \otimes_{\mathcal{A}} \mathcal{K}_{0,s}$. Thus, the action of $\rho_{s,t}(P_{s,t})$ is defined to be the identity, which is the same as the action of $\mathfrak{m}(\chi_{(s,t)})$. On the other hand, if $\widehat{\zeta}_1 \in \mathcal{N}_{s,t}$ or $\mathcal{N}_{t,+\infty}$, then z is in one of the summands

$$\mathcal{K}_{s,t}, \quad \mathcal{K}_{t,\infty}, \quad \mathcal{K}_{t,\infty} \otimes_{\mathcal{A}} \mathcal{K}_{s,t}.$$

In the first case, $\rho_{s,t}(P_{s,t})$ acts by zero because $P_{s,t}|_{\mathcal{K}_{s,t}} = 0$, while in the other two cases it acts by zero by construction of the monotone product. But the operator $\mathfrak{m}(\chi_{(0,s)})$ also acts by zero.

Therefore, we have

$$\rho_{s,t}((t-s)P_{s,t} + \mathfrak{m}(\phi_{s,t})) = (t-s)P + (t-s)\mathfrak{m}(\chi_{(0,s)}) + \mathfrak{m}(\phi_{s,t} \cdot \chi_{(s,t)}) = (t-s)P + \mathfrak{m}(\phi_{s,t}).$$

□

Proof of Proposition 7.3.10. Let us denote $\ell_{s,t} = \ell((1 \otimes 1)\chi_{(s,t)})$ and $\mathfrak{m}_{s,t} = \mathfrak{m}(X\chi_{(s,t)})$. It follows from the previous lemmas that

$$\begin{aligned} Y_{s,t} &= \ell_{s,t} + \ell_{s,t}^* + \mathfrak{m}_{s,t} + aQ_{s,t} \\ &= \ell_{s,t} + \ell_{s,t}^* + \mathfrak{m}_{s,t} + aQ_{s,t} \\ &= \rho_{s,t}\big(\ell(1 \otimes 1) + \ell(1 \otimes 1) + \mathfrak{m}(X) + a[(t-s)P_{s,t} + \mathfrak{m}(\phi_{s,t})]\big). \end{aligned}$$

Therefore, for $0 = t_0 < \cdots < t_N$, the operators Y_{t_{j-1},t_j} are monotone independent.

Now time-translation furnishes an isomorphism between $\widehat{\mathcal{N}}_{s,t}$ and $\widehat{\mathcal{N}}_{0,t-s}$ and between $\mathcal{N}_{s,t}$ and $\mathcal{N}_{0,t-s}$. Hence, there is an isomorphism $\mathcal{H}_{s,t} \cong \mathcal{H}_{0,t-s}$ which respects the creation, annihilation, and multiplication operators. It follows that $Y_{s,t}$ has the same law as $Y_{0,t-s}$. Thus, Y_t has monotone independent and stationary increments, so that the laws μ_t of Y_t for $t \in \mathbb{R}_+$ form a monotone convolution semigroup.

To show that the infinitesimal generator of this semigroup is $G_{\sigma,a}$, it suffices to differentiate F_{μ_t} at $t = 0$ as in the free and Boolean cases. Letting $\ell_t = \ell_{0,t}$ and $\mathfrak{m}_t = \mathfrak{m}_{0,t}$, we have $\operatorname{Im} z \geq \epsilon$ that

$$\begin{aligned} (z - Y_t)^{-1} = (z - \mathfrak{m}_t)^{-1} &+ (z - \mathfrak{m}_t)^{-1}(\ell_t + \ell_t^* + aT_t)(z - \mathfrak{m}_t)^{-1} \\ &+ (z - \mathfrak{m}_t)^{-1}\ell_t^*(z - \mathfrak{m}_t)^{-1}\ell_t(z - \mathfrak{m}_t)^{-1} + O_\epsilon(t^{3/2}). \end{aligned}$$

From this, we compute that

$$G_{\mu_t}(z) = \langle \xi, (z - Y_t)^{-1} \xi \rangle = z^{-1} + t \cdot z^{-1} G_{\sigma,a}(z) z^{-1} + O_\epsilon(t^{3/2}),$$

and hence $F_{\mu_t}(z) = z - tG_{\sigma,a}(z) + O_\epsilon(t^{3/2})$ as desired. □

Radius Estimates and Concluding Remarks

We have now completed the proof of the converse direction (3) in Theorem 7.1.2. To finish the proof of the theorem, we must establish the radius estimates (4).

Proof of Theorem 7.1.2 (4). The estimate $t\|a\| \leq \operatorname{rad}(\mu_t)$ is immediate since $ta = \mu_t(X)$.

Consider the second estimate, $\operatorname{rad}(\sigma) \leq C \operatorname{rad}(\mu_t)$. For the free case, we showed in Theorem 4.7.2 that R_{μ_t} is fully matricial on $B(0, (3 - 2\sqrt{2}) / \operatorname{rad}(\mu_t))$. Since $R_{\mu_t} = t\tilde{G}_{\sigma,a}$, this shows that $\operatorname{rad}(\sigma) \leq \operatorname{rad}(\mu_t)/(3 - 2\sqrt{2})$. For the Boolean case, we have $\operatorname{rad}(\sigma) \leq 2 \operatorname{rad}(\mu_t)$ by Theorem 3.5.3. For the monotone case, recall that if $F_{\mu_t}(z) = z - G_{\sigma_t, ta}(z)$, then $G_{\sigma,a} = \lim_{t \to 0} t^{-1} G_{\sigma_t, ta}$. As in the Boolean case, $\operatorname{rad}(\sigma_t) \leq 2 \operatorname{rad}(\mu_t)$. Also, $\operatorname{rad}(\sigma_t)$ is increasing in t. Thus, $\operatorname{rad}(\sigma) \leq 2 \operatorname{rad}(\mu_t)$.

The third estimate $\operatorname{rad}(\mu_t) \leq \operatorname{rad}(\sigma) + 2\sqrt{t\|\sigma(1)\|} + t\|a\|$ follows from the Fock space construction. Indeed, the law μ_t is realized by the operator

$$Y_t = \begin{cases} \mathfrak{m}(X\chi_{(0,t)}) + \ell((1 \otimes 1)\chi_{(0,t)}) + \ell((1 \otimes 1)\chi_{(0,t)})^* + ta, & \text{free case,} \\ \mathfrak{m}(X\chi_{(0,t)}) + \ell((1 \otimes 1)\chi_{(0,t)}) + \ell((1 \otimes 1)\chi_{(0,t)})^* + taP, & \text{Boolean case,} \\ \mathfrak{m}(X\chi_{(0,t)}) + \ell((1 \otimes 1)\chi_{(0,t)}) + \ell((1 \otimes 1)\chi_{(0,t)})^* + aT_{0,t}, & \text{monotone case.} \end{cases}$$

We have

$$\begin{aligned} \|\mathfrak{m}(X\chi_{(0,t)})\| &\leq \operatorname{rad}(\sigma), \\ \|\ell((1 \otimes 1)\chi_{(0,t)})\| &\leq \|(1 \otimes 1)\chi_{(0,t)}\| = \sqrt{t\|\sigma(1)\|}, \end{aligned}$$

and the last term can be estimated by $t\|a\|$. □

There are a few more facts about the Fock space construction that deserve comment. First, although there is no distinction between monotone and anti-monotone convolution semigroups, there is a distinction for processes with independent increments. Given a generalized law σ and a self-adjoint constant a, one can construct a process with anti-monotone independent and stationary increments with $G_{\sigma,a}$ as its infinitesimal generator. The construction is almost identical to the monotone case with the following changes.

1. The operator I is given by

   $$I(f,g)(t) = \int_0^t \langle f(s), g(s) \rangle_{\mathcal{M}} \, ds = \langle f, \chi_{(t,+\infty)} g \rangle_{\mathcal{N}}.$$

   rather than the integral from t to $+\infty$.

2. The function $\phi_{s,t}$ is replaced by $\int_0^x \chi_{(s,t)}(y)\, dy$.

3. In Lemma 7.3.11, for the first four statements, the roles of (s,t) and (s',t') are reversed.

4. These changes are carried through the proofs justifying the construction.

Now if we restrict our attention to a finite time interval $[0, T]$, the monotone and anti-monotone constructions are related in a natural way through time-reversal. Indeed, the space $\mathcal{H}_{s,t}^{\text{monotone}}$ can be mapped isomorphically onto $\mathcal{H}_{T-t,T-s}^{\text{anti-monotone}}$ by time reversal, and this respects the creation and annihilation operators. Under this isomorphism, $Y_{s,t}^{\text{monotone}}$ corresponds to $Y_{T-t,T-s}^{\text{anti-monotone}}$. More generally, if $(Y_t)_{t\in[0,T]}$ is a process with monotone independent and stationary increments, then $(Y_T - Y_{T-t})_{t\in[0,T]}$ is a process with anti-monotone independent and stationary increments.

The difference between the monotone and anti-monotone versions is also reflected in the two distinct differential equations for monotone convolution semigroups. In the monotone case, we can derive one of the equations as follows. From time t to time $t+\epsilon$, we observe a change from Y_t to $Y_{t+\epsilon}$ and hence from μ_t to $\mu_t \rhd \mu_\epsilon$. This leads to a change from F_{μ_t} to $F_{\mu_t} \circ F_{\mu_\epsilon}$, so that

$$F_{\mu_{t+\epsilon}}(z) - F_{\mu_t}(z) \approx DF_{\mu_t}(z)[F_{\mu_\epsilon}(z) - z] \approx \epsilon \cdot DF_{\mu_t}(z)[G_{\sigma,a}(z)].$$

Meanwhile, in the anti-monotone case, we observe a change from μ_t to $\mu_t \lhd \mu_\epsilon$ or from F_{μ_t} to $F_{\mu_\epsilon} \circ F_{\mu_t}$. Thus,

$$F_{\mu_{t+\epsilon}}(z) - F_{\mu_t}(z) \approx (F_{\mu_\epsilon} - \text{id}) \circ F_{\mu_t}(z) \approx \epsilon \cdot G_{\sigma,a} \circ F_{\mu_t}(z).$$

We also remark that the "drift term" corresponding to the constant a was handled in different ways for each type of independence. The operator $Y_{s,t}$ contained the term $(t-s)a$ in the free case, $(t-s)aP_\xi$ in the Boolean case, and $Q_{s,t} = (t-s)aP_\xi + a\mathfrak{m}(\phi_{s,t})$ in the (anti-)monotone case. However, there is a unifying point of view. One can modify the construction of the preceding sections to replace $\mathcal{M} = \mathcal{A}\langle X\rangle \otimes_\sigma \mathcal{A}$ by $\mathcal{M}' = \mathcal{A}\langle X\rangle \otimes_\sigma \mathcal{A} \oplus \mathcal{A}h$ for an $\mathcal{A}$-central unit vector h. Then considering the vector $h\chi_{(s,t)} \in L^2(\mathbb{R}_+, \mathcal{M}')$, we have

$$\ell(h\chi_{(s,t)})^* \ell(h\chi_{(s,t)}) = \begin{cases} (t-s), & \text{free case,} \\ (t-s)P_\xi, & \text{Boolean case,} \\ Q_{s,t}, & \text{(anti-)monotone case.} \end{cases} \tag{7.3.1}$$

7.4 The Central Limit Theorem Revisited

Brownian Motion

Recall that the semicircle, arcsine, and Bernoulli laws with mean zero and variance $\eta : \mathcal{A} \to \mathcal{A}$ were given by the cumulant generating function $K_{\nu_\eta}(z) = \eta(z)$ in the free, Boolean, and monotone cases respectively. Thus, $\nu_{t\eta}$ forms a convolution semigroup. The infinitesimal generator is the function $\eta(z^{-1})$ which is equal to the Cauchy-Stieltjes transform of the generalized law $\sigma_\eta : \mathcal{A}\langle X\rangle \to \mathcal{A}$ given by $p(X) \mapsto \eta(p(0))$.

The Fock space construction in the preceding section defines a process with independent and stationary increments with the law $\nu_{t\eta}$, which we call an *$\mathcal{A}$-valued free/Boolean/monotone/anti-monotone Brownian motion*. We leave it as an exercise for the reader to relate this to the earlier version of the Fock space construction in §6.2 which realized the semicircle, Bernoulli, and arcsine laws.

More generally, if we use the function $G_{\sigma_\eta,a}$ as the infinitesimal generator, we obtain a convolution semigroup of laws $\nu_{t\eta,ta}$ with cumulant generating function $\eta(z^{-1}) + a$. We call the law $\nu_{t\eta,ta}$ the *semicircle/Bernoulli/arcsine law of mean a and variance η* and we call the corresponding process with independent increments an *$\mathcal{A}$-valued free/Boolean/monotone/anti-monotone Brownian motion with drift.*

Coupling and the Central Limit Theorem

For a convolution semigroup μ_t with infinitesimal generator $G_{\sigma,a}$, we realized the μ_t by an operator on a Fock space, namely

$$Y_t = \begin{cases} \mathfrak{m}(X\chi_{(0,t)}) + \ell((1 \otimes 1)\chi_{(0,t)}) + \ell((1 \otimes 1)\chi_{(0,t)})^* + ta, & \text{free case,} \\ \mathfrak{m}(X\chi_{(0,t)}) + \ell((1 \otimes 1)\chi_{(0,t)}) + \ell((1 \otimes 1)\chi_{(0,t)})^* + taP, & \text{Boolean case,} \\ \mathfrak{m}(X\chi_{(0,t)}) + \ell((1 \otimes 1)\chi_{(0,t)}) + \ell((1 \otimes 1)\chi_{(0,t)})^* + aT_{0,t}, & \text{monotone case.} \end{cases}$$

Let

$$Z_t = \begin{cases} \ell((1 \otimes 1)\chi_{(0,t)}) + \ell((1 \otimes 1)\chi_{(0,t)})^* + ta, & \text{free case,} \\ \ell((1 \otimes 1)\chi_{(0,t)}) + \ell((1 \otimes 1)\chi_{(0,t)})^* + taP, & \text{Boolean case,} \\ \ell((1 \otimes 1)\chi_{(0,t)}) + \ell((1 \otimes 1)\chi_{(0,t)})^* + aT_{0,t}, & \text{monotone case.} \end{cases}$$

Then we claim that Z_t is a Brownian motion with drift, and thus we have formed a "coupling" between our original process and the Brownian motion.

The claim about Z_t is verified with the same arguments we used for Y_t. One checks first that $Z_t - Z_s$ is contained in $\rho_{s,t}(B(\mathcal{H}_{s,t}))$, so that Z_t is a process with independent increments, and stationarity follows from time translation. Moreover, for $\operatorname{Im} z \geq \epsilon$,

$$\langle \xi, (z - Z_t)^{-1}\xi \rangle = z^{-1} + t \cdot z^{-1}(\sigma(z^{-1}) + a)z^{-1} + O_\epsilon(t^{3/2}),$$

so that the infinitesimal generator of the semigroup of the laws of Z_t is precisely $G_{\sigma_\eta,a}$, where $\eta = \sigma|_{\mathcal{A}} = \operatorname{Var}_{\mu_1}$. Thus, the law of Z_t is $\nu_{t\eta,ta}$. Intuitively, the law of Z_t is obtained by replacing $\mathfrak{m}(X\chi_{(0,t)})$ by zero which amounts to replacing σ by σ_η.

Clearly, we have

$$\|Y_t - Z_t\| = \|\mathfrak{m}(X\chi_{(0,t)})\| = \operatorname{rad}(\sigma).$$

And more generally, $\|Y_{s,t} - Z_{s,t}\| \leq \operatorname{rad}(\sigma)$. Therefore, we have the the following result:

Theorem 7.4.1. *For free, Boolean, monotone, and anti-monotone independence, the following holds. Let μ_t be a convolution semigroup with infinitesimal generator $G_{\sigma,a}$. Let $\eta = \sigma|_{\mathcal{A}}$ and let $\nu_{t\eta,ta}$ be the semicircle/Bernoulli/arcsine law. Then there exists a non-commutative probability space $(\mathcal{B}, E)$ and operators Y_t and Z_t in $\mathcal{B}$ such that*

1. *Y_t is a process with independent and stationary increments with law μ_t.*
2. *Z_t is a Brownian motion with drift with $Z_t \sim \nu_{t\eta,ta}$.*
3. *Letting $Y_{s,t} = Y_t - Y_s$ and $Z_{s,t} = Z_t - Z_s$, we have $\|Y_{s,t} - Z_{s,t}\| \leq \operatorname{rad}(\sigma)$.*
4. *Given $0 = t_0 < t_1 < \cdots < t_N$, the algebras $\mathcal{A}\langle Y_{t_{j-1},t_j}, Z_{t_{j-1},t_j}\rangle_0$ for $j = 1, \ldots, N$ are independent.*

Proof. We let $\mathcal{H}$ be the Fock space, $\mathcal{B} = B(\mathcal{H})$, and Y_t and Z_t be as above. Because $\overline{\mathcal{B}\xi} = \mathcal{H}$, we see that $E = \langle \xi, \cdot \xi \rangle$ gives a faithful representation, so that $(\mathcal{B}, E)$ is an $\mathcal{A}$-valued probability space. We have already explained (1), (2), and (3), and claim (4) follows because $\mathcal{A}\langle Y_{t_{j-1},t_j}, Z_{t_{j-1},t_j}\rangle_0$ is contained in $\rho_{t_{j-1},t_j}(B(\mathcal{H}_{t_{j-1},t_j}))$. □

As a consequence, we have a version of the central limit theorem which describes the behavior of μ_t as $t \to \infty$. Since μ_N is the N-fold convolution power of μ_1, it follows from Theorem 6.5.5 (in the case $a = 0$) that if $\lambda_N = \operatorname{dil}_{N^{-1/2}}(\mu_N)$ and if $f \in C^3_{nc}(\mathcal{A}, R)$ for $R > (2 + N^{-1/2}) \operatorname{rad}(\mu_1)$, then

$$\|\lambda_N(f) - \nu_\eta(f)\| \leq \|f\|_{3,R},$$

where $\|f\|_{3,R}$ was a certain norm of $\Delta^3 f$, described in Definition 6.5.1. The coupling we have constructed here allows us to prove a similar estimate for all real t using the *first* derivatives of f.

Corollary 7.4.2. *For free, Boolean, or monotone independence, let μ_t be a convolution semigroup with mean ta and variance $t\eta$, let $\lambda_t = \operatorname{dil}_{t^{-1/2}}(\mu_t)$, and let $\nu_{\eta,a}$ be the semicircle/Bernoulli/arcsine law. Suppose that $t > 0$ and $R > 0$ such that*

$$R \geq t^{-1/2} \operatorname{rad}(\sigma) + 2\sqrt{\|\eta(1)\|} + t^{1/2}\|a\|$$

and that $f \in C^1_{nc}(\mathcal{A}, R)$. Then

$$\left\|\lambda_t(f) - \nu_{\eta,t^{1/2}a}(f)\right\| \leq t^{-1/2} \operatorname{rad}(\sigma)\|f\|_{1,R}.$$

In particular,

$$\sup_{\operatorname{Im} z \geq \epsilon} \left\|G_{\lambda_t}(z) - G_{\nu_{\eta,t^{1/2}a}}\right\| \leq \frac{\operatorname{rad}(\sigma)}{t^{1/2}\epsilon^2} \leq \frac{C \operatorname{rad}(\mu_1)}{t^{1/2}\epsilon^2}.$$

Proof. Let Y_t and Z_t be as above. Then $t^{-1/2}Y_t$ has the law λ_t and $t^{-1/2}Z_t$ has the law $\nu_{\eta,t^{1/2}a}$. Note that $\|t^{-1/2}Y_t\|$ and $\|t^{-1/2}Z_t\| \leq R$. Thus, for $f \in \mathcal{A}\langle X\rangle$, we have

$$\begin{aligned}
\left\|\lambda_t(f) - \nu_{\eta,t^{1/2}a}(f)\right\| &= \left\|E[f(t^{-1/2}Y_t) - f(t^{-1/2}Z_t)]\right\| \\
&= \left\|E[\Delta f(t^{-1/2}Y_t, t^{-1/2}Z_t)[t^{-1/2}Y_t - t^{-1/2}Z_t]\right\| \\
&\leq \|f\|_{1,R}\|t^{-1/2}Y_t - t^{-1/2}Z_t\| \\
&\leq \|f\|_{1,R} t^{-1/2} \operatorname{rad}(\sigma).
\end{aligned}$$

This proves the first claim for $f \in \mathcal{A}\langle X\rangle$ and this inequality extends to the completion $C^1_{nc}(\mathcal{A}, R)$. The estimate on the Cauchy-Stieltjes transform follows easily as in Proposition 6.5.7. □

7.5 Combinatorial Viewpoint on the Fock Space

In the last section, we defined operators Y_t with laws μ_t, and gave an analytic argument that the semigroup μ_t has $G_{\sigma,a}$ as its infinitesimal generator. Now we will give an alternative combinatorial proof by showing that $t\tilde{G}_{\sigma,a}$ is the cumulant generating function of μ_t. Although this argument is redundant at this point, we include it in order to clarify the relationship between the Hilbert-bimodule perspective and the combinatorial perspective on independence, as well as to make the connection with §6.2.

The Free Case

Lemma 7.5.1. *Consider the free Fock space on $\mathcal{N} = L^2(\mathbb{R}_+, \mathcal{A}\langle X\rangle \otimes_\sigma \mathcal{A})$ constructed in §7.3. Let*

$$T_j = \ell(\theta_j)^* + \ell(\zeta_j) + \mathfrak{m}(f_j) + a_j \in B(\mathcal{H}),$$

where θ_j, $\zeta_j \in \mathcal{N}$, $f_j \in L^\infty(\mathbb{R}_+, \mathcal{B})$, and $a_j \in \mathcal{A}$. Let K_n be the free cumulant with respect to the expectation given by the vacuum vector ξ. Then

$$K_n(T_1, \dots, T_n) = \begin{cases} a_j, & n = 1, \\ \langle \theta_1, f_2 \dots f_{n-1}\zeta_n\rangle_{\mathcal{N}}, & n > 1 \end{cases}$$

Proof. Note that θ_j, ζ_j, f_j, and a_j are uniquely determined by T_j. For every n and every $T_1, \dots, T_n$ as above, let $\Lambda_n[T_1, \dots, T_n]$ be the expression that we want to show is equal to $K_n[T_1, \dots, T_n]$. Note that the maps ℓ and $\mathfrak{m}$ are $\mathcal{A}$-$\mathcal{A}$-bimodule maps and hence the operators $\ell(\theta)^* + \ell(\zeta) + \mathfrak{m}(f) + a$ are an $\mathcal{A}$-$\mathcal{A}$-bimodule. Moreover, Λ_n is multilinear, and in fact $\mathcal{A}$-quasi-multilinear, in the T_j's. Thus, for a non-crossing partition π, we can define λ_π, the composition of the λ_n's according to the partition π. Because the cumulants are uniquely determined by the moment-cumulant formula, to prove our claim it suffices to show that

$$\langle \xi, T_1 \dots T_n \xi \rangle = \sum_{\pi \in \mathcal{NC}(n)} \Lambda_\pi(T_1, \dots, T_n).$$

To evaluate $\langle \xi, T_1 \dots T_n \xi \rangle$, we substitute $T_j = \ell(\theta_j)^* + \ell(\zeta_j) + \mathfrak{m}(f_j) + a_j$ and expand by multilinearity. The terms thus consist of strings of creation, annihilation and multiplication operators. As in the proof of Proposition 6.2.1, we will enumerate some of these terms using planar partitions and show that the other terms do not contribute to the expectation.

For each $\pi \in \mathcal{NC}(n)$, we define a string of creation, annihilation, and multiplication operators as follows. For each singleton block $\{j\}$ of π, we write a_j in the jth position of the string. For each block $\{j_1, \dots, j_k\}$ with $k > 1$, we write $\ell(\theta_{j_1})^*$ in the j_1 position, $\ell(\zeta_{j_k})$ in the j_k position, and $\mathfrak{m}(f_{j_i})$ in the j_i position for $1 < i < k$. This string is one of the terms in the product.

We claim that all other terms in the product have zero expectation. Because the creation operators map $\mathcal{N}^{\otimes_{\mathcal{A}} n}$ into $\mathcal{N}^{\otimes_{\mathcal{A}} (n+1)}$ and the annihilation operators do the reverse and also kill ξ, the creation and annihilation operators must be paired together in a planar way with an ℓ^* on the left side and an ℓ on the right side of each pairing, or else the string will have expectation zero, as in the proof of Proposition 6.2.1. Let $\tilde{\pi}$ be the planar partition in which these pairs form blocks, and the other elements of $[n]$ are singleton blocks.

Next, note that every multiplication operator $\mathfrak{m}(f_j)$ must be "inside" some creation-annihilation pair; otherwise, the multiplication operator would be applied to a vector in $\mathcal{A}\xi$ which would yield zero. We can form a partition π by joining each index j of a multiplication operator $\mathfrak{m}(f_j)$ to the pair $\tilde{\pi}$ immediately outside it (that is, the greatest block V of $\tilde{\pi}$ such that $\{j\} \succ V$). Then each block of π consists of a creation-annihilation pair together with all the multiplication operators which are immediately inside it or else is a single block corresponding to an element a_j. Thus, our string was produced by a partition π.

To complete the proof, one shows by induction on $|\pi|$ that the expectation of the corresponding string is $\Lambda_\pi(T_1, \dots, T_n)$. Indeed, every partition π must have an interval block $V = \{j, \dots, k\}$. If V is a singleton block, then it corresponds to $a_j \in \mathcal{A}$ and we can remove j from π and multiply T_{j+1} by a_j on the left. If $|V| > 1$, then observe that

$$\ell(\theta_j)^* \mathfrak{m}(f_{j+1}) \dots \mathfrak{m}(f_{k-1}) \ell(\zeta_k) = \langle \theta_j, f_{j+1} \dots f_{k-1} \zeta_k \rangle \in \mathcal{A},$$

and reduce to $\pi \setminus V$. □

Lemma 7.5.2. *For a measurable $\Omega \subseteq \mathbb{R}_+$, let $\mathcal{T}_\Omega$ be set of operators of the form $T = \ell(\theta)^* + \ell(\zeta) + \mathfrak{m}(f) + a$, where θ, ζ, and f are supported in Ω. If $\Omega_1, \dots, \Omega_N$ are disjoint and measurable, then the unital $\mathcal{A}$-algebras generated by $\mathcal{T}_{\Omega_1}, \dots, \mathcal{T}_{\Omega_N}$ are freely independent.*

Proof. By Lemma 7.5.1, the mixed cumulants between $\mathcal{T}_{\Omega_1}, \dots, \mathcal{T}_{\Omega_N}$ vanish. Therefore, by Remark 5.4.7, the algebras they generate are freely independent. □

Lemma 7.5.3. *Let* $Y_{s,t} = \ell((1 \otimes 1)\chi_{(s,t)})^* + \ell((1 \otimes 1)\chi_{(s,t)}) + \mathfrak{m}(X\chi_{(s,t)}) + (t-s)a$ *as in* §*7.3. Then the free cumulants of* $Y_{s,t}$ *are given by*

$$\mathrm{Cum}_n(Y_{s,t})[a_1, \dots, a_{n-1}] = \begin{cases} (t-s)a, & n = 1 \\ (t-s)\sigma(a_1 X a_2 \dots X a_{n-1}), & n > 1. \end{cases}$$

Moreover, if $0 = t_0 < \cdots < t_N$*, then the operators* Y_{t_{j-1},t_j} *are freely independent over* $\mathcal{A}$.

Proof. The first claim about the cumulants of $Y_{s,t}$ follows directly from Lemma 7.5.1. We see that the operators Y_{t_{j-1},t_j} are freely independent by taking $\Omega_j = (t_{j-1}, t_j)$ in Lemma 7.5.2. □

The Boolean Case

Lemma 7.5.4. *Consider the Boolean Fock space on* $\mathcal{N} = L^2(\mathbb{R}_+, \mathcal{A}\langle X \rangle \otimes_\sigma \mathcal{A})$ *constructed in* §*7.3. Let*

$$T_j = \ell(\theta_j)^* + \ell(\zeta_j) + \mathfrak{m}(f_j) + a_j P \in B(\mathcal{H}),$$

where θ_j, $\zeta_j \in \mathcal{N}$, $f_j \in L^\infty(\mathbb{R}_+, \mathcal{B})$, $a_j \in \mathcal{A}$, *and* P *is the projection onto* $\mathcal{A}\xi$. *Let* K_n *be the Boolean cumulant with respect to the expectation given by the vacuum vector* ξ. *Then*

$$K_n(T_1, \dots, T_n) = \begin{cases} a_j, & n = 1, \\ \langle \theta_1, f_2 \dots f_{n-1} \zeta_n \rangle_{\mathcal{N}}, & n > 1 \end{cases}$$

Proof. Note that θ_j, ζ_j, f_j, and a_j are uniquely determined by T_j. For every n and every $T_1, \dots, T_n$ as above, let $\Lambda_n(T_1, \dots, T_n)$ be the expression that we want to show is equal to $K_n(T_1, \dots, T_n)$. Note that λ_n is $\mathcal{A}$-quasi-multilinear. For an interval partition π, let Λ_π be the product of Λ_n's according to π. To prove our claim it suffices to show that

$$\langle \xi, T_1 \dots T_n \xi \rangle = \sum_{\pi \in \mathcal{I}(n)} \Lambda_\pi(T_1, \dots, T_n).$$

To evaluate $\langle \xi, T_1 \dots T_n \xi \rangle$, we substitute $T_j = \ell(\theta_j)^* + \ell(\zeta_j) + \mathfrak{m}(f_j) + a_j$ and expand by multilinearity. The terms thus consist of strings of creation, annihilation and multiplication operators. For each partition $\pi \in \mathcal{I}(n)$, we associate a string of creation, annihilation, and multiplication operators exactly as in the free case. To wit, for each singleton block $\{j\}$ of π, we write a_j in the jth position of the string. For each block $\{j, \dots, k\}$ with $k > j$, we write $\ell(\theta_j)^*$ in the j position, $\ell(\zeta_k)$ in the k position, and $\mathfrak{m}(f_i)$ in the i position for $j < i < k$. The expectation of this string is precisely $\Lambda_\pi(T_1, \dots, T_n)$.

Meanwhile, we claim that the other terms in the product have zero expectation. We have $\ell(\zeta)^*[a\xi] = 0$ as well as $\ell(\zeta)\mathfrak{m}(f)\ell(\theta) = 0$ and $P\mathfrak{m}(f)\ell(\zeta) = 0$. This implies that a creation operator must be applied before we can apply an annihilation operator, or else the string will be zero. We must also apply an annihilation operator between any two creation operators and vice versa. Thus, the creation and annihilation operators must alternate with the rightmost being a creation operator and the leftmost being an annihilation operator. Let us group these operators in creation-annihilation pairs (with the creation operator on the right of each pair and no other creation or annihilation operator intervening).

Since $P\mathfrak{m}(f)\ell(\zeta) = 0$, there cannot be any occurrences of $a_j P$ between a creation-annihilation pair, or else the string will have expectation zero. On the other hand, every multiplication operator $\mathfrak{m}(f_j)$ must occur between a creation-annihilation pair since otherwise it would be applied to a vector in $\mathcal{A}\xi$. Therefore, the only terms that contribute to the expectation are those that arise from interval partitions. □

The next two lemmas follow immediately by the same argument as in the free case.

Lemma 7.5.5. *For a measurable $\Omega \subseteq \mathbb{R}_+$, let $\mathcal{T}_\Omega$ be set of operators of the form $T = \ell(\theta)^* + \ell(\zeta) + \mathfrak{m}(f) + a$, where θ, ζ, and f are supported in Ω. If Ω_1, ..., Ω_N are disjoint and measurable, then the unital $\mathcal{A}$-algebras generated by $\mathcal{T}_{\Omega_1}$, ..., $\mathcal{T}_{\Omega_N}$ are Boolean independent.*

Lemma 7.5.6. *Let $Y_{s,t} = \ell((1 \otimes 1)\chi_{(s,t)})^* + \ell((1 \otimes 1)\chi_{(s,t)}) + \mathfrak{m}(X\chi_{(s,t)}) + (t-s)a$ as in §7.3. Then the Boolean cumulants of $Y_{s,t}$ are given by*

$$\operatorname{Cum}_n(Y_{s,t})[a_1, \dots, a_{n-1}] = \begin{cases} (t-s)a, & n = 1 \\ (t-s)\sigma(a_1 X a_2 \dots X a_{n-1}), & n > 1. \end{cases}$$

Moreover, if $0 = t_0 < \dots < t_N$, then the operators Y_{t_{j-1},t_j} are Boolean independent over $\mathcal{A}$.

The Monotone Case

Consider the monotone Fock space on $\mathcal{N} = L^2(\mathbb{R}_+, \mathcal{A}\langle X \rangle \otimes_\sigma \mathcal{A})$ constructed in §7.3, and recall that $\mathcal{C} = C([0,+\infty], \mathcal{A})$ and $\widehat{\mathcal{N}}$ is $\mathcal{N}$ equipped with the $\mathcal{C}$-valued inner product $\langle f, g \rangle(t) = \int_t^\infty \langle f(s), g(s) \rangle_{\mathcal{A}\langle X \rangle \otimes \mathcal{A}}\, ds$.

Although the monotone cumulants are not as well-behaved, there is still a good combinatorial description for the moments of operators of the form

$$T = \ell(\theta)^* + \ell(\zeta) + \mathfrak{m}(f) + \phi(0)P + \mathfrak{m}(\phi) \in B(\mathcal{H}),$$

where θ, $\zeta \in \mathcal{N}$, $f \in L^\infty(\mathbb{R}_+, \mathcal{B})$, and

$$\phi(t) = \int_t^\infty \psi(s)\, ds$$

for some $\psi \in L^1(\mathbb{R}_+, \mathcal{A})$. Note that this includes the operators $Y_{s,t}$ representing our process with independent and stationary increments.

In this case, the tuple (θ, ζ, f, ψ) is not uniquely determined by T because the possible values for the terms $\mathfrak{m}(f)$ and $\mathfrak{m}(\phi)$ overlap. For the sake of the computation for $Y_{s,t}$, it will be convenient to keep track of the tuple $\tau = (\theta, \zeta, f, \psi)$. If $\mathcal{T}$ is the space of such tuples, then $\mathcal{T}$ is an $\mathcal{A}$-$\mathcal{A}$-bimodule with the action given by

$$a_1 \tau a_2 = (a_2^* \theta a_1^*, a_1 \zeta a_2, a_1 f a_2, a_2 \phi a_2),$$

and the map $\tau \mapsto T$ is an $\mathcal{A}$-$\mathcal{A}$-bimodule map.

We define an $\mathcal{A}$-quasi-multilinear form $\Gamma_n : \mathcal{T}^n \to L^1(\mathbb{R}_+, \mathcal{A})$ by

$$\Gamma_n[\tau_1, \dots, \tau_n](t) = \begin{cases} \psi_1(t), & n = 1 \\ \langle \theta_1(t), f_2(t) \dots f_{n-1}(t) \zeta_n(t) \rangle_{\mathcal{A}\langle X \rangle \otimes_\sigma \mathcal{A}}, & n \geq 2. \end{cases}$$

The right hand side is in $L^1(\mathbb{R}_+, \mathcal{A})$ in the first case because $\psi_1 \in L^1(\mathbb{R}_+, \mathcal{A})$ and in the second case because θ_1 and $f_2 \dots f_{n-1}\zeta_n$ are in $L^2(\mathbb{R}_+^\pi, \mathcal{A})$.

For $\pi \in \mathcal{NC}(n)$, define $\Gamma_\pi[\tau_1, \dots, \tau_n] \in L^1(\mathbb{R}_+^\pi, \mathcal{A})$ by the recursive relation that if $V = \{j+1, \dots, k\}$ is an interval block of π, then for $t \in \mathbb{R}^\pi$,

$$\Gamma_\pi[\tau_1, \dots, \tau_n](t) = \Gamma_{\pi \setminus V}[\tau_1, \dots, \tau_j, \Gamma_{k-j}[\tau_{j+1}, \dots, \tau_k](t_V)\tau_{k+1}, \dots, \tau_n](t|_{\pi \setminus V}),$$

with the convention that if $k = n$, then $\Gamma_{n-j}[\tau_{j+1}, \dots, \tau_n]$ is factored out on the right hand side. This is well-defined for the same reason that Λ_π is well-defined for any sequence of $\mathcal{A}$-quasi-multilinear forms. To prove that this is in $L^1(\mathbb{R}_+^\pi, \mathcal{A})$, one evaluates Γ_π when θ_j, ζ_j, f_j, and ψ_j are simple functions of t and checks that the result is bounded in terms of the L^2 norms of θ_j and ζ_j, the L^∞ norms of f_j, and the L^1 norms of ψ_j. We leave the details to the reader since the computation is similar to the proof below, and at any rate, the claims about $Y_{s,t}$ only use the equality for simple functions.

Lemma 7.5.7. *With the setup and notation as above, we have*

$$\langle \xi, T_1 \dots T_n \xi \rangle = \sum_{\pi \in \mathcal{NC}(n)} \int_{\{t \in \mathbb{R}_+^\pi : t \models \pi\}} \Gamma_\pi[\tau_1, \dots, \tau_n](t)\, dt.$$

Proof. We substitute $T_j = \ell(\theta_j)^* + \ell(\zeta_j) + \mathfrak{m}(f_j) + \phi_j(0)P + \mathfrak{m}(\phi_j)$ where $\phi_j(t) = \int_t^\infty \psi_j(s)\, ds$. Then we expand $\langle \xi, T_1 \dots T_n \xi \rangle$ by multilinearity, resulting in a sum of terms given by strings of $\ell(\theta_j)^*$, $\ell(\zeta_j)$, $\mathfrak{m}(f_j)$, and $\phi_j(0)P + \mathfrak{m}(\phi_j)$.

For each partition π, there is a corresponding term in the sum given as follows: For each singleton block $\{j\}$ of π, the jth letter of the string is $\phi_j(0)P + \mathfrak{m}(\phi_j)$. If j is the lowest index of a block V with $|V| > 1$, then the jth letter of the string is $\ell(\theta_j)^*$. If j is the highest index of a block V with $|V| > 1$, then the jth letter of the string is $\ell(\zeta_j)$. Otherwise, the jth letter is $\mathfrak{m}(f_j)$.

The strings in the expansion of $\langle \xi, T_1 \dots T_n \xi \rangle$ which do not correspond to a non-crossing partition will not contribute to the expectation. The argument is the same as in the free case. In brief, the structure of the Fock space guarantees that the creation and annihilation operators must be paired in a planar way or else the expectation will be zero. Moreover, the multiplication operators $\mathfrak{m}(f_j)$ must occur inside a creation-annihilation pair or else they will multiply an element of $\mathcal{A}\xi$ and thus produce zero.

It remains to evaluate the expectation of each creation-annihilation-multiplication string given by a non-crossing partition π. Similar to the proof of Proposition 6.2.6, this involves to evaluating the integral over $\{t \in \mathbb{R}_+^\pi : t \models \pi\}$ as an iterated integral. We proceed by induction on $|\pi|$, including the base case within the general argument.

Suppose that $\pi \in \mathcal{NC}(n)$ and choose an interval block $V = \{j+1, \dots, k\}$ of π. Let

$$\gamma(t) = \Gamma_{|V|}[\tau_{j+1}, \dots, \tau_k](t)$$
$$\delta(t) = \int_t^\infty \gamma(s)\, ds$$

We claim the substring indexed by $\{j+1, \dots, k\}$ multiplies out to $\delta(0)P + \mathfrak{m}(\delta)$. In the case $|V| = 1$, this holds because

$$\phi_k(0)P + \mathfrak{m}(\phi_k) = \delta(0)P + \mathfrak{m}(\delta_0).$$

On the other hand, if $|V| > 1$, then

$$\ell(\theta_{j+1})^* \mathfrak{m}(f_{j+2}) \dots \mathfrak{m}(f_{k-1}) \ell(\zeta_k) = \langle \theta_{j+1}, f_{j+2} \dots f_{k-1} \zeta_k \rangle_{\widehat{\mathcal{N}}}(0) P + \mathfrak{m}(\langle \theta_{j+1}, f_{j+2} \dots f_{k-1} \zeta_k \rangle_{\widehat{\mathcal{N}}}).$$

If $|\pi| = 1$ (hence $j = 0$ and $k = 0$), this already completes the proof since the expectation of the string evaluates to

$$\delta(0) = \int_0^\infty \Gamma_\pi[\tau_1, \dots, \tau_n].$$

Otherwise, there are two cases.

First, suppose that the block V is outer (that is, minimal with respect to $\prec$). Then the operator $\delta(0)P + \mathfrak{m}(\delta)$ ends up being applied to a vector in $\mathcal{A}\xi$. Thus, we can replace it by the constant $\delta(0) \in \mathcal{A}$ without changing the expectation. Assuming that $k < n$, then applying the induction hypothesis to $\tau_1, \dots, \tau_j, \delta(0)\tau_{k+1}, \dots \tau_n$, the expectation of the string evaluates to

$$\begin{aligned}
&\int_{\{t \models \pi \setminus V\}} \Gamma_{\pi \setminus V}[\tau_1, \dots, \tau_j, \delta(0)\tau_{k+1}, \dots, \tau_n](t)\, dt \\
&= \int_{\{t \models \pi \setminus V\}} \int_{s>0} \Gamma_{\pi \setminus V}[\tau_1, \dots, \tau_j, \Gamma_{|V|}[\tau_{j+1}, \dots, \tau_k]\tau_{k+1}, \dots, \tau_n](t)\, ds\, dt \\
&= \int_{\{t \models \pi\}} \Gamma_\pi[\tau_1, \dots, \tau_n](t)\, dt.
\end{aligned}$$

If $k = n$, one proceeds similarly by multiplying $\delta(0)$ on the right of τ_n instead of the left of τ_{k+1}.

On the other hand, suppose that the block V is not outer. Then the operator $\delta(0)P + \mathfrak{m}(\delta)$ is applied to a vector in $(\mathcal{A}\xi)^\perp$. Thus, we can replace it by $\mathfrak{m}(\delta)$. Let $\tau = (0, 0, \delta, 0)$ and let π' be the partition of $[n - |V| + 1]$ obtained by replacing the block V by a singleton, and then joining this singleton block with the block W which is immediately outside of V in π to form one block W'. Then the expectation of the π string for $\tau_1, \dots, \tau_n$ is the same as the expectation of the π' string for $\tau_1, \dots, \tau_j, \tau, \tau_{k+1}, \dots, \tau_n$. Indeed, in the string for π', the jth element is the multiplication operator $\mathfrak{m}(\delta)$ from τ since this element is one of the elements in the middle of the block W'. So by the induction hypothesis, the expectation is

$$\begin{aligned}
&\int_{\{t \models \pi'\}} \Gamma_{\pi'}[\tau_1, \dots, \tau_j, \tau, \tau_{k+1}, \dots, \tau_n](t)\, dt \\
&= \int_{\{t \models \pi \setminus V\}} \Gamma_{\pi \setminus V}[\tau_1, \dots, \tau_j, \delta(t_W)\tau_{k+1}, \dots, \tau_n](t)\, dt \\
&= \int_{\{t \models \pi \setminus V\}} \int_{s > t_W} \Gamma_{\pi \setminus V}[\tau_1, \dots, \tau_j, \Gamma_{k-j}[\tau_{j+1}, \dots, \tau_k](s)\tau_{k+1}, \dots, \tau_n](t)\, ds\, dt \\
&= \int_{\{t \models \pi\}} \Gamma_\pi[\tau_1, \dots, \tau_n](t)\, dt,
\end{aligned}$$

which completes the proof. □

Lemma 7.5.8. *Fix $s < t$ and let $\mathcal{T}_{s,t}$ be the collection of operators of the form*

$$T = \ell(\theta \cdot \chi_{(s,t)})^* + \ell(\zeta \cdot \chi_{(s,t)}) + \mathfrak{m}(f \cdot \chi_{s,t}) + aQ_{s,t}$$

where $\theta, \zeta \in \mathcal{A}\langle X \rangle \otimes_\sigma \mathcal{A}$, $f \in \mathcal{B}$, $a \in \mathcal{A}$, and $Q_{s,t}$ is given as in Proposition 7.3.10. Let $T_1, \dots, T_n$ be operators in $\mathcal{T}_{s,t}$ given by θ_j, ζ_j, f_j, and a_j. Then their monotone cumulants are given by

$$K_n[T_1, \dots, T_n] = \begin{cases} (t-s)a_1, & n = 1 \\ (t-s)\langle \theta_1, f_2 \dots f_{n-1}\zeta \rangle_{\mathcal{A}\langle X \rangle \otimes_\sigma \mathcal{A}}, & n > 1. \end{cases}$$

Proof. Since s and t are fixed, the tuple $(\theta_j, \zeta_j, f_j, a_j)$ is uniquely determined by T_j in this case. Thus, we may safely define $\Lambda_n[T_1, \dots, T_n]$ to be the right hand side of the identity we want to prove. Note that if $\phi_{s,t}(x) = \int_x^\infty \chi_{(s,t)}(y)\, dy$, then $Q_{s,t} = \phi_{s,t}(0)P + \mathfrak{m}(\phi_{s,t})$. Applying the previous lemma to the tuples $\tau_j = (\theta_j \chi_{(s,t)}, \zeta_j \chi_{(s,t)}, f_j \chi_{(s,t)}, \chi_{(s,t)})$, we have

$$\begin{aligned}\langle \xi, T_1 \dots T_n \xi \rangle &= \sum_{\pi \in \mathcal{NC}(n)} \int_{\{u \in \mathbb{R}_+^\pi : u \models \pi\}} \Gamma_\pi[\tau_1, \dots, \tau_n](u)\, du \\ &= \sum_{\pi \in \mathcal{NC}(n)} \gamma_\pi \Lambda_\pi[T_1, \dots, T_n],\end{aligned}$$

where $\gamma_\pi = |\{u \in [0,1]^\pi : u \models \pi\}|$, because $\Gamma_\pi[\tau_1, \dots, \tau_n]$ is $(t-s)^{-|\pi|}\Lambda_\pi[T_1, \dots, T_n]$ times the indicator function of $\{u \in (s,t)^\pi : u \models \pi\}$. From the moment-cumulant relations, it follows that $K_n[T_1, \dots, T_n] = \Lambda_n[T_1, \dots, T_n]$. □

Lemma 7.5.9. *Let $0 = t_0 < t_1 < \dots < t_N$. The non-unital $\mathcal{A}$-algebras generated by $\mathcal{T}_{t_{j-1},t_j}$ are monotone independent.*

Proof. Consider a string of operators $T_1, \dots, T_n$ where $T_k \in \mathcal{T}_{t_{j_k-1}, t_{j_k}}$ is given by the tuple

$$\tau_k = (\theta_k \chi_{t_{j_k-1}, t_{j_k}}, \zeta_k \chi_{t_{j_k-1}, t_{j_k}}, f_k \chi_{t_{j_k-1}, t_{j_k}}, \chi_{t_{j_k-1}, t_{j_k}}).$$

We will argue that the expectation of $T_1 \dots T_k$ agrees with the expectation of the corresponding string in the monotone product of the algebras, which is given by Lemma 5.4.17. Let V_j be the set of indices k such that $j_k = j$ and let σ be the (not necessarily non-crossing) partition $\{V_1, \dots, V_N\}$. By applying Lemmas 7.5.7 to $\tau_1, \dots, \tau_n$, we obtain

$$\langle \xi, T_1 \dots T_k \xi \rangle = \sum_{\pi \in \mathcal{NC}(n)} \int_{\{t \in \mathbb{R}_+^\pi : t \models \pi\}} \Gamma_\pi[\tau_1, \dots, \tau_n](t)\, dt.$$

Now $\Gamma_\pi[\tau_1, \dots, \tau_n](t)$ is supported on the set of t values where for each index k in a block W, the coordinate t_W is in (t_{j_k-1}, t_{j_k}). In particular, Γ_π vanishes unless each block W is contained in a single V_j, or in other words $\pi \leq \sigma$. Thus, $\pi|_{V_j}$ is defined for each j. Now if W is a block of $\pi|_{V_j}$, then by 7.5.8

$$\Gamma_{|W|}[\tau_k : k \in W](t_W) = \chi_{(t_{j-1}, t_j)}(t_W)(t_j - t_{j-1})^{-1} K_{|W|}[\tau_k : k \in W].$$

Thus, by the inductive construction of Γ_π and K_π, we have

$$\Gamma_\pi[\tau_1, \dots, \tau_n](t) = \chi_{t|_{V_j} \in (t_j - t_{j-1})^{\pi|_{V_j}} \forall j}(t) \prod_{j=1}^N (t_j - t_{j-1})^{-|\pi|_{V_j}|} K_\pi[T_1, \dots, T_n].$$

So overall

$$\begin{aligned}\langle \xi, T_1 \dots T_k \xi \rangle &= \sum_{\pi \in \mathcal{NC}(n)} |\{t \in (t_0, t_1)^{\pi|_{V_1}} \times (t_{N-1}, t_N)^{\pi|_{V_N}} : t \models \pi\}| \prod_{j=1}^N (t_j - t_{j-1})^{-|\pi|_{V_j}|} K_\pi[T_1, \dots, T_n] \\ &= \sum_{\pi \in \mathcal{NC}(n)} |\{t \in (0,1)^{\pi|_{V_1}} \times (N-1, N)^{\pi|_{V_N}} : t \models \pi\}| \cdot K_\pi[T_1, \dots, T_n].\end{aligned}$$

By Lemma 5.4.17, this agrees with the expectation that this string would have in the situation of monotone independence. Therefore, the algebras must be monotone independent. □

The last two lemmas immediately show what we want to prove about $Y_{s,t}$.

Corollary 7.5.10. *The operators $Y_{s,t}$ satisfy*

$$\operatorname{Cum}_n(Y_{s,t})[a_1,\dots,a_{n-1}] = \begin{cases} (t-s)a, & n=1 \\ (t-s)\sigma(a_1 X a_2 \dots X a_{n-1}), & n>1. \end{cases}$$

Moreover, Y_{t_0,t_1}, ..., Y_{t_{N-1},t_N} are monotone independent for $0 = t_0 < t_1 < \dots < t_N$.

7.6 Infinitely Divisible Laws and the Bercovici-Pata Correspondence

In Theorem 7.1.2, we have given a description of convolution semigroups in terms of their infinitesimal generators. Now we turn to the question of *which laws* μ are included in a convolution semigroup $(\mu_t)_{t>0}$.

Definition 7.6.1. An $\mathcal{A}$-valued law μ is said to be

- *freely infinitely divisible* if for infinitely many n there exists $\mu_{1/n}$ such that $\mu = \mu_{1/n}^{\boxplus n}$;
- *Boolean infinitely divisible* if for infinitely many n there exists $\mu_{1/n}$ such that $\mu = \mu_{1/n}^{\uplus n}$;
- *monotone infinitely divisible* if for infinitely many n there exists $\mu_{1/n}$ such that $\mu = \mu_{1/n}^{\rhd n} = \mu_{1/n}^{\lhd n}$.

Theorem 7.6.2. *Every law is Boolean infinitely divisible. Moreover, for each type of independence, for an $\mathcal{A}$-valued law μ, the following are equivalent:*

1. *μ is infinitely divisible.*
2. *The cumulant generating function K_μ is given by $\tilde{G}_{\sigma,a}$ for some generalized law σ and self-adjoint $a \in \mathcal{A}$.*
3. *There exists a convolution semigroup μ_t with $\mu = \mu_1$.*

Proof. The implication (2) $\implies$ (3) was already established in Theorem 7.1.2, while the implication (3) $\implies$ (1) is immediate.

In the free case, we show that (1) $\implies$ (2) just as in Proposition 7.2.1. Let $n_k \to +\infty$ be a sequence of indices such that $\mu = \mu_{1/n_k}^{\oplus n_k}$. Since $\Phi_\mu = n_k \Phi_{\mu_{1/n_k}}$ is analytic on $\operatorname{Im} z \geq 2n_k^{-1/2} \|\operatorname{Var}_\mu(1)\|^{1/2}$, we see that Φ_μ extends to be analytic on the upper half-plane and hence equals $G_{\sigma,a}$ for some σ and a. Thus, the three statements are equivalent in the free case.

In the Boolean case, we have already seen that $B_\mu = G_{\sigma,a}$ for some σ and a, and thus (2) automatically holds. Hence, (3) and (1) also always hold in the Boolean case by the preceding argument.

It remains to show that (1) $\implies$ (2) in the monotone case. Let $n_k \to +\infty$ and $\mu = \mu_{1/n_k}^{\rhd n_k}$. Now $n_k B_{\mu_{1/n_k}} = G_{\sigma_k,a}(z)$ for some generalized law σ_k and the constant $a = \mu(X)$. Moreover, $\sigma_k(1) = \operatorname{Var}_\mu(1)$ and $\operatorname{rad}(\sigma_k)$ is bounded by $\operatorname{rad}(\sigma_0)$, where $B_\mu = G_{\sigma_0,a}$ by the same argument as in Lemma 7.2.6. Let $\lambda_{k,t}$ be the monotone convolution semigroup generated by $G_{\sigma_k,a}$. As in the proof of Proposition 7.3.10, we have for $\operatorname{Im} z \geq \epsilon$ that

$$\begin{aligned} F_{\lambda_{k,1/n_k}}(z) &= z - n_k^{-1} G_{\sigma_k,a}(z) + O_\epsilon(n_k^{-3/2}), \\ &= F_{\lambda_{k,1/n_k}}(z) + O_\epsilon(n_k^{-3/2}), \end{aligned}$$

where the error estimate O_ϵ is independent of k since $\operatorname{rad}(\sigma_k)$ and $\|\sigma_k(1)\|$ are uniformly bounded. Now by a telescoping series argument (as in Theorem 6.4.4), we have

$$\begin{aligned} \|F_\mu - F_{\lambda_{k,1}}\| &= \sum_{j=1}^{n_k} \left\| \left(F_{\mu_{1/n_k}^{\rhd (j-1)}} \circ F_{\mu_{1/n_k}} - F_{\mu_{1/n_k}^{\rhd j}} \circ F_{\lambda_{k,1/n_k}} \right) \circ F_{\lambda_{k,1/n_k}^{\rhd (n_k - j)}} \right\| \\ &\leq n_k \cdot O_\epsilon(n_k^{-3/2}) = O_\epsilon(n_k^{-1/2}). \end{aligned}$$

Thus, $F_{\lambda_{k,1}} \to F_\mu$. By Theorem 7.1.2 (4), we have $\operatorname{rad}(\lambda_{k,1}) \leq \operatorname{rad}(\sigma_0) + \sqrt{\|\sigma_0(1)\|} + \|a\|$, which is uniformly bounded, and therefore $\lambda_{k,1}$ converges in moments to μ by Proposition 3.6.6. This implies that the monotone cumulants of $\lambda_{k,1}$ converge and hence $G_{\sigma_k,a}$ converges to a function $G_{\sigma,a}$. Then $\tilde{G}_{\sigma,a}$ is the monotone cumulant generating function of μ and thus (2) holds. □

We have shown that infinitely divisible laws for each type of independence are in bijection with pairs (σ, a) where σ is a generalized law and a is a self-adjoint constant. In particular, this means that infinitely divisible laws for free, Boolean, and monotone independence are in bijection with each other. For instance, given a freely infinitely divisible law μ_{free}, there exist Boolean and monotone infinitely divisible laws μ_{Boolean} and μ_{monotone} such that

$$K_{\mu_{\text{free}}}^{\text{free}} = K_{\mu_{\text{Boolean}}}^{\text{Boolean}} = K_{\mu_{\text{monotone}}}^{\text{monotone}},$$

and the same holds if we start with a Boolean or monotone infinitely divisible law. The map $\mu_{\text{free}} \to \mu_{\text{Boolean}}$ will be called the *free-to-Boolean Bercovici-Pata bijection*, and we use similar terminology for each combination of two types of independence. We refer to these maps collectively as the *Bercovici-Pata correspondence*.

For infinitely divisible laws, arbitrary positive real convolution powers are defined, and moreover the Bercovici-Pata bijections preserves such convolution powers.

Definition 7.6.3. For each type of independence, if μ is a infinitely divisible law and $t > 0$, then the law with cumulant generating function tK_μ will be called the *t-th convolution power of* μ and will be denoted by $\mu^{\boxplus t}$ in the free case, $\mu^{\uplus t}$ in the Boolean case, and $\mu^{\rhd t} = \mu^{\lhd t}$ in the monotone case.

Observation 7.6.4. *If μ_{free}, $\mu_{Boolean}$, and $\mu_{monotone}$ are related by the Bercovici-Pata correspondence, then so are $\mu_{free}^{\boxplus t}$, $\mu_{Boolean}^{\uplus t}$, and $\mu_{monotone}^{\rhd t}$ for each $t > 0$.*

The space of infinitely divisible laws and the Bercovici-Pata correspondence have the following topological properties with respect to convergence in moments.

Proposition 7.6.5. *Consider the space Σ_M of $\mathcal{A}$-valued laws with $\operatorname{rad}(\mu) \leq M$ with the topology of convergence in moments. For each type of independence, the infinitely divisible laws are a closed (hence complete) subspace. Moreover, convergence in moments for the infinitely divisible laws is equivalent to uniform local convergence of the cumulant generating functions $\tilde{K}_\mu$ on the upper half-plane, and the set of cumulant generating functions for infinitely divisible $\mu \in \Sigma_M$ is complete.*

Proof. Suppose that μ_n is a sequence of infinitely divisible laws that converges in moments to a law μ. Let $\tilde{K}_{\mu_n} = G_{\sigma_n,a_n}$. Then $\operatorname{rad}(\sigma_n) \leq \operatorname{rad}(\mu_n) \leq R$. The cumulants of μ_n converge, which means that σ_n is Cauchy in moments, hence converges to a generalized law σ by Proposition 3.6.5. Also, $a_n \to a := \mu(X)$. Thus, $\tilde{K}_\mu = G_{\sigma,a}$, so μ is infinitely divisible. Thus, the infinitely divisible laws are a closed subspace.

The remaining claims follow from Proposition 3.6.5. Indeed, convergence in moments for infinitely divisible laws μ_n is equivalent to convergence in moments of σ_n and convergence of a_n, which is equivalent to uniform local convergence of G_{σ_n,a_n} on $\mathbb{H}_+(\mathcal{A})$. The same holds for Cauchyness of sequences, and we already know that Σ_M is complete, hence so is the space of infinitely divisible laws. □

Observation 7.6.6. *Each of the Bercovici-Pata bijections maps Σ_M continuously into Σ_{CM} for some constant C.*

Proof. The proof for each of the bijections is the same, but for concreteness of notation, consider the free-to-Boolean bijection. Let μ_{free} and μ_{Boolean} have the cumulant generating function $\tilde{G}_{\sigma,a}$. Note that

$$\begin{aligned}\operatorname{rad}(\mu_{\text{Boolean}}) &\leq \operatorname{rad}(\sigma) + 2\sqrt{\|\sigma(1)\|} + \|a\| \\ &\leq C \operatorname{rad}(\mu_{\text{free}}) + 2\sqrt{\mu_{\text{free}}(X^2)} + \|\mu_{\text{free}}(X)\| \\ &\leq C' \operatorname{rad}(\mu_{\text{free}}).\end{aligned}$$

Moreover, convergence in moments for μ_{free} is equivalent to convergence of σ and a, which is equivalent to convergence of μ_{Boolean}. □

The following theorem, stated in the spirit of the original paper [BP99, Theorem 1.2], shows how the Bercovici-Pata correspondence arises purely from the convolution operations.

Theorem 7.6.7. *Let λ_k be a sequence of $\mathcal{A}$-valued laws and $n_k \to \infty$. Then the following are equivalent:*

1. $\operatorname{rad}(\lambda_k^{\boxplus n_k})$ *is bounded and* $\lambda_k^{\boxplus n_k}$ *converges to a law* λ_{free} *as* $k \to \infty$.
2. $\operatorname{rad}(\lambda_k^{\uplus n_k})$ *is bounded and* $\lambda_k^{\uplus n_k}$ *converges to a law* λ_{Boolean} *as* $k \to \infty$.
3. $\operatorname{rad}(\lambda_k^{\rhd n_k})$ *is bounded and* $\lambda_k^{\rhd n_k}$ *converges to a law* $\lambda_{\text{monotone}}$ *as* $k \to \infty$.

Moreover, in this case, the laws λ_{free}, λ_{Boolean}, and $\lambda_{\text{monotone}}$ are infinitely divisible (for their respective types of independence) and are related by the Bercovici-Pata correspondence.

Proof. We will organize the proof into (1) $\implies$ (2), (2) $\implies$ (1), (3) $\implies$ (2), (2) $\implies$ (3), and prove the claims about infinite divisibility and the Bercovici-Pata correspondence along the way.

(1) $\implies$ (2). Suppose (1) holds. Let $\mu_k = \lambda_k^{\boxplus n_k}$ and assume $\operatorname{rad}(\mu_k) \leq M$. Using Theorem 4.7.2, R_{μ_k} is defined in a neighborhood of 0 and bounded, with estimates that only depend on M. Hence, so is $R_{\lambda_k} = (1/n_k)R_{\mu_k}$. Since $\tilde{G}_{\lambda_k}$ is the inverse function of $(z^{-1} + R_{\lambda_k}(z))^{-1}$, using the inverse function theorem (Theorem 2.8.1) and Theorem 3.4.1, we see that $\operatorname{rad}(\lambda_k) \leq M'$ where M' only depends on M.

Let $\Phi_{\lambda_k} = \tilde{R}_{\lambda_k}$ be the Voiculescu transform. We aim to show that $\Phi_{\lambda_k} = B_{\lambda_k} + O(n_k^{-3/2})$ on an appropriate domain. Let σ_k and a_k be given by $n_k B_{\lambda_k} = G_{\sigma_k,a_k}$, so that $F_{\lambda_k}(z) = z - n_k^{-1} G_{\sigma_k,a_k}$. Note that $\operatorname{rad}(\sigma_k) \leq 2\operatorname{rad}(\lambda_k) \leq 2M'$. Let $\Psi_{\lambda_k}(w) = F_{\lambda_k}^{-1}(w) = w + \Phi_{\lambda_k}(w)$. We showed in the proof of Theorem 4.7.2 that if $\delta > 2\sqrt{n_k^{-1}\|\sigma_k(1)\|}$, then Ψ_k defines a map $\mathbb{H}_{+,2\delta}(\mathcal{A}) \to \mathbb{H}_{+,\delta}(\mathcal{A})$. Now we have

$$(\operatorname{id} - n_k^{-1} G_{\sigma_k,a_k}) \circ \Psi_k = \operatorname{id}$$

hence for $\operatorname{Im} w \geq 2\delta$,

$$\|\Psi_{\lambda_k}(w) - w\| = \|n_k^{-1} G_{\sigma_k,a_k}(\Psi_{\lambda_k}(w))\| \qquad \leq \frac{\|\sigma_k(1)\|}{n_k \delta}.$$

Hence,

$$\begin{aligned}
\left\|\Phi_{\lambda_k}(w) - n_k^{-1} G_{\sigma_k,a_k}(w)\right\| &= \left\|\Psi_{\lambda_k}(w) - w - n_k^{-1} G_{\sigma_k,a_k}(w)\right\| \\
&= \left\|n_k^{-1} G_{\sigma_k,a_k}(\Psi_{\lambda_k}(w)) - n_k^{-1} G_{\sigma_k,a_k}(w)\right\| \\
&\leq \frac{\|\sigma_k(1)\|}{n_k \cdot 2\delta \cdot \delta} \|\Psi_{\lambda_k}(w) - w\| \\
&\leq \frac{\|\sigma_k(1)\|^2}{2n_k^2 \delta^3}.
\end{aligned}$$

Now we note that $\|\sigma_k(1)\| \leq \operatorname{rad}(\mu_k)^2 \leq M^2$. Thus,

$$\Phi_{\mu_k}(z) = n_k \Phi_{\lambda_k}(z) = G_{\sigma_k,a_k}(z) + O_{\delta,M}(n_k^{-1}),$$

provided that $n_k^{1/2} > 2M/\delta \geq 2\|\sigma_k(1)\|^{1/2}$.

We assumed that $\mu_k \to \lambda_{\text{free}}$ and hence R_{μ_k} converges to $R_{\lambda_{\text{free}}}$ in a neighborhood of 0, hence $\Phi_{\mu_k} \to \Phi_{\lambda_{\text{free}}}$ for $\operatorname{Im} z$ sufficiently large, by analytic continuation. Since $\operatorname{rad}(\sigma_k) \leq M'$ and $\|a_k\| \leq M$, we know that G_{σ_k,a_k} converges as $k \to \infty$ to a function $G_{\sigma,a}$. Then $G_{\sigma,a} = \Phi_{\lambda_{\text{free}}}$ and hence λ_{free} is freely infinitely divisible.

Furthermore, we have $G_{\sigma_k,a_k} = n_k B_{\lambda_k} = B_{\lambda_k^{\uplus n_k}}$. Now

$$\operatorname{rad}(\lambda_k^{\uplus n_k}) \leq \operatorname{rad}(\sigma_k) + 2\sqrt{\|\sigma_k(1)\|} + \|a_k\| \leq 4M'.$$

Thus, $\operatorname{rad}(\lambda_k^{\uplus n_k})$ is bounded as desired. Also, $B_{\lambda_k^{\uplus n_k}}$ converges and hence $\lambda_k^{\uplus n_k}$ converges in moments to some law λ_{Boolean}. Now

$$B_{\lambda_{\text{Boolean}}} = G_{\sigma,a} = \Phi_{\lambda_{\text{free}}},$$

and hence λ_{Boolean} and λ_{free} are related by the Bercovici-Pata correspondence.

(2) $\implies$ (1). As before, we will write $n_k B_{\lambda_k} = G_{\sigma_k,a_k}$. First, we must check that $\operatorname{rad}(\lambda_k^{\boxplus n_k})$ is bounded. Assume that $\operatorname{rad}(\lambda_k^{\boxplus n_k}) \leq M$. Then $\operatorname{rad}(\sigma_k) \leq 2M$ and $\|a_k\| \leq M$. Hence, $\operatorname{rad}(\lambda_k)$ is bounded by a constant M'. By Theorem 4.7.2, R_{λ_k} is analytic on $B(0, (3 - 2\sqrt{2})/M')$ and is bounded by

$$\|R_{\lambda_k^{\boxplus n_k}}\| = n_k \|R_{\lambda_k}\| \leq \frac{2\|\sigma_k(1)\| M'}{\sqrt{2} - 1} + \|a_k\| =: M',$$

By the inverse function theorem, this yields a uniform bound M'' on $\operatorname{rad}(\lambda_k^{\boxplus n_k})$ as $k \to \infty$.

Similar to the proof of (1) $\implies$ (2), we have

$$\Phi_{\lambda_k^{\boxplus n_k}}(z) = n_k \Phi_{\lambda_k}(z) = G_{\sigma_k,a_k}(z) + O_{\delta,M}(n_k^{-1}).$$

By our assumption that $\lambda_k^{\uplus n_k}$ converges, G_{σ_k,a_k} converges to some function $G_{\sigma,a}$. Thus, $\Phi_{\lambda_k^{\boxplus n_k}}$ converges, which implies that $\lambda_k^{\boxplus n_k}$ converges in moments to some law λ_{free}. As before, λ_{free} and λ_{Boolean} are related by the Bercovici-Pata correspondence since their cumulant generating function is $\tilde{G}_{\sigma,a}$.

(3) $\implies$ (2). Define (σ_k, a_k) by $n_k B_{\lambda_k} = G_{\sigma_k,a_k}$ and assume that $\operatorname{rad}(\lambda_k^{\rhd n_k}) \leq M$. Proceeding as in the proof of Theorem 7.6.2 for the monotone case, we have $\operatorname{rad}(\sigma_k) \leq 2\operatorname{rad}(\lambda_k^{\rhd n_k}) \leq 2M$, while $\sigma_k(1) = \operatorname{Var}_{\lambda_k^{\rhd n_k}}(1)$ and $a_k = \lambda_k^{\rhd n_k}(X)$ are bounded. Since $B_{\lambda_k^{\uplus n_k}} = G_{\sigma_k,a_k}$, we see that $\operatorname{rad}(\lambda_k^{\uplus n_k})$ is uniformly bounded.

Let λ_k' be the law with monotone cumulant generating function given by $n_k^{-1} G_{\sigma_k,a_k}$. Note that $\operatorname{rad}((\lambda_k')^{\rhd n_k})$ is uniformly bounded by Observation 7.6.6. Then for $\operatorname{Im} z \geq \epsilon$, we have

$$F_{\lambda_k'}(z) = z - n_k^{-1} G_{\sigma_k,a_k}(z) + O_{\epsilon,M}(n_k^{-3/2}) = F_{\lambda_k}(z) + O_{\epsilon,M}(n_k^{-3/2}).$$

By the telescoping series argument, we have

$$\left\| F_{(\lambda_k')^{\rhd n_k}}(z) - F_{\lambda_k^{\rhd n_k}}(z) \right\| = O_{\epsilon,M}(n_k^{-1/2}).$$

By assumption, $\lambda_k^{\rhd n_k}$ converges to some $\lambda_{\text{monotone}}$ and hence also $(\lambda_k')^{\rhd n_k}$ converges in moments to $\lambda_{\text{monotone}}$. The monotone cumulant generating function of $(\lambda_k')^{\rhd n_k}$ is $\tilde{G}_{\sigma_k,a_k}$, and thus G_{σ_k,a_k} converges to some function $G_{\sigma,a}$. Then $\tilde{G}_{\sigma,a}$ is the monotone cumulant generating function for $\lambda_{\text{monotone}}$ and hence $\lambda_{\text{monotone}}$ is monotone infinitely divisible.

But we also know that G_{σ_k,a_k} is the Boolean cumulant generating function of $\lambda_k^{\uplus n_k}$, and hence $\lambda_k^{\uplus n_k}$ converges to some law λ_{Boolean} with Boolean cumulant generating function $G_{\sigma,a}$, which is related to $\lambda_{\text{monotone}}$ by the Bercovici-Pata correspondence.

(2) $\implies$ (3). Define (σ_k, a_k) by $n_k B_{\lambda_k} = G_{\sigma_k,a_k}$ and assume that $\operatorname{rad}(\lambda_k^{\uplus n_k}) \leq M$ (and hence $\operatorname{rad}(\sigma_k) \leq 2M$). Now we show that $\operatorname{rad}(\lambda_k^{\rhd n_k})$ is uniformly bounded. Recall that

$$\tilde{G}_{\lambda_k}(z) = (z^{-1} - n_k^{-1}\tilde{G}_{\sigma_k,a_k}(z))^{-1} = z(1 - n_k^{-1}\tilde{G}_{\sigma_k,a_k}(z)z)^{-1}.$$

Because σ_k and a_k are bounded uniformly with estimates depending only on M, there exists R depending on M such that

$$\|z\| \leq R \implies \left\| \tilde{G}_{\sigma_k,a_k}(z) z \right\| \leq 1.$$

For such z, we have

$$\|\tilde{G}_{\lambda_k}(z)\| \leq \|z\|(1 - n_k^{-1})^{-1}.$$

This implies that $\tilde{G}_{\lambda_k}(z)$ maps $B(0,(1-n_k^{-1})r)$ into $B(0,r)$ for $r \leq R$. In particular, the n_k-fold composition of $\tilde{G}_{\lambda_k}$ maps $B(0,(1-n_k^{-1})^{n_k}R)$ into $B(0,R)$. Since monotone convolution corresponds to composition of the transforms $\tilde{G}$, we see that $\tilde{G}_{\lambda_k^{\rhd n_k}}$ is fully matricial on $B(0,(1-n_k^{-1})^{n_k}R)$, so that

$$\operatorname{rad}(\lambda_k^{\rhd n_k}) \leq (1 - n_k^{-1})^{-n_k} R^{-1}.$$

As $k \to \infty$, this converges to e/R and hence it is bounded as $k \to \infty$.

Let λ_k' be the law with monotone cumulant generating function $n_k^{-1} G_{\sigma_k,a_k}$. We assumed that $\lambda_k^{\uplus n_k}$ converges to λ_{Boolean}, hence G_{σ_k,a_k} converges to a function $G_{\sigma,a}$. This means that the monotone cumulants of $(\lambda_k')^{\rhd n_k}$ converge, so that $(\lambda_k')^{\rhd n_k}$ converges to a law $\lambda_{\text{monotone}}$. But we also have $\operatorname{rad}((\lambda_k')^{\rhd n_k})$ uniformly bounded and

$$\left\| F_{(\lambda_k')^{\rhd n_k}}(z) - F_{\lambda_k^{\rhd n_k}}(z) \right\| = O_{\epsilon,M}(n_k^{-1/2})$$

as in the proof of (3) $\implies$ (2). So $\lambda_k^{\rhd n_k}$ also converges to $\lambda_{\text{monotone}}$. Then $\lambda_{\text{monotone}}$ has the monotone cumulant generating function $\tilde{G}_{\sigma,a} = \tilde{B}_{\lambda_{\text{Boolean}}}$, and hence $\lambda_{\text{monotone}}$ is infinitely divisible and related to λ_{Boolean} by the Bercovici-Pata correspondence. $\square$

Remark 7.6.8. The semicircle, Bernoulli, and arcsine laws of variance η are related under the Bercovici-Pata correspondence because the cumulant generating function in each case is $\eta(z)$. This is also must be the case in light of the central limit theorem and Theorem 7.6.7. Indeed, if λ is any law of variance η and $n_k = k$ and $\lambda_k = \operatorname{dil}_{k^{-1/2}}(\lambda)$, then by the central limit theorem $\lambda_k^{\boxplus k}$, $\lambda_k^{\uplus k}$, and $\lambda_k^{\rhd k}$ converge to the semicircle, Bernoulli, and arcsine laws respectively. Thus, by Theorem 7.6.7, the semicircle, Bernoulli, and arcsine laws must be infinitely divisible and related by the Bercovici-Pata correspondence.

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