

ALGEBRAIC PROPERTIES OF GENERALIZED GRAPH LAPLACIANS: RESISTOR NETWORKS, CRITICAL GROUPS, AND HOMOLOGICAL ALGEBRA*

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Abstract. We propose an algebraic framework for generalized graph Laplacians which unifies the study of resistor networks, the critical group, and the eigenvalues of the Laplacian and adjacency matrices. Given a graph with boundary G together with a generalized Laplacian L with entries in a commutative ring R , we define a generalized critical group $\Upsilon_R(G, L)$. We relate $\Upsilon_R(G, L)$ to spaces of harmonic functions on the network using the Hom, Tor, and Ext functors of homological algebra. We study how these algebraic objects transform under combinatorial operations on the network (G, L) , including harmonic morphisms, layer-stripping, duality, and symmetry. In particular, we use layer-stripping operations from the theory of resistor networks to systematize discrete harmonic continuation. This leads to an algebraic characterization of the graphs with boundary that can be completely layer-stripped, an algorithm for simplifying computation of $\Upsilon_R(G, L)$, and upper bounds for the number of invariant factors in the critical group and the multiplicity of Laplacian eigenvalues in terms of geometric quantities.

Key words. graph Laplacian, resistor network, layer-stripping, critical group, discrete harmonic function, homological algebra

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1. Introduction. Motivated by questions from resistor networks as well as algebraic graph theory, we study algebraic properties of a generalized critical group. We relate spaces of harmonic functions to the generalized critical group using homological algebra. We study how these algebraic objects transform under modifications of the network, including harmonic morphisms, layer-stripping, duality, and symmetry.

1.1. Layer-stripping for resistor networks. Our first motivation comes from the theory of resistor networks developed by [15, 18, 13, 29, 27, 26]. A *graph with boundary* or *∂ -graph* is a graph (V, E) together with a specified partition of V into a set ∂V of *boundary vertices* and a set V° of *interior vertices*. The boundary vertices are the vertices where we will allow a net flow of current into or out of the network. A *resistor network* is an edge-weighted ∂ -graph, where each weight or conductance $w(e)$ is strictly positive. An *electrical potential* is a function $u: V \rightarrow \mathbb{R}$. The *net current* at a vertex x is given by the *weighted Laplacian*

$$Lu(x) = \sum_{y \sim x} w(x, y)[u(x) - u(y)].$$

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A potential function is *harmonic* if the net current vanishes at each interior vertex.

The discrete electrical inverse problem studied by [15, 13, 27, 29, 26] asks whether the conductances of a network can be recovered by performing boundary measurements of harmonic functions. We measure how the potentials $u|_{\partial V}$ and net currents $Lu|_{\partial V}$ relate for a harmonic function u , and we encode this information in a *response matrix* Λ (for precise definition, see [18, section 3.2]). The inverse problem asks whether we can uniquely determine w knowing only G and Λ . In other words, for fixed G , we want to reverse the transformation $w \mapsto \Lambda$.

The electrical inverse problem cannot be solved for all graphs, but many graphs can be recovered via *layer-stripping*, a technique in which the edge weights are recovered iteratively, working inward from the boundary. At each step, one recovers the conductance of a near-boundary edge, then removes that edge by a *layer-stripping operation* of deletion or contraction, and thus reduces the problem to a smaller graph [18, section 6.5].

Layer-stripping operations have intrinsic algebraic and combinatorial interest as well. For instance, if a ∂ -graph can be completely layer-stripped to nothing, then one can construct its response matrix iteratively through simple transformations corresponding to the layer-stripping operations [18, section 6]. This process parametrizes the response matrices associated for resistor networks which are circular planar (i.e., able to be embedded in the disk). Furthermore, as observed by [30], the action of layer-stripping operations on these response matrices generates a group isomorphic to the symplectic group. In [1], circular planar networks (up to Y - Δ equivalence) are given the structure of a poset with $G' \geq G$ if G' can be layer-stripped down to G .

Let us call a ∂ -graph *layerable* if it can be completely layer-stripped to the empty graph. In this paper, we will construct an algebraic invariant to test layerability. Our strategy is to replace edge weights in \mathbb{R}_+ with edge weights in an arbitrary commutative ring R . Then we consider the weighted Laplacian as an operator on functions $u : V \rightarrow M$, where M is a given R -module. We examine algebraic properties of the module $\mathcal{U}(G, L, M)$ of harmonic functions and the module $\mathcal{U}_0(G, L, M)$ of harmonic functions such that u and Lu both vanish on the boundary of G .

We show that if a ∂ -graph is layerable and we assign edge weights which are units in a ring R , then $\mathcal{U}_0(G, L, M) = 0$. That is, if u is harmonic with $u|_{\partial V} = 0$ and $Lu|_{\partial V} = 0$, then u is identically zero. The idea is to start with the values on the boundary and work one's way inward following the sequence of layer-stripping operations. At each step, we deduce that another edge has zero current or that another vertex has zero potential. In essence, this is a discrete version of harmonic continuation.

The condition that $\mathcal{U}_0(G, L, M) = 0$ for all L and M does not quite characterize layerable ∂ -graphs. If we have a ∂ -graph G and $\mathcal{U}_0(G, L, M) = 0$ for every Laplacian L obtained by assigning unit edge weights in any ring R , then G may not be layerable. However, it must be *completely reducible*, that is, it can be reduced to nothing using layer-stripping and another operation which splits apart two subgraphs that are glued together at a common boundary vertex (see Theorem 7.13).

Moreover, we can characterize layerability algebraically by generalizing L to allow arbitrary diagonal entries. As shown in Theorem 5.16, G is layerable if and only if $\mathcal{U}_0(G, L, M) = 0$ for every generalized Laplacian L of the form $D - A$, where D is a diagonal matrix and A is the adjacency matrix weighted by units in R .

Remark 1.1. It is important to point out that our invariants do *not* test whether the inverse problem can be solved. Solving the inverse problem would require not

only deleting and contracting a sequence of edges, but also being able to determine the weight of each edge from the boundary behavior of the network. For a treatment of the inverse problem through layer-stripping, see [27, 26].

Remark 1.2. It is straightforward to test whether a specific ∂ -graph is layerable by repeatedly iterating over the boundary vertices searching for edges that can be removed, and this can be done in polynomial time. The advantage of an algebraic invariant is that it can be used to test layerability for whole classes of networks by relating it to other more global properties (see, e.g., Proposition 7.14). It also gives us significant information about nonlayerable graphs with boundary.

1.2. Harmonic functions and the critical group. The modules $\mathcal{U}(G, L, M)$ and $\mathcal{U}_0(G, L, M)$ of harmonic functions turn out to be related to another R -module, which we call the *fundamental module* Υ . The module Υ is a generalization of the *critical group* of a graph (also known as the *sandpile group*, *Jacobian*, or *Picard group*), which has received significant attention from physicists, combinatorialists, probabilists, algebraic geometers, and number theorists.

The critical group can be produced through several different combinatorial models. The abelian sandpile model was introduced in statistical physics by Dhar [19], who was motivated by the study of self-organized criticality. Grains of sand are placed on the vertices of a graph. If a vertex has at least as many grains of sand as its degree, the vertex is allowed to *topple* by sending one grain of sand to each of its neighbors. The elements of the sandpile group are the critical configurations of sand [23, section 14], [9]. There are other combinatorial models which produce the same group. Extending work of [40] on the balancing game, [10] introduced the chip-firing game and uncovered its connection to greedoids. The dollar game appeared in [8] and was analyzed extensively using the methods of algebraic potential theory.

Sandpile theory has since expanded into other areas of combinatorics, graph theory, and even algebraic geometry. Graph theorists study the sandpile group in the guise of the quotient of the chain group by the submodule generated by cycles and bonds [8, sections 26–29]. Probabilists study the abelian sandpile model due to its intimate connections with generating uniformly random spanning trees [24, 32]; the sandpile group acts freely and transitively on the set of spanning trees of the graph [24, section 7], [9, Theorem 7.3]. Viewing sand configurations as divisors on the graph, [33, 5] interpreted the sandpile group as the Jacobian variety of a degenerate curve and proved a Riemann–Roch theorem for graphs.

For such a fruitful object with deep and diverse connections, the critical group has a surprisingly simple algebraic characterization. For a connected graph G , if $\mathbb{Z}V$ is the group of 0-chains on the vertices and $L : \mathbb{Z}V \rightarrow \mathbb{Z}V$ is the graph Laplacian, then $\text{coker}(L) \cong \text{Crit}(G) \oplus \mathbb{Z}$ (see [9, Theorem 4.2]). This construction of $\text{Crit}(G)$ easily generalizes to ∂ -graphs with edge weights in an arbitrary ring. In the general case, we define $\Upsilon(G, L)$ as the cokernel of the generalized Laplacian L viewed as a map from 0-chains on the interior vertices to 0-chains on V (in a similar manner to [4, 21]).

We relate Υ with harmonic functions by observing that

$$\mathcal{U}(G, L, M) = \text{Hom}(\Upsilon(G, L), M);$$

in other words, $\Upsilon(G, L)$ is the *representing object* for the functor $\mathcal{U}(G, L, -)$ on R -modules. We also show that $\mathcal{U}_0(G, L, M) = \text{Tor}_1(\Upsilon(G, L), M)$ (for nondegenerate networks). In other words, M -valued harmonic functions that are not detectable from boundary measurements indicate *torsion* of the fundamental module Υ . These

algebraic facts lead to several equivalent algebraic characterizations of layerability and complete reducibility in terms of Υ (Theorems 5.16 and 7.13).

As a special case of our theory, for a graph without boundary with edge weights 1, we have $\Upsilon \cong \text{Crit}(G) \oplus \mathbb{Z}$. Moreover,

$$\mathcal{U}(G, L, \mathbb{R}/\mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}}(\text{Crit}(G), \mathbb{R}/\mathbb{Z}) \times \mathbb{R}/\mathbb{Z} \cong \text{Crit}(G) \times \mathbb{R}/\mathbb{Z},$$

where the isomorphism $\text{Hom}_{\mathbb{Z}}(\text{Crit}(G), \mathbb{R}/\mathbb{Z}) \cong \text{Crit}(G)$ follows from Pontryagin duality because $\text{Crit}(G)$ is a finite abelian group. Thus, we recover the observation of [39, section 2] and [24, p. 11] that $\text{Crit}(G)$ is isomorphic to the group of \mathbb{R}/\mathbb{Z} -valued harmonic functions modulo constants.

This harmonic-function perspective makes the computation of Υ (and hence $\text{Crit}(G)$) accessible to the powerful technique of discrete harmonic continuation, which has proved extremely useful to the resistor network community—for instance, see [18, sections 4.1–4.5], [17, section 4], [27], [28, section 2.3]. We illustrate this technique in section 3, using it to compute Υ for a family of ∂ -graphs embedded on the cylinder.

In section 5 we present a systematic approach which uses layer-stripping as a geometric model for harmonic continuation. As an application, we have the following result (a special case of Theorem 5.23): Suppose G is a graph without boundary and G' is obtained from G by assigning s vertices to be boundary vertices. If G' is layerable, then $\text{Crit}(G)$ has at most $s - 1$ invariant factors. In fact, these invariant factors can be found from the Smith normal form of an $s \times s$ matrix computed explicitly from the sequence of layer-stripping operations.

1.3. Discrete differential geometry and complex analysis. The generalized critical group Υ serves as a link between the combinatorial properties of a ∂ -graph and the algebraic properties of harmonic functions, not unlike the way that homology links the topology of a Riemannian manifold to harmonic differential forms. In light of Hodge theory, harmonic differential forms on a manifold represent elements of the de Rham cohomology groups. On the other hand, these groups are characterized as $\text{Hom}_{\mathbb{Z}}(H^n, \mathbb{R})$ by the de Rham theorem, where H^n is the homology of a chain complex defined using formal linear combinations of simplices. In a similar way, the module $\mathcal{U}(G, L, M)$ of harmonic functions on an R -network can be represented as $\text{Hom}(\Upsilon, M)$, where Υ is obtained by considering formal linear combinations of vertices.

In fact, the analogy between Riemannian geometry and weighted graphs can be made quite precise. We shall sketch the connection here in a similar way to [4, section 2.1] and [36]. We remark also that [21] has generalized the critical group to higher dimensions using an analogue of the Hodge Laplacian.

Given a ∂ -graph G and a commutative ring R , we define chain groups

$$C_0 := RV, \quad C_1 := RE/\{-e = \bar{e}\}_{e \in E},$$

that is, the free R -modules on the vertex and edge sets, respectively, after identifying the negative of an oriented edge with its reverse orientation. Dual to chains, we have modules $\Omega^j(G, L, M)$ consisting of M -valued j -forms:

$$\Omega^j(G, L, M) := \text{Hom}(C_j, M), \quad j = 0, 1.$$

The boundary map $\partial: C_1 \rightarrow C_0$ given by $\partial e = e_+ - e_-$ induces the discrete gradient $d: \Omega^0 \rightarrow \Omega^1$ given by $df(e) = f(e_+) - f(e_-)$. The coboundary map $\partial^*: C_0 \rightarrow C_1$ given by $x \mapsto \sum_{e: e_+ = x} e$ induces the discrete divergence $d^*: \Omega^1 \rightarrow \Omega^0$ given by $d^* \omega(x) = \sum_{e: e_+ = x} \omega(e)$. The weighted chain Laplacian $\partial w \partial^*: C_0 \rightarrow C_0$ induces the

weighted Laplacian on cochains or functions $d^*wd: \Omega^0 \rightarrow \Omega^0$. (Here w denotes the map $e \mapsto w(e)e$.)

For a graph without boundary, the module $\mathcal{U}(G, L, M)$ arises from (weighted) cohomology theory as the kernel of $d^*wd: \Omega^0 \rightarrow \Omega^0$. On the other hand, $\Upsilon(G, L)$ arises from (weighted) homology theory as the cokernel of $\partial w \partial^*: C_0 \rightarrow C_0$. In this discrete setting, the de Rham–like duality $\mathcal{U}(G, L, M) \cong \text{Hom}_R(\Upsilon(G, L), M)$ follows immediately from properties of quotient modules (see Lemma 2.9).

There is an even better developed analogy between graphs and Riemann surfaces [6, 42, 36, 37, 11], and we will continue to draw inspiration from complex analysis and topology even as we build a purely combinatorial and algebraic theory. We shall describe analogues of holomorphic maps (section 4.2), harmonic continuation (section 5), and harmonic conjugates (section 8).

1.4. Overview. The paper is organized as follows:

Section 2, on the fundamental module $\Upsilon(G, L)$, defines the generalized critical group $\Upsilon(G, L)$ and interprets $\text{Hom}(\Upsilon(G, L), -)$ and $\text{Tor}_1(\Upsilon(G, L), -)$ in terms of harmonic functions (Lemma 2.9 and Proposition 2.11). We give applications to the special case of principal ideal domains (PIDs) and $\text{Crit}(G)$.

In section 3, on chain link fence networks, we compute Υ for an infinite family of networks with nontrivial boundary which played a key role in the electrical inverse problem [29]. This computation illustrates and motivates ideas we develop systematically later (harmonic continuation, covering spaces, sub- ∂ -graphs). As a preview of the computation, Figure 1.1 shows a $(\mathbb{Z}/64)$ -valued harmonic function on one of the chain link fence networks. This function has $u = 0$ and $Lu = 0$ on the boundary of the network. One interesting corollary of our analysis is that the \mathbb{Z} -module of such harmonic functions breaks up into the direct sum of harmonic functions which are zero on the first column of vertices and harmonic functions which are zero on the second column.

In section 4, we describe categories of ∂ -graphs and R -networks, adapted from the ideas of [42, 6, 41]. We show Υ is a covariant functor from R -networks to R -modules. We give applications to the critical group and eigenvectors of the Laplacian.

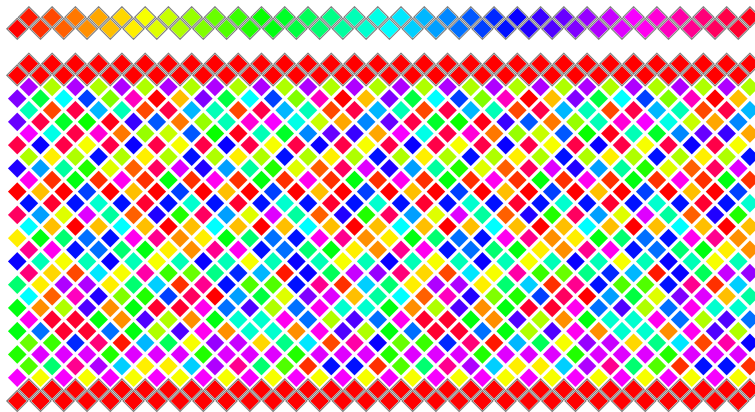


FIG. 1.1. A $(\mathbb{Z}/64)$ -valued harmonic function on a graph we will study in section 3. The squares represent vertices, and edges exist between squares which share a side. The squares on the right and left sides are identified. The color represents the value of the function; the top bar lists the colors for 0 through 63 from left to right. (In the printed grayscale version, there is some ambiguity about which color represents which value; for a more accurate picture, see the online color version.)

In section 5, on layer-stripping and harmonic continuation, we describe the process of layer-stripping borrowed from the theory of resistor networks [15, 13, 29, 26]. This leads to an algebraic characterization of the finite ∂ -graphs that can be completely layer-stripped (Theorem 5.16). We use layer-stripping as a geometric model for harmonic continuation. A systematic approach to harmonic continuation leads to an algorithm for simplifying the computation of Υ (Theorem 5.23). Corollaries include bounds on the number of invariant factors in the sandpile group and the multiplicity of eigenvalues for graph Laplacians.

In section 6, on functorial properties of layer-stripping, we relate layer-stripping with the morphisms of ∂ -graphs from section 4. We show that if $f: G' \rightarrow G$ is a ∂ -graph morphism and if G can be completely layer-stripped to the empty graph, then so can G' (Lemma 6.4). The ability to pull back the layer-stripping process leads to a clean description of how far layer-stripping operations can simplify a finite ∂ -graph in the general case (Theorem 6.11).

Section 7 defines a class of completely reducible networks which can be reduced to nothing by layer-stripping operations and splitting apart networks that are glued together at a common boundary vertex. We prove an algebraic characterization of complete reducibility which is analogous to the one for layerability, and we apply our theory to boundary-interior bipartite networks.

In section 8, we show that a network and its dual have isomorphic fundamental modules Υ , generalizing an earlier duality result for the critical group [8, 14]. The corresponding statement for harmonic functions is that every harmonic function on G has an essentially unique harmonic conjugate on G^\dagger . Harmonic conjugates provide an alternative approach to a critical group computation of [8] for a simple family of wheel graphs.

In section 9, on covering maps and symmetry, we sketch potential applications of symmetry and group actions for understanding the algebraic structure of Υ with special focus on the torsion primes of the critical group.

The concluding section 10, on open problems, hints at further possible applications and generalizations.

2. The fundamental module $\Upsilon(G, L)$. We shall generalize the graph Laplacian and critical group in several ways, adapting existing ideas in the literature, especially those of [21], [4], and [15]. Briefly, we will work over an arbitrary ring R rather than \mathbb{Z} or \mathbb{R} , assign weights in R to the edges, modify the diagonal terms of L arbitrarily, and choose some *boundary* vertices at which we will not enforce harmonicity. We will give our general definitions and then describe the examples we have in mind in section 2.1.

In the remainder of section 2, we will characterize various algebraic properties of the fundamental module, including its Tor and Ext functors, in terms of harmonic functions. In particular, in the case where R is a PID, we will describe several ways of computing the torsion submodule of the generalized critical group in terms of harmonic functions, which we will use later on in the paper.

We assume familiarity with basic terminology for graphs, categories, rings, and modules, as well as basic homological algebra. For background, refer to [2], [44, Chapters 1–3], [35, Chapters I–V], [43]. We shall also use the theory of modules over a PID, including the classification of finitely generated modules and the Smith normal form for morphisms from $R^n \rightarrow R^m$ (see [20, section 12]).

2.1. Definitions: ∂ -graphs, generalized Laplacians, and the module Υ . We will take the word *graph* to mean a countable, locally finite, undirected

multi-graph. We write V or $V(G)$ for the vertex set of the graph G and E or $E(G)$ for the set of *oriented* edges. If e is an oriented edge, e_+ and e_- refer to its starting and ending vertices, and \bar{e} refers to its reverse orientation. We use the notation $\mathcal{E}(x) = \{e : e_+ = x\}$ for the set of oriented edges exiting x . The *degree* of a vertex x is the number of such edges, that is, $\deg(x) = |\mathcal{E}(x)|$.

A *graph with boundary* (abbreviated to ∂ -graph) is a graph with a specified partition of V into two sets V° and ∂V , called the *interior* and *boundary vertices*, respectively. We will use the letter G to denote ∂ -graphs as well as graphs. We will sometimes view a graph without boundary as a ∂ -graph by taking $V^\circ = V$ and $\partial V = \emptyset$.

Let G be a ∂ -graph and R a commutative ring. Then RV will denote the free R -module with basis V ; in the language of topology, RV is the module of 0-chains or formal R -linear combinations of vertices. Similarly, RV° will denote the free module with basis V° , which is a submodule of RV .

DEFINITION 2.1. *A generalized Laplacian for G over R is an R -module morphism $L : RV \rightarrow RV$ of the form*

$$Lx = d(x)x + \sum_{e \in \mathcal{E}(x)} w(e)(x - e_-) = \text{for } x \in V,$$

where w is a function $E \rightarrow R$ satisfying $w(\bar{e}) = w(e)$ and d is a function $V \rightarrow R$.

Here $d(x)$ does not represent the diagonal entry of L at x but rather the difference of the diagonal entry from the standard weighted Laplacian. Note that Lx can also be written

$$Lx = \left(d(x) + \sum_{e \in \mathcal{E}(x)} w(e) \right) x - \sum_{e \in \mathcal{E}(x)} w(e)e_-.$$

Our usage of the term “generalized Laplacian” is consistent with [23, section 13.9].

DEFINITION 2.2. *An R -network is a pair (G, L) , where G is a ∂ -graph and L is an associated generalized Laplacian over R . We call (G, L) an R^\times -network if $w(e)$ is in the group of units R^\times for every edge e ; note we do not assume $d(x) \in R^\times$.*

DEFINITION 2.3. *For an R -network (G, L) , we define the fundamental R -module $\Upsilon_R(G, L)$ as the R -module*

$$\Upsilon(G, L) = RV/L(RV^\circ).$$

When it is helpful to emphasize the ring R , we will write $\Upsilon_R(G, L)$.

Example 2.4. For a ∂ -graph G , the standard graph Laplacian L_{std} over $R = \mathbb{Z}$ corresponds to the case where $d(x) = 0$ and $w(e) = 1$. Let G be a finite connected graph (without boundary) considered as a ∂ -graph by setting $V^\circ = V$. Then $\Upsilon_{\mathbb{Z}}(G, L_{\text{std}})$ is the cokernel of $L_{\text{std}} : \mathbb{Z}V \rightarrow \mathbb{Z}V$, which is known to be isomorphic to $\text{Crit}(G) \oplus \mathbb{Z}$. See [9, Theorem 4.2], [23, Theorem 14.13.3], and [21, section 2].

Example 2.5. Lorenzini [33, p. 481] considers the generalized critical group of *arithmetical graphs* constructed by taking $R = \mathbb{Z}$, taking $w(e) = 1$, and choosing $d(x)$ such that the diagonal entries of L are positive and $\ker L$ contains some vector $r : V \rightarrow \mathbb{N}$ with positive entries. Such graphs arise in algebraic geometry.

Example 2.6. For a weighted graph or resistor network as in [18, section 3.1], [15], we consider $R = \mathbb{R}$, let $w(e) > 0$ be the conductance of the edge, and let $d(x) = 0$.

Then L represents the linear map that sends a potential function in \mathbb{R}^V to the function in \mathbb{R}^V giving the net current induced at each vertex. If G is connected and has at least one boundary vertex, then the submatrix of L with rows and columns indexed by the interior vertices will be invertible [18, Lemma 3.8]. Therefore, $L : \mathbb{R}^{V^\circ} \rightarrow \mathbb{R}^V$ has the maximal rank $|V^\circ|$, so $\Upsilon(G, L)$ will be a vector space over \mathbb{R} of dimension $|\partial V|$.

Example 2.7. Let G be a finite graph without boundary and $R = \mathbb{C}[z]$. Then $zI - L_{\text{std}}$ is obtained by taking $w(e) = -1$ and $d(x) = z$. We can relate $\Upsilon_{\mathbb{C}[z]}(G, zI - L_{\text{std}})$ to the eigenspaces and characteristic polynomial of L_{std} as follows: Recall that L_{std} is symmetric and hence can be written as $S\Lambda S^{-1}$, where S is unitary and Λ is a diagonal matrix with diagonal entries given by the eigenvalues $\lambda_1, \dots, \lambda_n$ of L_{std} . Then we have

$$\begin{aligned} \Upsilon_{\mathbb{C}[z]}(G, zI - L_{\text{std}}) &= \text{coker}_{\mathbb{C}[z]}(zI - L_{\text{std}}) \\ &= \text{coker}_{\mathbb{C}[z]}(S(zI - \Lambda)S^{-1}) \\ &\cong \bigoplus_{j=1}^n \mathbb{C}[z]/(z - \lambda_j). \end{aligned}$$

The summands $\mathbb{C}[z]/(z - \lambda_j)$ correspond to the eigenspaces of L_{std} . In the theory of modules over a PID, this is an elementary-divisor decomposition of $\Upsilon_{\mathbb{C}[z]}(G, zI - L_{\text{std}})$ over $\mathbb{C}[z]$ (for further algebraic explanation see [20, section 12]). The product of the elementary divisors $(z - \lambda_j)$ is the characteristic polynomial $\det(zI - L_{\text{std}})$. The characteristic polynomial of the adjacency matrix relates to our theory in a similar way. We will develop this example further in Example 2.16 and Proposition 4.14.

2.2. Duality between $\Upsilon(G, L)$ and harmonic functions. We mentioned in the introduction that \mathbb{R}/\mathbb{Z} -valued harmonic functions for L_{std} are related to $\text{Crit}(G)$ through Pontryagin duality. We will adapt this observation to our more general setting by interpreting $\text{Hom}(\Upsilon, -)$ and $\text{Tor}_1(\Upsilon, -)$ in terms of harmonic functions. We begin with the Hom functor.

Recall that if M and N are R -modules, then $\text{Hom}_R(M, N)$ is the set of R -module morphisms $M \rightarrow N$. For an R -module M , let M^V be the R -module of functions $V \rightarrow M$ (or 0-cochains in the language of topology). Recall M^V is naturally isomorphic to $\text{Hom}_R(RV, M)$. The map $L : RV \rightarrow RV$ induces a map in the reverse direction $L^* = \text{Hom}(L, M) : M^V \rightarrow M^V$. Explicitly, if $u : V \rightarrow M$, then $L^*u : V \rightarrow M$ is given by

$$L^*u(x) = u(Lx) = d(x)u(x) + \sum_{e \in \mathcal{E}(x)} w(e)(u(x) - u(e_-)).$$

Observe that L_{std}^*u corresponds to the standard Laplacian on functions $V \rightarrow \mathbb{Z}$.

If we express L with a matrix using the standard basis for RV , then the $L^* : M^V \rightarrow M^V$ is given by the transposed matrix. However, the matrix of L in the standard basis is symmetric, and thus $L^* : M^V \rightarrow M^V$ is given by the same matrix as L . Hence, for a finite ∂ -graph, if we identify R^V with RV , then L and L^* are the same operator.

In light of this fact, it does not seem necessary for our notation to distinguish between L and L^* . Henceforth, we will denote them both by L . However, we will preserve the distinction between the chain module RV and the cochain module R^V (and of course the cochain module M^V for each R -module M). The domain and codomain for the various operators denoted by L will be made clear in context.

DEFINITION 2.8. *Let (G, L) be an R -network. We say that $u : V \rightarrow M$ is harmonic if $Lu(x) = 0$ for every $x \in V^\circ$. We denote the R -module of harmonic functions by*

$$\mathcal{U}(G, L, M) = \{u \in M^V : Lu|_{V^\circ} \equiv 0\}.$$

Note that $\mathcal{U}(G, L, -)$ is a covariant functor $R\text{-mod} \rightarrow R\text{-mod}$. The significance of harmonic functions to the study of Υ comes from the following module-theoretic duality between $\Upsilon(G, L)$ and $\mathcal{U}(G, L, M)$.

LEMMA 2.9. *For every R -network (G, L) , there is a natural R -module isomorphism*

$$\mathcal{U}(G, L, M) \cong \text{Hom}_R(\Upsilon(G, L), M).$$

Proof. A function $u : V \rightarrow M$ is equivalent to an R -module morphism $u : RV \rightarrow M$, and $(Lu)(x) = (L^*u)(x) = u(Lx)$. Thus, u is harmonic if and only if

$$(Lu)(x) = u(Lx) = 0 \text{ for each } x \in V^\circ.$$

In other words, u is harmonic if and only if it vanishes on $L(RV^\circ)$. Thus, harmonic functions are equivalent to R -module morphisms $RV/L(RV^\circ) \rightarrow M$, and $\Upsilon(G, L)$ was defined as $RV/L(RV^\circ)$. □

2.3. Torsion and degeneracy. In this section, we will describe $\text{Tor}_j(\Upsilon(G, L), M)$ in terms of harmonic M -valued functions. To motivate the discussion, recall that for a graph without boundary with edge weights 1, we have $\Upsilon_{\mathbb{Z}}(G, L_{\text{std}}) \cong \text{Crit}(G) \oplus \mathbb{Z}$. In this case, the most interesting part of $\Upsilon_{\mathbb{Z}}(G, L_{\text{std}})$ is its torsion submodule $\text{Crit}(G)$. Similarly, when R is a PID and (G, L) is a finite R -network, the nontrivial task will be to understand the torsion submodule of $\Upsilon(G, L)$ (see Proposition 2.20 below for further explanation).

The Tor functor is a standard homological tool which can (among other things) detect the torsion part of a finitely generated module over a PID. The Tor functors are defined as the left-derived functors of the tensor-product functor $N \otimes -$. An R -module N is called *flat* if $N \otimes -$ is exact, which is equivalent to $\text{Tor}_j(N, -) = 0$ for $j > 0$ (see [2, Exercise 2.25]). If R is a PID, then N is flat if and only if it is torsion-free (see [20, Exercise 10.4.26]).

The functor $\text{Tor}_1(\Upsilon(G, L), -)$ turns out to have an easy description in terms of harmonic functions.

DEFINITION 2.10. *Define*

$$\mathcal{U}_0(G, L, M) = \{\text{finitely supported } u \in M^V : Lu \equiv 0 \text{ and } u|_{\partial V} \equiv 0\}.$$

PROPOSITION 2.11. *Suppose R is a commutative ring and (G, L) is an R -network. If $\mathcal{U}_0(G, L, R) = 0$, then we have a natural R -module isomorphism*

$$\mathcal{U}_0(G, L, M) \cong \text{Tor}_1^R(\Upsilon(G, L), M),$$

and $\text{Tor}_j^R(\Upsilon(G, L), M) = 0$ for $j > 1$. In the case where $\mathcal{U}_0(G, L, R) \neq 0$, we still have a natural surjection

$$\mathcal{U}_0(G, L, M) \twoheadrightarrow \text{Tor}_1^R(\Upsilon(G, L), M).$$

Proof. Note that RV° can be interpreted as the module of finitely supported R -valued functions that vanish on ∂V . Thus, $\mathcal{U}_0(G, L, R)$ is the kernel of the map $L : RV^\circ \rightarrow RV$. Since we assumed $\mathcal{U}_0(G, L, R) = 0$, we know that

$$\cdots \rightarrow 0 \rightarrow RV^\circ \xrightarrow{L} RV \rightarrow \Upsilon \rightarrow 0$$

is a free resolution of Υ . Thus, $\text{Tor}_j(\Upsilon, M)$ is the homology of the sequence

$$\cdots \rightarrow 0 \rightarrow RV^\circ \otimes M \xrightarrow{L \otimes \text{id}} RV \otimes M \rightarrow 0.$$

Thus, $\text{Tor}_j(\Upsilon, M) = 0$ for $j > 1$, and $\text{Tor}_1(\Upsilon, M)$ is the kernel of the map $L \otimes \text{id} : RV^\circ \otimes M \rightarrow RV \otimes M$. We can identify $RV^\circ \otimes M$ with the module of finitely supported functions $u : V \rightarrow M$ with $u|_{\partial V} = 0$, and then $L \otimes \text{id}$ is simply the generalized Laplacian. Hence, the kernel of $L \otimes \text{id} : RV^\circ \otimes M \rightarrow RV \otimes M$ is precisely $\mathcal{U}_0(G, L, M)$. Thus, $\text{Tor}_1^R(\Upsilon(G, L)) \cong \mathcal{U}_0(G, L, M)$, and the naturality of the isomorphism with respect to M is easy to verify from the construction.

In the general case, we have a free resolution

$$\cdots \rightarrow F_3 \rightarrow F_2 \rightarrow RV^\circ \xrightarrow{L} RV \rightarrow \Upsilon \rightarrow 0.$$

Then $\text{Tor}_1(\Upsilon(G, L), M)$ is obtained as a quotient of the kernel of $L \otimes \text{id} : RV^\circ \otimes M \rightarrow RV \otimes M$, so there is a surjection $\mathcal{U}_0(G, L, M) \twoheadrightarrow \text{Tor}_1^R(\Upsilon(G, L), M)$. \square

We will call an R -network *nondegenerate* if $\mathcal{U}_0(G, L, R) = 0$ and *degenerate* if $\mathcal{U}_0(G, L, R) \neq 0$. Thus, Proposition 2.11 shows that if (G, L) is nondegenerate, then $\mathcal{U}_0(G, L, M) \cong \text{Tor}_1^R(\Upsilon(G, L), M)$. We shall now show how nondegeneracy holds whenever the edge weights are positive as elements of an ordered ring (like the edge weights in a resistor network).

DEFINITION 2.12. *An ordered ring [31, Chapter 6] is a ring R together with a (transitive) total order $<$ given on R such that, for all elements $a, b, c \in R$,*

$$\begin{aligned} a < b &\implies a + c < b + c, \\ 0 < a \text{ and } 0 < b &\implies 0 < ab. \end{aligned}$$

PROPOSITION 2.13. *Suppose R is an ordered commutative ring and (G, L) is an R -network. Assume $w(e) > 0$ for all $e \in E$ and $d(x) \geq 0$ for all $x \in V$. Assume G is connected and one of the following holds: (A) $\partial V \neq \emptyset$, (B) V is infinite, or (C) there exists $x \in V$ with $d(x) > 0$.*

- (1) *If u is a finitely supported harmonic function and $u|_{\partial V} = 0$, then $u = 0$.*
- (2) *If u is a finitely supported harmonic function and $Lu|_{\partial V} = 0$, then u is constant. Moreover, if (B) or (C) holds, then $u = 0$.*
- (3) *We have $\mathcal{U}_0(G, L, R) = 0$, so the network is nondegenerate.*

Proof. This is a standard argument; one version can be found in [18, section 3.4 and Lemma 3.8]. For finitely supported functions $u, v : V \rightarrow R$, denote $\langle u, v \rangle = \sum_{x \in V} u(x)v(x)$. Observe that

$$\langle u, Lu \rangle = \sum_{x \in V} u(x) \left(d(x)u(x) + \sum_{e \in \mathcal{E}(x)} w(e)[u(x) - u(e_-)] \right).$$

If we let E' be a set of oriented edges containing exactly one of the two orientations for each edge, then algebraic manipulation yields

$$\langle u, Lu \rangle = \sum_{x \in V} d(x)u(x)^2 + \sum_{e \in E'} w(e)[u(e_+) - u(e_-)]^2.$$

This is a sum of all nonnegative terms. Thus, if $\langle u, Lu \rangle = 0$, we must have $u(e_+) - u(e_-) = 0$ for all $e \in E'$, which implies u is constant since G is connected. Moreover, if (B) holds, then since u is finitely supported, u must be zero at some vertex and so $u \equiv 0$. If (C) holds and $d(x) > 0$, then $u(x) = 0$ as well and hence $u \equiv 0$.

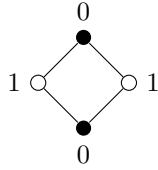


FIG. 2.1. A \mathbb{Z} -network with 2-torsion; $\bullet \in \partial V$, $\circ \in V^\circ$, and $w(e) = 1$. The numbers depict a nonzero element of $\mathcal{U}_0(G, L_{\text{std}}, \mathbb{Z}/2)$.

To prove (1), suppose u is harmonic (that is, $Lu|_{V^\circ} = 0$) and $u|_{\partial V} = 0$. Then

$$\langle u, Lu \rangle = \sum_{x \in V^\circ} u(x)Lu(x) + \sum_{x \in \partial V} u(x)Lu(x) = 0.$$

This implies u is constant. If (A) holds, then $u|_{\partial V} = 0$ implies that $u \equiv 0$, and if (B) or (C) holds, then the preceding argument already implies $u \equiv 0$.

To prove (2), suppose u is harmonic and $Lu|_{\partial V} = 0$. This amounts to saying $Lu = 0$ on all of V , which of course implies that $\langle u, Lu \rangle = 0$. Thus, (2) follows from the preceding argument. Moreover, (3) is an immediate consequence of either (1) or (2). \square

In the language of PDE, (1) is a uniqueness principle for the Dirichlet problem and (2) is a uniqueness principle for the Neumann problem. For a function u to be in \mathcal{U}_0 , it must violate *both* uniqueness principles simultaneously. While this is impossible for R -valued functions under the hypotheses of Proposition 2.13, it is possible for functions taking values in an R -module M . In fact, Proposition 2.11 says that torsion of $\Upsilon(G, L)$ corresponds to failure of the uniqueness principles for M -valued harmonic functions. In electrical language, for a nondegenerate network, $\text{Tor}_1(\Upsilon, M) \neq 0$ if and only if there are harmonic M -valued functions that are not detectable from boundary measurements of potential and current.

Example 2.14. Figure 2.1 shows a nondegenerate \mathbb{Z} -network such that

$$\text{Tor}_1(\Upsilon_{\mathbb{Z}}(G, L_{\text{std}}), \mathbb{Z}/2) = \mathcal{U}_0(G, L_{\text{std}}, \mathbb{Z}/2) \neq 0.$$

Nondegeneracy follows from Proposition 2.13, and a nonzero element of $\mathcal{U}_0(G, L_{\text{std}})$ is shown in Figure 2.1. In fact, $\Upsilon_{\mathbb{Z}}(G, L_{\text{std}}) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2$ (see Example 2.21).

Since torsion of $\Upsilon(G, L)$ and degeneracy of (G, L) are both measured by conditions of the form $\mathcal{U}_0(G, L, M) \neq 0$, it is not surprising that they are related. As a corollary of Proposition 2.11, we can show that *torsion* of Υ for a nondegenerate network over R is equivalent to *degeneracy* of networks over quotient rings of R . Given an R -network (G, L) and an ideal $\mathfrak{a} \subset R$, define $(G, L/\mathfrak{a})$ as the R/\mathfrak{a} -network obtained by reducing the edge weights modulo \mathfrak{a} .

COROLLARY 2.15. *Let (G, L) be a nondegenerate R -network. If \mathfrak{a} is an ideal of R , then $\text{Tor}_1(\Upsilon(G, L), R/\mathfrak{a}) = 0$ if and only if $(G, L/\mathfrak{a})$ is nondegenerate. Hence, $\Upsilon(G, L)$ is flat if and only if $(G, L/\mathfrak{a})$ is nondegenerate for every proper ideal \mathfrak{a} . Moreover, it suffices to check prime ideals or maximal ideals.*

Proof. Note that for every function $u: V \rightarrow R/\mathfrak{a}$, we have $Lu = (L/\mathfrak{a})u$, and hence

$$\text{Tor}_1^R(\Upsilon_R(G, L), R/\mathfrak{a}) \cong \mathcal{U}_0^R(G, L, R/\mathfrak{a}) = \mathcal{U}_0^{R/\mathfrak{a}}(G, L/\mathfrak{a}, R/\mathfrak{a}).$$

The first claim follows. For the second claim, recall that an R -module N is flat if and only if $\text{Tor}_1(N, R/\mathfrak{a}) = 0$ for all proper ideals $\mathfrak{a} \subset R$, and it suffices to check prime ideals or maximal ideals [2, Exercise 2.26], [44, Corollary 3.2.13]. \square

Example 2.16. As in Example 2.7, let G be a finite graph without boundary and consider the $\mathbb{C}[z]$ -network $(G, zI - L_{\text{std}})$. Since $\det(zI - L_{\text{std}}) \neq 0$, we know $zI - L_{\text{std}}$ is an injective map $\mathbb{C}[z]^V \rightarrow \mathbb{C}[z]^V$ and hence the network is nondegenerate. Recall that the maximal ideals of $\mathbb{C}[z]$ are $\{(z - \lambda) : \lambda \in \mathbb{C}\}$. For each $\lambda \in \mathbb{C}$, the quotient $\mathbb{C}[z]/(z - \lambda)$ is a field isomorphic to \mathbb{C} via the obvious map $\mathbb{C} \rightarrow \mathbb{C}[z]/(z - \lambda)$. For each λ ,

$$\text{Tor}_1(\Upsilon_{\mathbb{C}[z]}(G, zI - L_{\text{std}}), \mathbb{C}[z]/(z - \lambda)) \cong \mathcal{U}_0(G, zI - L_{\text{std}}, \mathbb{C}[z]/(z - \lambda)).$$

If we reinterpret the right-hand side in terms of the quotient network over $\mathbb{C}[z]/(z - \lambda)$ and apply the standard isomorphism $\mathbb{C}[z]/(z - \lambda) \cong \mathbb{C}$, we see that there is a vector space isomorphism

$$\mathcal{U}_0(G, zI - L_{\text{std}}, \mathbb{C}[z]/(z - \lambda)) \cong \mathcal{U}_0(G, \lambda I - L_{\text{std}}, \mathbb{C}),$$

where the right-hand side is computed over $R = \mathbb{C}$. This is precisely the λ -eigenspace of L_{std} if λ is an eigenvalue and zero otherwise.

In the next example, for a finite ∂ -graph G and a field F , we will model a general F^\times -network by assigning indeterminates as edge weights. This construction will be used in our algebraic characterization of layerability (Theorem 5.16).

DEFINITION 2.17. *Let G be a finite ∂ -graph and let F be a field. We define the ring $R^*(G, F)$ and generalized Laplacian $L^*(G, F)$ as follows. Let $R^*(G, F)$ be the polynomial algebra over F generated by indeterminates $\{t_x : x \in V\}$ and $\{t_e^{\pm 1} : e \in E\}$, where $t_e = t_{\bar{e}}$. The generalized Laplacian L^* over R^* is given by the functions $w^*(e) = t_e$ and $d^*(x) = t_x$.*

PROPOSITION 2.18. *The $(R^*)^\times$ -network (G, L^*) defined above is nondegenerate. If $\Upsilon(G, L^*)$ is a flat R^* -module, then every F^\times -network on the ∂ -graph G is nondegenerate. Moreover, the converse holds if F is algebraically closed.*

Proof. First we prove nondegeneracy. Note that $\det L^*$ is clearly a nonzero polynomial, hence $L^* : R^*V \rightarrow R^*V$ is injective. If $u \in \mathcal{U}_0(G, L^*, R^*)$, then we have $L^*u \equiv 0$ (that is, $L^*u(x) = 0$ both for $x \in V^\circ$ and $x \in \partial V$), and therefore, $u \equiv 0$ by injectivity of L^* .

Next, we show that flatness of $\Upsilon(G, L^*)$ implies non-degeneracy of every F^\times -network (G, L) . Suppose that L is a generalized Laplacian over F given by $w : E \rightarrow F^\times$ and $d : V \rightarrow F$. Let \mathfrak{a} be the ideal in R^* generated by $t_e - w(e)$ for $e \in E$ and $t_x - d(x)$ for $x \in V$. Then \mathfrak{a} is a maximal ideal and R^*/\mathfrak{a} is a field isomorphic to F , where t_e is identified with $w(e) \in F^\times$ and t_x is identified with $d(x)$. Therefore, $(G, L^*/\mathfrak{a})$ corresponds to the F^\times -network (G, L) . Thus, by Corollary 2.15, we have a vector space isomorphism

$$\text{Tor}_1^{R^*}(\Upsilon(G, L^*), R^*/\mathfrak{a}) \cong \mathcal{U}_0(G, L^*/\mathfrak{a}, R^*/\mathfrak{a}) \cong \mathcal{U}_0(G, L, F).$$

In particular, flatness of $\Upsilon_{R^*}(G, L^*)$ implies that every F^\times -network on the ∂ -graph G is nondegenerate.

Furthermore, the converse holds if F is algebraically closed. Indeed, in this case, every maximal ideal \mathfrak{a} of R^* has the form

$$\mathfrak{a} = (t_e - w(e) : e \in E; t_x - d(x) : x \in V)$$

for some $w : E \rightarrow F$ and $d : V \rightarrow F$. (This can be deduced by noting that R^* is a localization of the polynomial algebra $F[t_e : e \in E; t_x : x \in V]$, and every proper ideal in the polynomial algebra $F[x_1, \dots, x_n]$ must have a common zero by Hilbert’s Nullstellensatz [2, Exercise 5.17], [20, section 15.3, Corollary 33].) Then since t_e is a unit in R^* by construction, we deduce that $w(e)$ is nonzero. Thus, w and d define an F^\times -network on G , which corresponds to $(G, L/\mathfrak{a})$. Hence, if every F^\times -network on G is nondegenerate, then $\text{Tor}_1^{R^*}(\Upsilon(G, L^*), R^*/\mathfrak{a}) = 0$ for every maximal ideal and hence $\Upsilon(G, L^*)$ is flat. \square

2.4. Exactness of $\mathcal{U}(G, L, -)$. In section 2.3, we interpreted *flatness* of $\Upsilon(G, L)$ in terms of harmonic functions, and now we will (quite briefly) do the same for *projectiveness* of $\Upsilon(G, L)$, which is detected by the functors $\text{Ext}^j(\Upsilon(G, L), -)$. Once again, we are largely motivated by the situation where $\Upsilon(G, L)$ is a finitely generated module over a PID R , in which case the functors $\text{Ext}^j(\Upsilon(G, L), -)$ completely describe the torsion submodule of $\Upsilon(G, L)$.

Recall that for an R -module N , $\text{Hom}(N, -)$ is always left exact, and N is called *projective* if it is also right exact. The failure of N to be projective is measured by the functors $\text{Ext}^j(N, -)$, which are the right-derived functors of $\text{Hom}(N, -)$. Moreover, N is projective if and only if $\text{Ext}^1(N, -) = 0$. Free modules are always projective. If R is a PID and N is a finitely generated R -module, then N is torsion-free if and only if it is projective (as one can deduce from the classification of finitely generated modules over a PID).

The fundamental module Υ is projective if and only if $\text{Hom}(\Upsilon(G, L), -) = \mathcal{U}(G, L, -)$ is right exact. Thus, concretely, right exactness asks, Given a surjective map $M \rightarrow N$ between R -modules, is $\mathcal{U}(G, L, M) \rightarrow \mathcal{U}(G, L, N)$ a surjection? In other words, does every N -valued harmonic function on (G, L) lift to an M -valued harmonic function?

Example 2.19. $\mathcal{U}(G, L, -)$ fails to be right exact for the \mathbb{Z} -network in Figure 2.2. Consider the surjection $\mathbb{Z}/4 \rightarrow \mathbb{Z}/2$. The corresponding map $\mathcal{U}(G, L, \mathbb{Z}/4) \rightarrow \mathcal{U}(G, L, \mathbb{Z}/2)$ is not surjective. If $u \in \mathcal{U}(G, L, \mathbb{Z}/4)$, then $2u(B) = u(A) + u(D) = 2u(C) \pmod{4}$, and hence $u(B) = u(C) \pmod{2}$. However, not all $\mathbb{Z}/2$ -valued harmonic functions satisfy $u(B) = u(C)$; for instance, the indicator function $1_B : V \rightarrow \mathbb{Z}/2$ is harmonic.

2.5. Summary of homological properties. Our results thus far provide a lexicon giving “harmonic” or “electrical” interpretations of the homological properties of Υ for nondegenerate networks:

- (1) As remarked in the proof of Proposition 2.11, $\Upsilon(G, L)$ has a free resolution given by $0 \rightarrow RV^\circ \xrightarrow{L} RV \rightarrow \Upsilon(G, L) \rightarrow 0$.
- (2) $\text{Hom}(\Upsilon(G, L), M) = \mathcal{U}(G, L, M)$ is the module of M -valued harmonic functions.

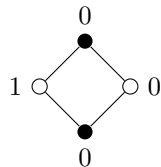


FIG. 2.2. A \mathbb{Z} -network with 2-torsion; $\bullet \in \partial V$, $\circ \in V^\circ$, and $w(e) = 1$. The numbers depict a $\mathbb{Z}/2$ -valued harmonic function that does not lift to a $\mathbb{Z}/4$ -valued harmonic function.

- (3) $\text{Tor}_1(\Upsilon(G, L), M) = \mathcal{U}_0(G, L, M)$ is the module of finitely supported harmonic functions with vanishing potential and current on the boundary.
- (4) $\text{Ext}^1(\Upsilon(G, L), -)$ is the right-derived functor of $\mathcal{U}(G, L, -) = \text{Hom}_R(\Upsilon(G, L), -)$. It measures the failure of N -valued harmonic functions to lift to M -valued harmonic functions when $M \rightarrow N$ is surjective.
- (5) Using our free resolution of $\Upsilon(G, L)$, we can also compute $\text{Ext}^1(\Upsilon(G, L), M)$ as the cokernel of $L: M^V \rightarrow M^{V^\circ}$. In other words, it is the module of M -valued functions on V° modulo those that arise as the generalized Laplacian (or net current) of M -valued potentials on V .

These observations lead to many different ways of computing Υ when the ring R is a PID such as \mathbb{Z} or $\mathbb{C}[z]$ (Proposition 2.20). We recall the following terminology and facts about PIDs. A ring is called a PID if every ideal is generated by a single element. The classification of finite abelian groups (or \mathbb{Z} -modules) generalizes to PIDs: if R is a PID, then any finitely generated R -module is isomorphic to one of the form

$$R^{\oplus m} \oplus \bigoplus_{j=1}^n R/f_j,$$

where $f_j | f_{j+1}$. The numbers of f_j are called the *invariant factors* of M . We call $R^{\oplus m}$ and $\bigoplus_{j=1}^n R/f_j$ respectively the *free submodule* and *torsion submodule* of M . For details, see [20, section 12.1].

PROPOSITION 2.20. *Let R be a PID, F its field of fractions, and (G, L) a finite nondegenerate R -network. Then the free submodule of $\Upsilon(G, L)$ has rank $|\partial V|$. Moreover, the following are (noncanonically) isomorphic:*

- (1) *The torsion submodule of $\Upsilon(G, L)$.*
- (2) *The cokernel of $L: R^V \rightarrow R^{V^\circ}$.*
- (3) *$\mathcal{U}(G, L, F/R)$ modulo the image of $\mathcal{U}(G, L, F)$.*
- (4) *$\mathcal{U}_0(G, L, F/R) = \text{Tor}_1(\Upsilon(G, L), F/R)$.*

Thus, in particular,

$$(2.1) \quad \Upsilon(G, L) \cong R^{|\partial V|} \oplus \mathcal{U}_0(G, L, F/R).$$

Proof. To show that the free rank is $|\partial V|$, note that since (G, L) is nondegenerate, we have a short exact sequence

$$0 \rightarrow RV^\circ \xrightarrow{L} RV \rightarrow \Upsilon(G, L) \rightarrow 0.$$

Let $\Upsilon(G, L) \cong R^n \oplus N$, where N is the torsion submodule. Since F is a flat R -module, we have a short exact sequence

$$0 \rightarrow F \otimes RV^\circ \xrightarrow{L} F \otimes RV \rightarrow (F \otimes R^n) \oplus (F \otimes N) \rightarrow 0.$$

However, $F \otimes N = 0$. Thus, our sequence becomes $0 \rightarrow FV^\circ \rightarrow FV \rightarrow F^n \rightarrow 0$, and the rank-nullity theorem implies $n = |V| - |V^\circ| = |\partial V|$.

Next, we prove that (1)–(4) are isomorphic. Note (2) and (3) are two different ways of evaluating $\text{Ext}^1(\Upsilon(G, L), R)$; (2) uses the projective resolution of $\Upsilon(G, L)$ and (3) uses the injective resolution of R given by $0 \rightarrow R \rightarrow F \rightarrow F/R \rightarrow 0$. To show that the torsion submodule of $\Upsilon(G, L)$ is isomorphic to $\text{Tor}_1(\Upsilon(G, L), F/R)$ and $\text{Ext}^1(\Upsilon(G, L), R)$, decompose $\Upsilon(G, L)$ as the direct sum of cyclic modules.

To check (2.1), recall $\Upsilon(G, L) \cong R^n \oplus N$, where R^n is the free submodule and N is the torsion submodule. The preceding argument showed that $n = |\partial V|$ and $N \cong \mathcal{U}_0(G, L, F/R)$. □

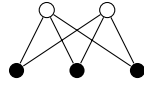


FIG. 2.3. The complete boundary-interior bipartite ∂ -graph $K_{3,2}$.

Example 2.21 (complete bipartite graphs). Consider the complete bipartite graph $K_{m,n}$ whose partite sets consist of m boundary vertices and n interior vertices, respectively (see Figure 2.3 for an example). The standard Laplacian $L_{\text{std}}: R^V \rightarrow R^{V^\circ}$ is

$$L_{\text{std}} = \left(\begin{array}{cccc|cccc} -1 & \cdots & \cdots & -1 & m & & & \\ \vdots & \ddots & \ddots & \vdots & & \ddots & & \\ \vdots & & \ddots & \vdots & & & \ddots & \\ -1 & \cdots & \cdots & -1 & & & & m \end{array} \right)$$

$\underbrace{\hspace{10em}}_{\partial V} \quad \underbrace{\hspace{10em}}_{V^\circ}$

and its cokernel is isomorphic to $(\mathbb{Z}/m)^{n-1}$. Dually,

$$\mathcal{U}_0(G, L_{\text{std}}, \mathbb{Q}/\mathbb{Z}) = \left\{ u \in (\mathbb{Q}/\mathbb{Z})^{V^\circ} : mu = 0, \sum_{x \in V^\circ} u(x) = 0 \right\} \cong (\mathbb{Z}/m)^{n-1}.$$

Hence, by (2.1) in Proposition 2.20, we have

$$\Upsilon(G, L_{\text{std}}) \cong \mathbb{Z}^m \oplus (\mathbb{Z}/m)^{n-1}.$$

2.6. Application to critical group. Let us examine how our algebraic constructions work out in the case of the critical group.

PROPOSITION 2.22. *Let G be a connected graph without boundary, considered as a ∂ -graph with zero boundary vertices.*

- (1) $\text{Crit}(G)$ is the torsion submodule of $\Upsilon_{\mathbb{Z}}(G, L_{\text{std}})$.
- (2) For every vertex x , the map $\mathbb{Z}x \rightarrow \Upsilon_{\mathbb{Z}}(G, L_{\text{std}})$ is injective and we have an internal direct sum $\Upsilon_{\mathbb{Z}}(G, L_{\text{std}}) = \mathbb{Z}x \oplus \text{Crit}(G)$.
- (3) We have $\mathcal{U}(G, L_{\text{std}}, \mathbb{Q}/\mathbb{Z}) \cong \text{Crit}(G) \times \mathbb{Q}/\mathbb{Z}$.
- (4) We have $\mathcal{U}_0(G, L_{\text{std}}, \mathbb{Q}/\mathbb{Z}) = \mathcal{U}(G, L_{\text{std}}, \mathbb{Q}/\mathbb{Z})$.

Proof. Let $\epsilon: \mathbb{Z}V \rightarrow \mathbb{Z}$ be the map given by $x \mapsto 1$ for every $x \in V$. Note that $L_{\text{std}}(\mathbb{Z}V) \subseteq \ker \epsilon$. Moreover, it is well-known that L_{std} has rank $|V| - 1$ when G is connected (see, e.g., [18, Lemma 3.8]), so that $\ker \epsilon / \text{im } L_{\text{std}}$ is a torsion \mathbb{Z} -module, and this is known to be isomorphic to $\text{Crit}(G)$ [9, Theorem 4.2], [21, Definition 2.2]. For each vertex x , we have $\mathbb{Z}V = \mathbb{Z}x \oplus \ker \epsilon$, which implies that

$$\mathbb{Z}V / \text{im } L_{\text{std}} = \mathbb{Z}x \oplus \text{Crit}(G).$$

This establishes (1) and (2). Next, (3) follows by applying $\text{Hom}(-, \mathbb{Q}/\mathbb{Z})$ to (2) and using Lemma 2.9, and (4) follows from the definition of \mathcal{U} and \mathcal{U}_0 because there are no boundary vertices. \square

Several constructions of the critical group involve designating a “sink” vertex x . In a similar way, we can choose a boundary vertex x when computing $\text{Crit}(G)$.

PROPOSITION 2.23. *Let G be a connected graph without boundary, and let G' be obtained from G by assigning one boundary vertex x .*

- (1) The network (G', L_{std}) is nondegenerate.
- (2) We have $\Upsilon_{\mathbb{Z}}(G', L_{\text{std}}) = \Upsilon_{\mathbb{Z}}(G, L_{\text{std}})$, hence (1), (2), and (3) of Proposition 2.22 hold with G replaced by G' .
- (3) We have $\mathcal{U}_0(G, L_{\text{std}}, \mathbb{Q}/\mathbb{Z}) \cong \text{Tor}_1(\Upsilon(G', L_{\text{std}}), \mathbb{Q}/\mathbb{Z}) \cong \text{Crit}(G)$.

Proof. Nondegeneracy follows from Proposition 2.13. Let V° denote $V^\circ(G') = V(G) \setminus \{x\}$. To prove that $\Upsilon(G, L_{\text{std}}) = \Upsilon(G', L_{\text{std}})$, it suffices to show that $L_{\text{std}}(\mathbb{Z}V^\circ) = L_{\text{std}}(\mathbb{Z}V)$. Recall that the constant vector c_0 is in the kernel of L_{std} . If $w \in L_{\text{std}}(\mathbb{Z}V)$, then $w = L_{\text{std}}z$ for some $z \in \mathbb{Z}V$. By subtracting a multiple of the c_0 , we can assume that the coordinate of z corresponding to the vertex x is zero. This means $z \in \mathbb{Z}V^\circ$, so $w \in L_{\text{std}}(\mathbb{Z}V^\circ)$.

From $\Upsilon(G, L_{\text{std}}) = \Upsilon(G', L_{\text{std}})$, it immediately follows that $\text{Crit}(G)$ is the torsion submodule of $\Upsilon(G', L_{\text{std}})$. From the application of $\text{Hom}_{\mathbb{Z}}(-, M)$, we see that $\mathcal{U}(G, L_{\text{std}}, M) = \mathcal{U}(G', L_{\text{std}}, M)$ for every \mathbb{Z} -module M , and hence $\mathcal{U}(G', L_{\text{std}}, \mathbb{Q}/\mathbb{Z}) \cong \mathcal{U}(G, L_{\text{std}}, \mathbb{Q}/\mathbb{Z}) \cong \text{Crit}(G) \times \mathbb{Q}/\mathbb{Z}$. Finally, (3) follows from Proposition 2.20. \square

3. A family of “chain link fence” networks.

3.1. Motivation and set-up. To date, the theory of the critical group has mainly focused on graphs without boundary. In this section, we will analyze an infinite family of ∂ -graphs with nontrivial boundary. These ∂ -graphs resemble a chain link fence which embeds either on the cylinder or on the Möbius band (depending on parity). This family is a variant of the “purely cylindrical” graphs described in [29] which play a key role in the electrical inverse problem. Though self-contained, our computation here will illustrate and motivate techniques that we will develop systematically later in the paper—including discrete harmonic continuation, symmetry and covering spaces, and subgraphs.

Consider a ∂ -graph $\text{CLF}(m, n)$ with $V = \mathbb{Z}/m \times \{0, \dots, n\}$ and $\partial V = \mathbb{Z}/m \times \{0\}$ and edges defined by

$$\begin{aligned} (j, k) &\sim (j + 1, n - k + 1) \text{ for } k \geq 1, \\ (j, k) &\sim (j + 1, n - k) \text{ for } k \geq 0, \end{aligned}$$

as shown in Figure 3.1. If m is even, then the network is one of the Lam and Pilyavskyy’s “purely cylindrical graphs” and it embeds into the cylinder or the annulus (see Figure 3.2). If m is odd, then it resembles a chain-link fence twisted into a Möbius band.

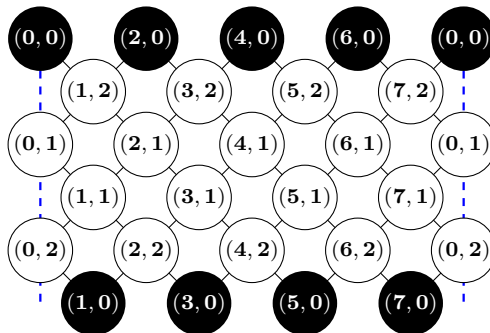


FIG. 3.1. The ∂ -graph $\text{CLF}(8, 2)$. Boundary vertices are black, interior vertices are white, and the vertices on the left and right sides are identified along the dashed lines.

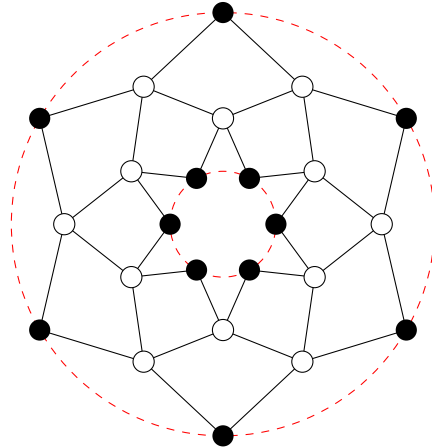


FIG. 3.2. The ∂ -graph $\text{CLF}(12, 1)$ embedded in the annulus rather than the cylinder.

Consider the \mathbb{Z} -network $(\text{CLF}(m, n), L_{\text{std}})$. Since the network is nondegenerate (Proposition 2.13), we have by (2.1) in Proposition 2.20 that

$$\Upsilon_{\mathbb{Z}}(\text{CLF}(m, n), L_{\text{std}}) \cong \mathbb{Z}^{|\partial V|} \oplus \mathcal{U}_0(\text{CLF}(m, n), L_{\text{std}}, \mathbb{Q}/\mathbb{Z}).$$

We will compute the torsion summand $\mathcal{U}_0(\text{CLF}(m, n), L_{\text{std}}, \mathbb{Q}/\mathbb{Z})$, showing the following.

THEOREM 3.1. *For the \mathbb{Z} -network $(\text{CLF}(m, n), L_{\text{std}})$, we have*

$$\mathcal{U}_0(\text{CLF}(m, n), L_{\text{std}}, \mathbb{Q}/\mathbb{Z}) \cong \begin{cases} (\mathbb{Z}/2)^n, & m \text{ odd,} \\ (\mathbb{Z}/2)^{2n}, & m \equiv 2 \pmod{4}, \\ \left(\bigoplus_{j=1}^n \mathbb{Z}/\gcd(4^j, 2m) \right)^{\oplus 2}, & m \equiv 0 \pmod{4}. \end{cases}$$

In section 3.2, we use harmonic continuation to write our module in a simple form in terms of a $2n \times 2n$ matrix \mathbf{T} (Lemma 3.2). Then in section 3.3, we work algebraically to find the invariant factor decomposition. Finally, in section 3.4, we will bootstrap our computation to handle a slightly different family of ∂ -graphs.

3.2. Harmonic continuation computation. Since we will deal with vectors in \mathbb{Z}^n as well as \mathbb{Z}^{2n} , we establish the following notational conventions:

- Vectors in \mathbb{Z}^n or $(\mathbb{Q}/\mathbb{Z})^n$ will be lowercase regular type.
- $n \times n$ matrices will be uppercase regular type.
- Vectors in \mathbb{Z}^{2n} or $(\mathbb{Q}/\mathbb{Z})^{2n}$ will be lowercase bold.
- $2n \times 2n$ matrices will be uppercase bold.
- “.” denotes the dot product.
- e_1, \dots, e_n and $\mathbf{e}_1, \dots, \mathbf{e}_{2n}$ denote the standard basis vectors.
- Vectors are assumed to be column vectors by default.

Moreover, we will abbreviate $\mathcal{U}_0(\text{CLF}(m, n), L_{\text{std}}, \mathbb{Q}/\mathbb{Z})$ to $U(m, n)$.

Our goal is to compute the \mathbb{Q}/\mathbb{Z} -valued harmonic functions with $u = Lu = 0$ on ∂V . We start by understanding the harmonic functions with $u = 0$ on the boundary using harmonic continuation around the circumference of the cylinder. Assume

$u(j, 0) = 0$ and let

$$a_j = \begin{pmatrix} u(j, 1) \\ \vdots \\ u(j, n) \end{pmatrix} \in (\mathbb{Q}/\mathbb{Z})^n.$$

The idea is to solve for a_{j+1} in terms of a_j and a_{j-1} such that partial function defined by a_{j-1} , a_j , and a_{j+1} will be harmonic on the j th column of vertices. Thus, we start with a_1 and a_0 , then find a_2, a_3, \dots . Recall the index j for vertices in the graph is reduced modulo m . The a_j 's to yield a well-defined harmonic function on $\text{CLF}(m, n)$, we require that $a_m = a_0$ and $a_{m+1} = a_1$.

In terms of the a_j 's, harmonicity amounts to

$$4a_j = Ea_{j-1} + Ea_{j+1},$$

where E is the $n \times n$ matrix with 1's on and directly above the skew-diagonal and zeros elsewhere—for instance,

$$E = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad n = 5.$$

Thus, the vectors a_j satisfy the recurrence relation

$$\begin{pmatrix} a_{j+1} \\ a_j \end{pmatrix} = \begin{pmatrix} 4E^{-1} & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} a_j \\ a_{j-1} \end{pmatrix}.$$

Let \mathbf{T} be the $2n \times 2n$ “propagation matrix” of harmonic continuation, that is,

$$\mathbf{T} = \begin{pmatrix} 4E^{-1} & -I \\ I & 0 \end{pmatrix}.$$

Note that $\det \mathbf{T} = \pm 1$, so \mathbf{T} is invertible over \mathbb{Z} . Multiplying by \mathbf{T}^{-1} corresponds to harmonic continuation in the opposite direction around the circumference of the cylinder.

Let us denote $\mathbf{a} = (a_1, a_0)^T \in (\mathbb{Q}/\mathbb{Z})^{2n}$. Through harmonic continuation, we can see that

$$\{u \in \mathcal{U}(\text{CLF}(m, n), \mathbb{Q}/\mathbb{Z}) : u|_{\partial V} = 0\} \cong \{\mathbf{a} \in (\mathbb{Q}/\mathbb{Z})^{2n} : \mathbf{T}^m \mathbf{a} = \mathbf{a}\}.$$

Next, we must determine when a fixed point of \mathbf{T}^m will yield a harmonic function u with $Lu|_{\partial V} = 0$, which amounts to writing all the net current conditions in terms of the first two columns of vertices. The net current at a boundary vertex $(j, 0)$ is

$$Lu(j, 0) = -u(j - 1, n) - u(j + 1, n) = -\mathbf{e}_{2n} \cdot \mathbf{T}^{j-1} \mathbf{a} - \mathbf{e}_{2n} \cdot \mathbf{T}^{j+1} \mathbf{a}.$$

We need to choose \mathbf{a} so that this holds for $j = 1, \dots, m$, but since we also require \mathbf{a} to be a fixed point of \mathbf{T}^m , we might as well require $\mathbf{e}_{2n} \cdot (\mathbf{T}^{j-1} + \mathbf{T}^{j+1}) \mathbf{a} = 0$ for all $j \in \mathbb{Z}$. Therefore, we have the following.

LEMMA 3.2.

$$U(m, n) \cong \{\mathbf{a} \in (\mathbb{Q}/\mathbb{Z})^{2n} : \mathbf{T}^m \mathbf{a} = \mathbf{a} \text{ and } \mathbf{e}_{2n} \cdot (\mathbf{T}^{j-1} + \mathbf{T}^{j+1}) \mathbf{a} = 0 \text{ for } j \in \mathbb{Z}\}.$$

3.3. An explicit basis for \mathcal{U}_0 . A key insight in the remaining computation is to consider the two conditions $\mathbf{T}^m \mathbf{a} = \mathbf{a}$ and $\mathbf{e}_{2n} \cdot (\mathbf{T}^{j-1} + \mathbf{T}^{j+1}) \mathbf{a} = 0$ separately. We denote

$$M_1 = \{\mathbf{a} \in (\mathbb{Q}/\mathbb{Z})^{2n} : \mathbf{e}_{2n} \cdot (\mathbf{T}^{j-1} + \mathbf{T}^{j+1}) \mathbf{a} = 0 \text{ for } j \in \mathbb{Z}\},$$

$$M_2 = \{\mathbf{a} \in (\mathbb{Q}/\mathbb{Z})^{2n} : \mathbf{T}^m \mathbf{a} = \mathbf{a}\}.$$

Observe that $U(m, n) \cong M_1 \cap M_2$.

Remark 3.3. Here is a geometric interpretation of M_1 and M_2 . Note that the universal cover of the cylinder or Möbius band is an infinite strip, and $\text{CLF}(m, n)$ is covered by a corresponding graph $\text{CLF}(\infty, n)$ with vertex set $\mathbb{Z} \times \{0, \dots, n\}$. The condition defining M_1 says that harmonic continuation with initial values \mathbf{a} defines a harmonic function on $\text{CLF}(\infty, n)$ with $u = 0$ and $L_{\text{std}} u = 0$ on the boundary. The condition defining M_2 says that $u(j, k)$ is periodic in j , and hence u corresponds to a harmonic function on $\text{CLF}(m, n)$.

We will compute M_1 first, using two auxiliary lemmas. In the following, $\mathbb{Z}[4E^{-1}]$ will denote the subring of $M_{n \times n}(\mathbb{Z})$ generated by $4E^{-1}$ and $\mathbb{Z}[\mathbf{T}, \mathbf{T}^{-1}]$ will denote the subring of $M_{2n \times 2n}(\mathbb{Z})$ generated by \mathbf{T} and \mathbf{T}^{-1} . We use $(I, 0) \cdot \mathbb{Z}[\mathbf{T}, \mathbf{T}^{-1}]$ to denote the \mathbb{Z} -submodule of $M_{n \times 2n}(\mathbb{Z})$ consisting of matrices of the form $(I, 0) \cdot \mathbf{S}$, where $\mathbf{S} \in \mathbb{Z}[\mathbf{T}, \mathbf{T}^{-1}]$ and $(I, 0) \in M_{n \times 2n}(\mathbb{Z})$ is written as a matrix with two $n \times n$ blocks. The notation $\mathbb{Z}[4E^{-1}] \cdot (I, 0)$ is to be interpreted similarly.

LEMMA 3.4. *We have an equality of \mathbb{Z} -modules*

$$(I, 0) \cdot \mathbb{Z}[\mathbf{T}, \mathbf{T}^{-1}] = \mathbb{Z}[4E^{-1}] \cdot (I, 0) + \mathbb{Z}[4E^{-1}] \cdot (0, I).$$

Proof. The inclusion \subseteq is straightforward since \mathbf{T} and \mathbf{T}^{-1} are block 2×2 matrices with block entries in $\mathbb{Z}[4E^{-1}]$. To prove the opposite inclusion, first observe that

$$\mathbf{T}^{-1} = \begin{pmatrix} 0 & I \\ -I & 4E^{-1} \end{pmatrix}.$$

Then note that

$$(I, 0) = (I, 0)\mathbf{I} \in (I, 0) \cdot \mathbb{Z}[\mathbf{T}, \mathbf{T}^{-1}](0, I) = (I, 0)\mathbf{T}^{-1} \in (I, 0) \cdot \mathbb{Z}[\mathbf{T}, \mathbf{T}^{-1}].$$

Moreover,

$$\mathbf{T} + \mathbf{T}^{-1} = \begin{pmatrix} 4E^{-1} & 0 \\ 0 & 4E^{-1} \end{pmatrix} = 4\mathbf{F},$$

where $\mathbf{F} := \text{diag}(E^{-1}, E^{-1})$. Thus, $4\mathbf{F} \in \mathbb{Z}[\mathbf{T}, \mathbf{T}^{-1}]$. This implies

$$(4^j E^{-j}, 0) = (I, 0)4^j \mathbf{F}^j \in (I, 0) \cdot \mathbb{Z}[\mathbf{T}, \mathbf{T}^{-1}],$$

$$(0, 4^j E^{-j}) = (0, I)4^j \mathbf{F}^j \in (I, 0) \cdot \mathbb{Z}[\mathbf{T}, \mathbf{T}^{-1}],$$

and thus all of $\mathbb{Z}[4E^{-1}] \cdot (I, 0) + \mathbb{Z}[4E^{-1}] \cdot (0, I)$ is contained in $(I, 0) \cdot \mathbb{Z}[\mathbf{T}, \mathbf{T}^{-1}]$. \square

LEMMA 3.5. *The row vectors $\{e_n^t E^{-j}\}_{j=0}^{n-1}$ are a basis for \mathbb{Z}^n .*

Proof. Because E is invertible over \mathbb{Z} , it suffices to show that $\{e_n^t E^{-j} E^{n-1}\}_{j=0}^{n-1} = \{e_n^t E^j\}_{j=0}^{n-1}$ is a basis for \mathbb{Z}^n . Moreover, to show it is a basis, it suffices to show that it spans \mathbb{Z}^n ; linear independence will follow automatically because our collection of

vectors has n elements. Let W_k be the \mathbb{Z} -span of $e_n^t, e_n^t E, \dots, e_n^t E^{k-1}$. We can show by induction on k that W_k includes the first k vectors from the ordered basis

$$e_n^1, e_1^t, e_{n-1}^t, e_2^t, e_{n-2}^t, e_3^t, \dots$$

The general procedure is clear from the first few steps:

- We have $e_n^t \in W_1$ trivially.
- Because $e_n^t \in W_1$ and $e_n^t E = e_1^t$, we have $e_1^t \in W_2$.
- Next, because $e_1^t \in W_2$, we have $e_1^t E = e_n^t + e_{n-1}^t \in W_3$. Moreover, $e_n^t \in W_1 \subseteq W_3$, so that $e_{n-1}^t \in W_3$.
- Next, because $e_{n-1}^t \in W_3$, we have $e_{n-1}^t E = e_1^t + e_2^t \in W_4$, which implies that $e_2^t \in W_4$.

At the last step of the induction, we obtain $W_n = \mathbb{Z}^n$ as desired. □

LEMMA 3.6.

$$M_1 \cong \left(\bigoplus_{j=1}^n \mathbb{Z}/4^j \right)^{\oplus 2}$$

Proof. Noting that $\mathbf{e}_{2n}^t \mathbf{T} = \mathbf{e}_n^t$, we have

$$M_1 = \{ \mathbf{a} \in (\mathbb{Q}/\mathbb{Z})^{2n} : \mathbf{e}_n \cdot (\mathbf{T}^{j-1} + \mathbf{T}^{j+1}) \mathbf{a} = 0 \text{ for } j \in \mathbb{Z} \}.$$

As in the proof of Lemma 3.4, we have $\mathbf{T} + \mathbf{T}^{-1} = 4\mathbf{F}$, and hence,

$$M_1 = \{ \mathbf{a} \in (\mathbb{Q}/\mathbb{Z})^{2n} : 4\mathbf{e}_n \cdot \mathbf{T}^j \mathbf{F} \mathbf{a} = 0 \text{ for } j \in \mathbb{Z} \}.$$

Since \mathbf{F} is invertible, we can view it as a change of coordinates on $(\mathbb{Q}/\mathbb{Z})^{2n}$ and replace $\mathbf{F} \mathbf{a}$ by \mathbf{a} , so that

$$M_1 \cong \{ \mathbf{a} \in (\mathbb{Q}/\mathbb{Z})^{2n} : 4\mathbf{e}_n^t \mathbf{T}^j \mathbf{a} = 0 \text{ for } j \in \mathbb{Z} \}.$$

We can rewrite $\mathbf{e}_n^t \mathbf{T}^j$ as $e_n^t(I, 0)\mathbf{T}^j$, where I and 0 are $n \times n$ identity and zero matrices, respectively, as mentioned above. Let $N \subseteq \mathbb{Z}^{2n}$ be the module of row vectors in \mathbb{Z}^{2n} given by

$$N = 4e_n^t(I, 0)(\mathbb{Z}[\mathbf{T}, \mathbf{T}^{-1}]) = \{ 4e_n^t(I, 0)\mathbf{S} : \mathbf{S} \in \mathbb{Z}[\mathbf{T}, \mathbf{T}^{-1}] \}.$$

Then we have

$$M_1 \cong \{ \mathbf{a} \in (\mathbb{Q}/\mathbb{Z})^{2n} : \mathbf{n}^t \mathbf{a} = 0 \text{ for all } \mathbf{n}^t \in N \}.$$

The remainder of the proof will use Lemmas 3.4 and 3.5 to exhibit a convenient basis for N from which the invariant factors decomposition of M_1 will be obvious.

From Lemma 3.5, we deduce that

$$\{ e_n^t E^{-j}(I, 0) \}_{j=0}^{n-1} \cup \{ e_n^t E^{-j}(0, I) \}_{j=0}^{n-1} \text{ is a basis for } \mathbb{Z}^{2n},$$

where vectors in \mathbb{Z}^{2n} are viewed as row vectors. Denote this new basis by $\{ \mathbf{w}_1^t, \dots, \mathbf{w}_{2n}^t \}$. Meanwhile, substituting the result of Lemma 3.4 into the definition of N shows that

$$N \text{ is spanned by } \{ 4e_n^t(4E^{-1})^j(I, 0) \}_{j=0}^{n-1} \cup \{ 4e_n^t(4E^{-1})^j(0, I) \}_{j=0}^{n-1},$$

These vectors are scalar multiples of the basis vectors for \mathbb{Z}^{2n} given in the previous equation, hence independent, and thus

$$\{ 4\mathbf{w}_1^t, \dots, 4^n \mathbf{w}_n^t, 4\mathbf{w}_{n+1}^t, \dots, 4\mathbf{w}_{2n}^t \} \text{ is a basis for } N.$$

Let $\mathbf{S} : \mathbb{Z}^{2n} \rightarrow \mathbb{Z}^{2n}$ be the change of basis matrix such that $\mathbf{w}_j^t \mathbf{S} = \mathbf{e}_j^t$. Then changing coordinates by \mathbf{S} on $(\mathbb{Q}/\mathbb{Z})^{2n}$ yields

$$\begin{aligned} M_1 &\cong \{ \mathbf{a} \in (\mathbb{Q}/\mathbb{Z})^{2n} : \mathbf{n}^t \mathbf{a} = 0 \text{ for all } \mathbf{n}^t \in N \} \\ &\cong \{ \mathbf{a} \in (\mathbb{Q}/\mathbb{Z})^{2n} : \mathbf{n}^t \mathbf{S} \mathbf{a} = 0 \text{ for all } \mathbf{n}^t \in N \} \\ &= \{ \mathbf{a} \in (\mathbb{Q}/\mathbb{Z})^{2n} : 4\mathbf{e}_1^t \mathbf{a} = 0, \dots, 4^n \mathbf{e}_n^t \mathbf{a} = 0, 4\mathbf{e}_{n+1}^t \mathbf{a}, \dots, 4^n \mathbf{e}_{2n}^t \mathbf{a} \} \\ &\cong \left(\bigoplus_{j=1}^n (\mathbb{Z}/4^j) \right)^{\oplus 2}. \end{aligned} \quad \square$$

Having computed M_1 , we now turn to M_2 . Although M_2 itself is difficult to compute, we now know that M_1 is a 2-torsion module. Thus, we only have to compute the 2-torsion submodule of M_2 , which greatly simplifies the task. Let $\mathbb{Z}[1/2]$ denote subring of \mathbb{Q} generated by \mathbb{Z} and $1/2$, viewed as a \mathbb{Z} -module, or equivalently the \mathbb{Z} -module of rational numbers whose denominators are powers of 2. Let

$$M'_2 = M_2 \cap (\mathbb{Z}[1/2]/\mathbb{Z})^{2n} = \{ \mathbf{a} \in (\mathbb{Z}[1/2]/\mathbb{Z})^{2n} : (\mathbf{T}^m - \mathbf{I})\mathbf{a} = \mathbf{0} \}.$$

Since $M_1 \subseteq (\mathbb{Z}[1/2]/\mathbb{Z})^{2n}$, we know that

$$\mathcal{U}_0(\text{CLF}(m, n), \mathbb{Q}/\mathbb{Z}) \cong M_1 \cap M_2 = M_1 \cap M'_2.$$

We will determine the 2-torsion properties of $\mathbf{T}^m - \mathbf{I}$ by finding an accurate enough 2-adic expansion of it.

LEMMA 3.7. *Suppose $m = r2^s$ with r odd. Then*

$$M'_2 \cong \begin{cases} (\mathbb{Z}/2)^n, & m \text{ odd,} \\ (\mathbb{Z}/2)^{2n}, & m \equiv 2 \pmod{4}, \\ (\mathbb{Z}/2^{s+1})^{2n}, & m \equiv 0 \pmod{4}. \end{cases}$$

Proof. First, consider the case where m is not divisible by 4, or equivalently $s \leq 2$. Note that

$$\mathbf{T} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \pmod{4}.$$

Hence, when $m = 1 \pmod{4}$,

$$\mathbf{T}^m - \mathbf{I} = \begin{pmatrix} -I & -I \\ I & -I \end{pmatrix} \pmod{4}.$$

The kernel of this map on $(\mathbb{Z}[1/2]/\mathbb{Z})^{2n}$ is therefore isomorphic to $(\mathbb{Z}/2)^n$. The case where $m = 3 \pmod{4}$ is similar. When $m = 2 \pmod{4}$, then $\mathbf{T}^m - \mathbf{I} = -2\mathbf{I} \pmod{4}$, so we get $M'_2 \cong (\mathbb{Z}/2)^{2n}$.

To handle the case where $m = 0 \pmod{4}$, we compute by hand that

$$\mathbf{T}^4 = \begin{pmatrix} -I & -4E^{-1} \\ 4E^{-1} & -I \end{pmatrix}^2 = \begin{pmatrix} I & 8E^{-1} \\ -8E^{-1} & I \end{pmatrix} \pmod{16}.$$

From here, one can verify by induction that

$$\mathbf{T}^{2^s} = \mathbf{I} + 2^{s+1} \begin{pmatrix} 0 & E^{-1} \\ -E^{-1} & 0 \end{pmatrix} \pmod{2^{s+2}} \text{ for } s \geq 2.$$

Hence, if r is odd, then using binomial expansion, we obtain

$$\mathbf{T}^{r2^s} = \mathbf{I} + r2^{s+1} \begin{pmatrix} 0 & E^{-1} \\ -E^{-1} & 0 \end{pmatrix} = \mathbf{I} + 2^{s+1} \begin{pmatrix} 0 & E^{-1} \\ -E^{-1} & 0 \end{pmatrix} \pmod{2^{s+2}} \text{ for } s \geq 2.$$

Since E^{-1} is invertible, this implies that the kernel of $\mathbf{T}^{r2^s} - \mathbf{I}$ over $\mathbb{Z}[1/2]/\mathbb{Z}$ is isomorphic to $(\mathbb{Z}/2^{s+1})^{2n}$. \square

Proof of Theorem 3.1. Recall that $U(m, n) \cong M_1 \cap M'_2 \subseteq (\mathbb{Q}/\mathbb{Z})^{2n}$. Note that $(\mathbb{Q}/\mathbb{Z})^{2n}$ has a unique submodule isomorphic to $(\mathbb{Z}/2)^{2n}$, so we can regard $(\mathbb{Z}/2)^{2n} \subseteq (\mathbb{Q}/\mathbb{Z})^{2n}$. In the case where m is not divisible by 4, we have

$$M'_2 \subseteq (\mathbb{Z}/2)^{2n} \subseteq M_1,$$

and therefore $M_1 \cap M'_2 = M'_2$, and this yields the asserted formula in Theorem 3.1. Now suppose that m is divisible by 4, and $m = r2^s$, where r is odd. Then M'_2 is the unique submodule of $(\mathbb{Q}/\mathbb{Z})^{2n}$ isomorphic to $(\mathbb{Z}/2^{s+1})^{2n}$, while $M_1 \cong (\bigoplus_{j=1}^n \mathbb{Z}/4^j)^{\oplus 2}$. Thus, the only possibility is that

$$M_1 \cap M'_2 \cong \left(\bigoplus_{j=1}^n \mathbb{Z}/\gcd(4^j, 2^{s+1}) \right)^{\oplus 2} = \left(\bigoplus_{j=1}^n \mathbb{Z}/\gcd(4^j, 2m) \right)^{\oplus 2}. \quad \square$$

Our proof technique in this section allows us to analyze other algebraic properties of $U(m, n)$. For instance, we have the following lemma, which we will use in the next section: Let $U_1 = U_1(m, n)$ be the submodule of $\mathcal{U}_0(\text{CLF}(m, n), L_{\text{std}}, \mathbb{Q}/\mathbb{Z})$ consisting of functions that vanish on the vertices $(j, 0)$, and let U_2 be the submodule of functions vanishing on the vertices $(j, 1)$. Then we have the following.

LEMMA 3.8. *If m is even, then*

$$U(m, n) = U_1(m, n) \oplus U_2(m, n),$$

where

$$U_1(m, n) \cong U_2(m, n) \cong \begin{cases} (\mathbb{Z}/2)^n, & m \equiv 2 \pmod{4}, \\ \bigoplus_{j=1}^n \mathbb{Z}/\gcd(2m, 4^j), & m \equiv 0 \pmod{4}. \end{cases}$$

Proof. Recall that we expressed a function $u \in \mathcal{U}_0(\text{CLF}(m, n), L_{\text{std}}, \mathbb{Q}/\mathbb{Z})$ in terms of the two vectors a_0 and a_1 , representing its values on the first two columns of vertices. The submodules U_1 and U_2 correspond to the conditions $a_0 = 0$ and $a_1 = 0$, respectively. From the proof of Lemma 3.6, we have

$$M_1 = \left\{ \mathbf{a} = \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \in (\mathbb{Q}/\mathbb{Z})^{2n} : \mathbf{n}^t \cdot \mathbf{F}\mathbf{a} = 0 \text{ for } \mathbf{n}^t \in N \right\}.$$

Then we used the change of coordinates \mathbf{S} to write the basis for N in a simpler form. The proof thus showed that

$$\mathbf{F}^{-1}\mathbf{S}^{-1}M_1 = \{ \mathbf{a} \in (\mathbb{Q}/\mathbb{Z})^{2n} : 4\mathbf{e}_1^t \mathbf{a} = 0, \dots, 4^n \mathbf{e}_n^t \mathbf{a} = 0, 4\mathbf{e}_{n+1}^t \mathbf{a}, \dots, 4^n \mathbf{e}_{2n}^t \mathbf{a} \}.$$

The latter module clearly decomposes as the direct sum of the submodule where $a_1 = 0$ and the submodule where $a_0 = 0$. However, the changes of coordinates \mathbf{F} and \mathbf{S} both respect the decomposition of $(\mathbb{Q}/\mathbb{Z})^{2n}$ into $(\mathbb{Q}/\mathbb{Z})^n \times 0^n$ and $0^n \times (\mathbb{Q}/\mathbb{Z})^n$. Thus, M_1 has the same direct sum decomposition. If m is even, then by Lemma

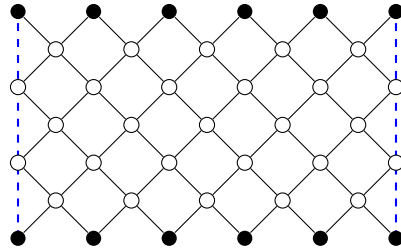


FIG. 3.3. The ∂ -graph $\text{CLF}'(5, 5)$.

3.7 we know that for some k , M'_2 is the unique submodule of $(\mathbb{Q}/\mathbb{Z})^{2n}$ isomorphic to $(\mathbb{Z}/2^k)^{2n}$ which is invariant under change of coordinates. Hence,

$$M_1 \cap M'_2 = [M_1 \cap ((\mathbb{Z}/2^k)^n \times 0^n)] \oplus [M_1 \cap (0^n \times (\mathbb{Z}/2^k)^n)].$$

In other words, the submodule of $\mathbf{a} \in (\mathbb{Q}/\mathbb{Z})^{2n}$ corresponding to \mathcal{U}_0 breaks up into a direct sum of vectors with $a_0 = 0$ and vectors with $a_1 = 0$. The same argument as in the proof of Theorem 3.1 shows that each summand is isomorphic to $(\mathbb{Z}/2)^n$ when $m \equiv 2 \pmod 4$ and otherwise, it is $\bigoplus_{j=1}^n \mathbb{Z}/\gcd(4^j, 2m)$. \square

3.4. Chain link fence variants. The family $\text{CLF}(m, n)$ is a variant of the family of ∂ -graphs described in [29]. We let $\text{CLF}'(m, n)$ be the graph from [29] described as follows: The vertex set is

$$V(\text{CLF}'(m, n)) = \{(x, y) \in \mathbb{Z}/2m \times \{0, \dots, n + 1\} : x + y \text{ is even}\}.$$

In the condition “ $x + y$ is even,” we are implicitly reducing x and y mod 2 using the canonical maps $\mathbb{Z}/2m \rightarrow \mathbb{Z}/2$ and $\mathbb{Z} \rightarrow \mathbb{Z}/2$. The boundary vertices are

$$\partial V(\text{CLF}'(m, n)) = V(\text{CLF}'(m, n)) \cap \mathbb{Z}/2m \times \{0, n + 1\}.$$

The edges are given by $(x, y) \sim (x + 1, y \pm 1)$ whenever y and $y \pm 1$ are both in $\{0, \dots, n\}$. See Figure 3.3.

Observe that there is a ∂ -graph isomorphism $\text{CLF}(2m, n) \rightarrow \text{CLF}'(m, 2n)$ given by

$$(j, k) \mapsto \begin{cases} (j, k), & i \text{ is even,} \\ (j, n - k), & i \text{ is odd.} \end{cases}$$

Thus, we have already computed $\mathcal{U}_0(\text{CLF}'(m, n), L_{\text{std}}, \mathbb{Q}/\mathbb{Z})$ for even values of n in Theorem 3.1. We will show the following.

THEOREM 3.9. Denote $U'(m, n) = \mathcal{U}_0(\text{CLF}'(m, n), L_{\text{std}}, \mathbb{Q}/\mathbb{Z})$. Then

$$U'(m, n) \cong \begin{cases} (\mathbb{Z}/2)^n, & m \text{ odd,} \\ \bigoplus_{j=1}^{\lfloor n/2 \rfloor} \mathbb{Z}/\gcd(4^j, m) \oplus \bigoplus_{j=1}^{\lfloor n/2 \rfloor} \mathbb{Z}/\gcd(4^j, m), & m \text{ even.} \end{cases}$$

The reader should verify that this agrees with Theorem 3.1 when n is even. We will deduce the odd case directly from Lemma 3.8 using elementary reasoning with subgraphs.

There is also a canonical inclusion $f_{m,n} : \text{CLF}'(m, n) \rightarrow \text{CLF}'(m, n + 1)$ given by mapping a vertex in $\text{CLF}'(m, n)$ to the vertex in $\text{CLF}'(m, n + 1)$ with the same

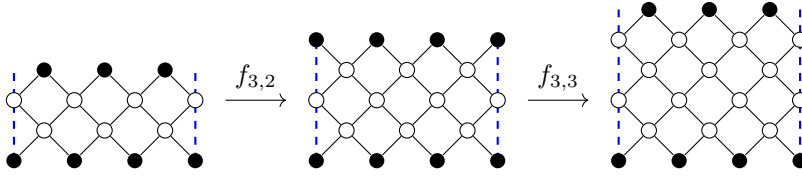


FIG. 3.4. Inclusion maps $CLF'(3, 2) \rightarrow CLF'(3, 3) \rightarrow CLF'(3, 4)$.

coordinates. Thus, we can think of $CLF'(m, n + 1)$ as being obtained from $CLF'(m, n)$ by adding another row of vertices at the top and changing the previous top row to interior vertices (see Figure 3.4). Next, if $u \in \mathcal{U}_0(CL F'(m, n), L_{std}, \mathbb{Q}/\mathbb{Z})$, then define $(f_{m,n})_* u$ on $CLF'(m, n + 1)$ by extending u to be zero on the top row (or row $n + 2$) of vertices in $CLF'(m, n + 1)$. Then have the following.

LEMMA 3.10. *The map $(f_{m,n})_*$ defines an injection $U'(m, n) \rightarrow U'(m, n + 1)$. The image of $(f_{m,n})_*$ consists of functions v which vanish on row $n + 1$ in $CLF'(m, n + 1)$. Moreover, if $v \in U'(m, n + 1)$ vanishes on one vertex in row $n + 1$, then it vanishes on all vertices in row $n + 1$.*

Proof. We must verify that $v := (f_{m,n})_* u$ is actually in $\mathcal{U}_0(CL F'(m, n + 1), L_{std}, \mathbb{Q}/\mathbb{Z})$. By construction $v = 0$ on the boundary rows 0 and $n + 2$ in $CLF'(m, n + 1)$. We also have $L_{std} v = L_{std} u = 0$ on rows 0 through n . Because v is zero on rows $n + 1$ and $n + 2$ we have $L_{std} v = L_{std} u = 0$ on row $n + 1$. And finally, v being zero on rows $n + 1$ and $n + 2$ implies that the Laplacian is zero on row $n + 2$.

The injectivity of $(f_{m,n})_*$ is obvious, and clearly the image functions all vanish on row n . Conversely, suppose $v \in \mathcal{U}_0(CL F'(m, n + 1), L_{std}, \mathbb{Q}/\mathbb{Z})$ vanishes on the n th row, and let u be the restriction to $CLF'(m, n)$. Since v vanishes on rows $n + 1$ and $n + 2$, the edges between these rows make no contribution to $L_{std} v$, and hence $L_{std} u = L_{std} v = 0$ on the n th row. This shows $u \in \mathcal{U}_0(CL F'(m, n), L_{std}, \mathbb{Q}/\mathbb{Z})$ as desired.

For the final claim, suppose $v \in U'(m, n + 1)$ vanishes on a vertex $(i, n + 1)$ in row $n + 1$. Then the conditions $v(i + 1, n + 2) = 0$ and $L_{std} v(i + 1, n + 2) = 0$ force $v(i + 2, n + 1) = 0$. Similarly, $v(i + 2, n + 1) = 0$ implies $v(i + 4, n + 1) = 0$ and so on, so that v vanishes on all of row $n + 1$. (Recall there are no vertices at coordinates $(i + 1, n + 1), (i + 3, n + 1), \dots$) \square

As in Lemma 3.8, define

$$U'_1(m, n) = \{u \in U'(m, n): u(0, j) = 0 \text{ for all } j\},$$

$$U'_2(m, n) = \{u \in U'(m, n): u(1, j) = 0 \text{ for all } j\}.$$

LEMMA 3.11. *We have $U'(m, n) = U'_1(m, n) \oplus U'_2(m, n)$.*

Proof. The case for even n follows from Lemma 3.8. Suppose n is odd. To show that $U'_1(m, n) \cap U'_2(m, n) = 0$, note that $(f_{m,n})_*$ maps $U'_j(m, n)$ into $U'_j(m, n + 1)$. Since $U'_1(m, n + 1) \cap U'_2(m, n + 1) = 0$ by the even case and since $(f_{m,n})_*$ is injective, we deduce that $U'_1(m, n) \cap U'_2(m, n) = 0$.

To show that $U'(m, n) = U'_1(m, n) + U'_2(m, n)$, let $u \in U'(m, n)$ and let $v = (f_{m,n})_* u$. From the even case, we know $v = v_1 + v_2$, where $v_1 \in U'_1(m, n + 1)$ and $v_2 \in U'_2(m, n + 1)$. Now v_1 vanishes on $(0, n + 1)$ by definition of $U'_1(m, n + 1)$; then the last claim of Lemma 3.10 implies that v_1 vanishes on row $n + 1$. Since v vanishes on row $n + 1$ by assumption, we know $v_2 = v - v_1$ also vanishes on row $n + 1$. Therefore,

v_1 and v_2 are in the image of $(f_{m,n})_*$, that is, $v_1 = (f_{m,n})_*u_1$ and $v_2 = (f_{m,n})_*u_2$ for some $u_1, u_2 \in U'(m, n)$. Then clearly $u = u_1 + u_2$ and $u_1 \in U'_1(m, n)$ and $u_2 \in U'_2(m, n)$. \square

Proof of Theorem 3.9. The case where n is even has already been handled in Theorem 3.1. Now suppose n is odd. Note that $(f_{m,n})_*$ gives an injection $U'_1(m, n) \rightarrow U'_1(m, n+1)$. But in fact, this is an isomorphism because any function $v \in U'_1(m, n+1)$ vanishes on $(0, n+1)$, hence vanishes on all of row $n+1$, hence comes from a function u in $U'_1(m, n)$. A similar argument shows that $U'_2(m, n) \cong U'_2(m, n-1)$. Therefore,

$$\begin{aligned} U'(m, n) &= U'_1(m, n) \oplus U'_2(m, n) \\ &= U'_1(m, n+1) \oplus U'_2(m, n-1) \\ &= U_1(2m, \frac{n+1}{2}) \oplus U_2(2m, \frac{n-1}{2}), \end{aligned}$$

and the proof is completed by applying Lemma 3.8. \square

4. The categories of ∂ -graphs and R -networks.

4.1. Motivation. The example of the CLF networks already illustrated the usefulness of covering spaces (Remark 3.3) and subgraphs (Lemmas 3.10 and 3.11, proof of Theorem 3.9). We will now describe general morphisms of R -networks, adapting the ideas of [42, 6, 41] to ∂ -graphs, as well as giving applications to spanning tree counts and eigenvectors.

Harmonic morphisms of graphs were first defined by Urakawa [42, Definition 2.2]. Baker and Norine extended the definition to multigraphs and showed that a harmonic morphism $\phi : G' \rightarrow G$ defines a map ϕ_* from the critical group of G' to that of G as well as a map ϕ^* from the critical group of G to that of G' [6, section 2.3]. In other words, the critical group (a.k.a. sandpile group or Jacobian) can be viewed either as a covariant or as a contravariant functor from the category of graphs and harmonic morphisms to the category of abelian groups. Special cases of Baker and Norine’s construction were defined earlier (2002) in the undergraduate thesis of Treumann [41].

We will construct categories of ∂ -graphs and R -networks and show that $\Upsilon(G, L)$ and $\mathcal{U}_0(G, L, M)$ for fixed M are covariant functors from R -networks to R -modules, and $\mathcal{U}(G, L, M)$ is a contravariant functor. It makes sense for $\mathcal{U}_0(G, L, M)$ to be covariant and $\mathcal{U}(G, L, M)$ to be contravariant with respect to (G, L) because $\mathcal{U}(G, L, M) \cong \text{Hom}(\Upsilon(G, L), M)$ by Lemma 2.9 and $\mathcal{U}_0(G, L, M) \cong \text{Tor}_1(\Upsilon(G, L), M)$ for non-degenerate networks by Proposition 2.11.

4.2. The category of ∂ -graphs. Before defining morphisms of R -networks and verifying the functorial properties, we must record and explain the purely combinatorial definition of a ∂ -graph morphism, based on Baker and Norine’s definition of harmonic morphism [6, section 2.3]. For a vertex x in a ∂ -graph G , recall that we use the notation $\mathcal{E}(x) = \{e \in E(G) : e_+ = x\}$ for the set of oriented edges exiting x .

DEFINITION 4.1. *A ∂ -graph morphism $f : G_1 \rightarrow G_2$ is a map $f : V_1 \sqcup E_1 \rightarrow V_2 \sqcup E_2$ such that*

- (1) *f maps vertices to vertices,*
- (2) *f maps interior vertices to interior vertices,*
- (3) *if $f(e)$ is an oriented edge, then $f(e_+) = (f(e))_+$ and $f(e_-) = (f(e))_-$ and $f(\bar{e}) = \overline{f(e)}$,*
- (4) *if $f(e)$ is a vertex, then $f(\bar{e}) = f(e)$ and $f(e_\pm) = f(e)$,*

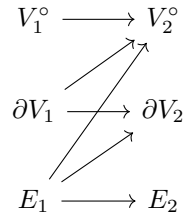


FIG. 4.1. Where a ∂ -graph morphism is allowed to map the sets V° , ∂V , and E .

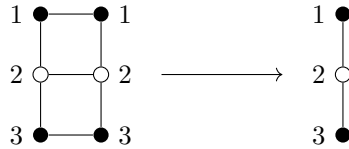


FIG. 4.2. A ∂ -graph morphism. The horizontal edges are collapsed and mapped to the vertices 1, 2, 3 on the right. The vertical edges on the left graph are mapped to the vertical edges on the right.

- (5) for every $x \in V_1^\circ$, the restricted map $\mathcal{E}(x) \cap f^{-1}(E_2) \xrightarrow{f} \mathcal{E}(f(x))$ has constant fiber size. In other words, it is n -to-1 for some integer $n \geq 0$ (depending on x).

Figure 4.1 shows where a ∂ -graph morphism can map the edges and vertices. The statement of condition (5) implicitly uses the fact that f restricts to a map $\mathcal{E}(x) \cap f^{-1}(E_2) \rightarrow \mathcal{E}(f(x))$, which follows from (3). The integer n associated to a vertex $x \in V_1^\circ$ in condition (5) will be called the *degree of f at x* and denoted by $\text{deg}(f, x)$. Note that (5) implies

$$\text{for all } x \in V_1^\circ, \text{ for all } e \in \mathcal{E}(f(x)), \quad |\mathcal{E}(x) \cap f^{-1}(e)| = \text{deg}(f, x).$$

It will also be convenient to extend the definition of $\text{deg}(f, x)$ to $x \in \partial V_1$ by setting

$$\text{deg}(f, x) = \max_{e \in \mathcal{E}(f(x))} |\mathcal{E}(x) \cap f^{-1}(e)|.$$

Conditions (1), (3), and (4) say that f is like a graph homomorphism (see [23, section 6] for background) except that it allows an edge to be collapsed to a vertex as in Figure 4.2. In other words, if we view G_1 and G_2 as cell complexes, then f is a continuous cellular map.

Condition (5) says that f restricts to an n -fold covering of the neighborhood $\mathcal{E}(x)$ of x onto the neighborhood $\mathcal{E}(f(x))$ of $f(x)$, after ignoring collapsed edges (note that ignoring collapsed edges is exactly the effect of the taking the intersection of $\mathcal{E}(x)$ with $f^{-1}(E_2)$ in (5)). For examples, see Figures 4.3 and 4.4. The next lemma is the first step in establishing our functoriality properties.

LEMMA 4.2. *∂ -graphs form a category. Moreover, if $f : G_1 \rightarrow G_2$ and $g : G_2 \rightarrow G_3$ are ∂ -graph morphisms, then*

$$\begin{aligned} \text{deg}(g \circ f, x) &= \text{deg}(f, x) \text{deg}(g, f(x)) \text{ for all } x \in V_1^\circ, \\ \text{deg}(g \circ f, x) &\leq \text{deg}(f, x) \text{deg}(g, f(x)) \text{ for all } x \in \partial V_1. \end{aligned}$$

Proof. To verify the category axioms, it suffices to show that if $f : G_1 \rightarrow G_2$ and $g : G_2 \rightarrow G_3$ are ∂ -graph morphisms, then so is $g \circ f$. Clearly, (1) and (2) are preserved

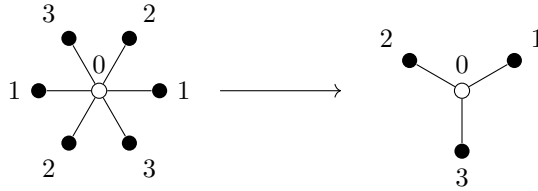


FIG. 4.3. A ∂ -graph morphism. The numbers show where each vertex is mapped.

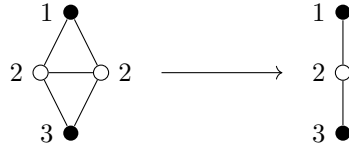


FIG. 4.4. A ∂ -graph morphism. The horizontal edge is collapsed into the vertex 2 on the right, while the slanted edges on the left map to the vertical edges on the right.

by composition. To check $g \circ f$ satisfies (3), note that if $g \circ f(e)$ is an oriented edge, then $f(e)$ must be an oriented edge by (1), and hence we can apply (3) to f at the edge e and (3) to g at the edge $f(e)$.

To check (4), suppose $g \circ f(e)$ is a vertex. If $f(e)$ is a vertex, then apply (4) to f . If $f(e)$ is an edge, then apply (3) to f and (4) to g .

To check (5), suppose $x \in V_1^\circ$ and $e \in \mathcal{E}(g \circ f(x))$. Any element of $f^{-1}(g^{-1}(e))$ must be mapped into $\mathcal{E}(f(x))$ by f , and thus

$$\mathcal{E}(x) \cap f^{-1}(g^{-1}(e)) = \sqcup_{e' \in \mathcal{E}(f(x)) \cap g^{-1}(e)} \mathcal{E}(x) \cap f^{-1}(e').$$

Using the fact that $f(x) \in V_2^\circ$, we see that this is a disjoint union of $\text{deg}(g, f(x))$ sets of size $\text{deg}(f, x)$. This implies that

$$|\mathcal{E}(x) \cap f^{-1}(g^{-1}(e))| = \text{deg}(f, x) \text{deg}(g, f(x)) \text{ for all } e \in \mathcal{E}(g \circ f(x)),$$

and hence $g \circ f$ satisfies (5) and hence is a ∂ -graph morphism.

Moreover, the last computation showed that $\text{deg}(g \circ f, x) = \text{deg}(f, x) \text{deg}(g, f(x))$ and a similar argument shows that $\text{deg}(g \circ f, x) \leq \text{deg}(f, x) \text{deg}(g, f(x))$ for $x \in \partial V_1$. \square

As in [6, 42, 36], we can think of ∂ -graph morphisms as a discrete analogue of holomorphic maps between Riemann surfaces with boundary. The first, perhaps trivial, analogy is that both ∂ -graph morphisms and holomorphic functions are closed under composition. Moreover, in the next section, we will show that if $f: G_1 \rightarrow G_2$ is a ∂ -graph morphism and u is harmonic on G_2 , then $u \circ f$ is harmonic on G_1 .

Just as with Riemann surfaces, the simplest type of ∂ -graph morphism is a covering map, which completely preserves local structure. In the discrete setting, we define covering maps as follows. Note that this agrees with topological definition if we view ∂ -graphs as a cell complexes and forget the distinction between interior and boundary vertices.

DEFINITION 4.3. A covering map is a ∂ -graph morphism $f: \tilde{G} \rightarrow G$ such that f defines a surjection $V(\tilde{G}) \sqcup E(\tilde{G}) \rightarrow V(G) \sqcup E(G)$, f maps interior vertices to interior vertices, f maps boundary vertices to boundary vertices, f maps edges to edges, and the restricted map $\mathcal{E}(x) \rightarrow \mathcal{E}(f(x))$ is a bijection for every $x \in \tilde{V}$.

We have already seen a covering map in Remark 3.3. Moreover, the standard construction of the bipartite double cover for a graph easily adapts to ∂ -graphs. We shall say more about covering spaces in section 10.

Another important type of morphism is the inclusion of sub- ∂ -graphs. We have already used sub- ∂ -graphs in section 3.4. Later, in section 5, we will consider restricting harmonic functions to sub- ∂ -graphs and extending them from sub- ∂ -graphs using a discrete analogue of harmonic continuation. We now record the precise definition of a sub- ∂ -graph for future use.

DEFINITION 4.4. *Assume that G_1 and G_2 are ∂ -graphs, such that (V_1, E_1) is a subgraph of (V_2, E_2) . Then we say G_1 is a sub- ∂ -graph of G_2 if the inclusion map $G_1 \rightarrow G_2$ is a ∂ -graph morphism. One can verify from Definition 4.1 that a subgraph G_1 will be a sub- ∂ -graph if and only if $x \in V_1^\circ$ implies that $x \in V_2^\circ$ and $\mathcal{E}_{G_1}(x) = \mathcal{E}_{G_2}(x)$.*

Like a holomorphic function, a ∂ -graph morphism $f: G_1 \rightarrow G_2$ may exhibit *ramification* when a star $\mathcal{E}(x)$ in G_1 is an n -fold cover of a star $\{e : e_+ = f(x)\}$ in G_2 for $n > 1$. For example, see Figure 4.3. This is a discrete model of the behavior of the map $z \mapsto z^n$ in a neighborhood of the origin in \mathbb{C} . The formula $\deg(g \circ f, x) = \deg(f, x) \deg(g, f(x))$ also mimics the way that local degrees of holomorphic maps are multiplicative under composition. The behavior of a ∂ -graph morphism is unconstrained by condition (5) at the boundary, just as an analytic function on a Riemann surface need not be n -to-1 in the neighborhood of a boundary point.

Recall that for compact connected Riemann surfaces without boundary, every nonconstant holomorphic map is surjective as a consequence of the open mapping theorem. Now we will prove an analogous statement in the discrete case, which applies even to infinite ∂ -graphs. We will view graphs without boundary as the subclass of ∂ -graphs with no boundary vertices. Continuing the terminology of [42, 6], we will refer to ∂ -graph morphisms of boundary graphs without boundary as *harmonic morphisms*. An example of such a morphism is shown in [6, Figure 1]. An alternative proof of the following proposition can also be found in [6, Lemmas 2.4 and 2.7].

PROPOSITION 4.5. *Let $f: G_1 \rightarrow G_2$ be a harmonic morphism of nonempty connected graphs without boundary. Either f maps $V_1 \sqcup E_1$ to a single vertex x of G_2 or f is a surjection $V_1 \sqcup E_1 \rightarrow V_2 \sqcup E_2$.*

Proof. Let $A = \{x \in f(V_1) : \deg(f, y) > 0 \text{ for some } y \in f^{-1}(x)\}$. We claim that if $x \in A$, then all neighbors of x are also in A . Suppose $x \in A$ and x' is joined to x by an edge e . Since $x \in A$, there exists $y \in f^{-1}(x)$ with $\deg(f, y) > 0$. This implies that y has some edge \tilde{e} which maps to e , so y has some neighbor y' which maps to x' . Since the edge \tilde{e} incident to y' maps to an edge in G_2 , we must have $\deg(f, y') > 0$. Therefore, $x' \in A$.

Since G_2 is connected, either $A = V_2$ or $A = \emptyset$. In the first case, f must be surjective onto V_2 , and then by definition of A and $\deg(f, y)$, we deduce that f is surjective onto E_2 . In the second case, we have $\deg(f, y) = 0$ for all $y \in V_1$, which implies that all edges in G_2 are collapsed to vertices. Then since G_2 is connected, f must be constant. \square

Though ∂ -graph morphisms are much like holomorphic maps, the ability of ∂ -graph morphisms to collapse an edge into a vertex seems to have no direct analogue in complex analysis. In a neighborhood of a vertex x where $\deg(f, x) = 1$ and some edges in $\mathcal{E}(x)$ are collapsed, a ∂ -graph f behaves more like an orthogonal projection (recall that that if $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is an orthogonal projection and $u: \mathbb{R}^n \rightarrow \mathbb{R}$ is harmonic,

then $u \circ f$ is harmonic). A more precise analogy is between the projection maps associated to a product of Riemannian manifolds and the projection maps associated to a *box product* of ∂ -graphs (also known as the *Cartesian product*), which is defined as follows.

Let G_1 and G_2 be ∂ -graphs. Then we define the *box product* $G = G_1 \square G_2$ by

$$V = V_1 \times V_2, \quad E = E_1 \times V_2 \cup V_1 \times E_2, \quad V^\circ = V_1^\circ \times V_2^\circ.$$

Then if $(e, x) \in E_1 \times V_2$, we define $\overline{(e, x)} = (\bar{e}, x)$, $(e, x)_+ = (e_+, x)$, and $(e, x)_- = (e_-, x)$ and make a similar definition for $(x, e) \in V_1 \times E_2$. In particular, two vertices (x, y) and (x', y') are adjacent if $x = x'$ and $y \sim y'$ or if $x \sim x'$ and $y = y'$. Then the obvious projection map $f_1 : G \rightarrow G_1$ is a ∂ -graph morphism. Note that f_1 has degree 1 at each vertex and collapses all edges in $V_1 \times E_2$ to vertices.

In general, the local behavior of a ∂ -graph morphism at an interior vertex combines ramification and collapsing, that is, it combines the behavior of branched covering maps and projections. What is unique about the discrete setting is that f may behave like a holomorphic map $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ near one vertex and behave like a projection map $\mathbb{R}^3 \rightarrow \mathbb{R}^2$ at another vertex; this cannot happen for manifolds because the dimension does not vary from point to point.

4.3. The category of R -networks. Recall that an R -network is given by a pair (G, L) , where L is given by a weight function $w : E \rightarrow R$ and a diagonal term $d : V \rightarrow R$.

DEFINITION 4.6. *An R -network morphism $f : (G_1, L_1) \rightarrow (G_2, L_2)$ is given by a ∂ -graph morphism $f : G_1 \rightarrow G_2$ satisfying*

$$w_1(e) = w_2(f(e)) \text{ for every } e \in f^{-1}(E_2)$$

and

$$d_1(x) = \deg(f, x)d_2(f(x)) \text{ for every } x \in V_1^\circ.$$

It is straightforward to verify that R -networks form a category, using the fact that ∂ -graphs form a category and $\deg(g \circ f, x) = \deg(f, x) \deg(g, f(x))$ for an interior vertex x . We denote this category by **R -net**.

Remark 4.7. Note that the second condition of Definition 4.6 is trivially satisfied in the case where $d_j = 0$. In particular, if $f : G_1 \rightarrow G_2$ is a ∂ -graph morphism, then f automatically defines a \mathbb{Z} -network morphism $(G_1, L_{\text{std},1}) \rightarrow (G_2, L_{\text{std},2})$, where L_{std} is the standard Laplacian with edge weights 1. Thus, $G \mapsto (G, L_{\text{std}})$ is a functor from ∂ -graphs to \mathbb{Z} -networks.

The functoriality properties we will prove rely on the following observation.

LEMMA 4.8. *Let $f : (G_1, L_1) \rightarrow (G_2, L_2)$ be an R -network morphism. Note that f extends linearly to a map $RV_1 \rightarrow RV_2$. If $x \in V_1^\circ$, then*

$$f(L_1x) = \deg(f, x)L_2(f(x)).$$

Proof. Note that

$$f(L_1x) = d_1(x)f(x) + \sum_{e \in \mathcal{E}(x)} w_1(e)(f(x) - f(e_-)).$$

By Definition 4.6, $d_1(x) = \deg(f, x)d_2(x)$. Moreover, by Definition 4.1, each $e \in \mathcal{E}(x)$ will map either to $f(x)$ or to some $e' \in \mathcal{E}(f(x))$. Thus, the sum over the edges becomes

$$\sum_{e \in \mathcal{E}(x) \cap f^{-1}(f(x))} w_1(e)(f(x) - f(e_-)) + \sum_{e' \in \mathcal{E}(f(x))} \sum_{e \in \mathcal{E}(x) \cap f^{-1}(e')} w_1(e)(f(x) - f(e_-)).$$

The first term vanishes since if e is collapsed into x , then $f(e_-) = f(x)$. In the second term, note $w_1(e) = w_2(f(e))$ by Definition 4.6 and the number of terms corresponding to each e' is $\deg(f, x)$. Hence,

$$\begin{aligned} \sum_{e \in \mathcal{E}(x)} w_1(e)(f(x) - f(e_-)) &= \sum_{e' \in \mathcal{E}(f(x))} \sum_{e \in \mathcal{E}(x) \cap f^{-1}(e')} w_2(e')(f(x) - f(e_-)) \\ &= \sum_{e' \in \mathcal{E}(f(x))} \deg(f, x)w_2(e')(f(x) - e'_-). \end{aligned}$$

Therefore,

$$\begin{aligned} f(L_1x) &= \deg(f, x)d_2(f(x))f(x) + \deg(f, x) \sum_{e' \in \mathcal{E}(f(x))} w_2(e')(f(x) - e'_-) \\ &= \deg(f, x)L_2(f(x)). \end{aligned} \quad \square$$

Now we can establish the promised functoriality properties. We remark that special cases of Lemma 4.9 were proved in [41] and the case of Lemma 4.10 where $R = \mathbb{Z}$ and $L = L_{\text{std}}$ was proved in [6, Proposition 2.8].

LEMMA 4.9. *The map $(G, L) \mapsto \Upsilon(G, L)$ is a functor $R\text{-net} \rightarrow R\text{-mod}$, where the definition on morphisms is as follows: If $f : (G_1, L_1) \rightarrow (G_2, L_2)$ is an R -network morphism, then Υf is given by*

$$\Upsilon f : \Upsilon(G_1, L_1) \rightarrow \Upsilon(G_2, L_2) : x + L(RV_1^\circ) \mapsto f(x) + L(RV_2^\circ).$$

Proof. Let us verify that the map Υf is well-defined. Recall

$$\Upsilon(G, L) = RV(G)/L(RV^\circ(G)).$$

The map $f : (G_1, L_1) \rightarrow (G_2, L_2)$ defines a map $f : RV_1 \rightarrow RV_2$. By Lemma 4.8, f maps $L_1(RV_1^\circ)$ into $L_2(RV_2^\circ)$. This implies f yields a well-defined map on the quotient. Checking that $\Upsilon(g \circ f) = \Upsilon g \circ \Upsilon f$ is straightforward. \square

LEMMA 4.10. *The map $((G, L), M) \mapsto \mathcal{U}(G, L, M)$ is a functor $R\text{-net}^{op} \times R\text{-mod} \rightarrow R\text{-mod}$, where the definition on morphisms is given as follows: If $f : (G_1, L_1) \rightarrow (G_2, L_2)$ and $\phi : M \rightarrow M'$ is an R -module morphism, then we have*

$$\mathcal{U}(f, \phi) : \mathcal{U}(G_2, L_2, M) \rightarrow \mathcal{U}(G_1, L_1, M') : u \mapsto \phi \circ u \circ f.$$

The isomorphism $\mathcal{U}(G, L, M) \cong \text{Hom}_R(\Upsilon(G, L), M)$ given by Lemma 2.9 is natural in $((G, L), M)$.

Proof. To check that the map $\mathcal{U}(f, \phi)$ actually maps into $\mathcal{U}(G_1, L_1, M')$, let $u \in \mathcal{U}(G_2, L_2, M)$. If $x \in V_1^\circ$, then $f(x) \in V_2^\circ$, and by Lemma 4.8, we have

$$L_1(\phi \circ u \circ f)(x) = \phi \circ u \circ f(L_1x) = \deg(f, x)\phi \circ u(L_2f(x)) = 0.$$

Functoriality is straightforward to check, and the naturality of the isomorphism $\mathcal{U}(G, L, M) \cong \text{Hom}_R(\Upsilon(G, L), M)$ is checked easliy from the proof of Lemma 2.9. \square

LEMMA 4.11. *The map $((G, L), M) \mapsto \mathcal{U}_0(G, L, M)$ is a functor $R\text{-net} \times R\text{-mod} \rightarrow R\text{-mod}$, where the definition on morphisms is given as follows: If $f: (G_1, L_1) \rightarrow (G_2, L_2)$ and $\phi: M \rightarrow M'$, then we have*

$$[\mathcal{U}_0(f, \phi)u](y) = \sum_{x \in f^{-1}(y) \cap V_1} \deg(f, x) \phi \circ u(x).$$

The surjection $\mathcal{U}_0(G, L, M) \rightarrow \text{Tor}_1^R(\Upsilon(G, L), M)$ given by Proposition 2.11 is natural in $((G, L), M)$.

Proof. As in the proof of Proposition 2.11, for an R -network (G, L) , we can view $RV^\circ \otimes M$ as the module of finitely supported functions $u: V^\circ \rightarrow M$ by the identification

$$u \leftrightarrow \sum_{x \in V^\circ} x \otimes u(x).$$

Then

$$\mathcal{U}_0(G, L, M) = \ker(L \otimes \text{id}: RV^\circ \otimes M \rightarrow RV \otimes M).$$

By Lemma 4.8, if $f: (G_1, L_1) \rightarrow (G_2, L_2)$, then the left diagram below commutes and hence the right diagram below commutes:

$$\begin{array}{ccc} RV_1^\circ & \xrightarrow{L_1} & RV_1 \\ \text{deg}(f, \cdot) f \downarrow & & \downarrow f \\ RV_2^\circ & \xrightarrow{L_2} & RV_2 \end{array} \quad \begin{array}{ccc} RV_1^\circ \otimes M & \xrightarrow{L_1} & RV_1 \otimes M \\ \text{deg}(f, \cdot) f \otimes \phi \downarrow & & \downarrow f \otimes \phi \\ RV_2^\circ \otimes M' & \xrightarrow{L_2} & RV_2 \otimes M' \end{array}$$

In the diagram at right, the kernels of the horizontal maps are $\mathcal{U}_0(G_1, L_1, M)$ and $\mathcal{U}_0(G_2, L_2, M')$, respectively, so we obtain a map $\mathcal{U}_0(G_1, L_1, M) \rightarrow \mathcal{U}_0(G_2, L_2, M')$ satisfying

$$\sum_{x \in V_1^\circ} x \otimes u(x) \mapsto \sum_{x \in V_1^\circ} \text{deg}(f, x) f(x) \otimes \phi(u(x)).$$

The asserted formula follows from grouping the terms in the right-hand sum by the value of $f(x)$. Again, functoriality of the construction is straightforward to check, and so is the naturality of the transformation in Proposition 2.11. \square

4.4. Applications of functoriality. We now give some immediate applications of functoriality to the critical group and the characteristic polynomial of L . The following application to the critical group and spanning trees is a generalization of [7, Theorem 5.7] and [41, Proposition 19 and Corollary 20]. We do not know of a combinatorial proof of Corollary 4.13 below and suggest it as a question for future research.

PROPOSITION 4.12. *Viewed as the torsion submodule of $\Upsilon_{\mathbb{Z}}(G, L_{\text{std}})$, the critical group is a covariant functor from finite connected graphs to \mathbb{Z} -modules, where the morphisms between graphs are the harmonic morphisms. If $f: G_1 \rightarrow G_2$ is a non-constant harmonic morphism between connected graphs, then $\text{Crit}(f): \text{Crit}(G_1) \rightarrow \text{Crit}(G_2)$ is surjective, and if f is constant, then it is zero.*

Proof. Note that f induces a map $\Upsilon_{\mathbb{Z}} f: \Upsilon_{\mathbb{Z}}(G_1, L_{\text{std},1}) \rightarrow \Upsilon_{\mathbb{Z}}(G_2, L_{\text{std},2})$. Also, recall from Proposition 2.22 that $\text{Crit}(G_j)$ is the torsion part of $\Upsilon_{\mathbb{Z}}(G_j, L_{\text{std},j})$, and for each vertex $x \in V_1$, we have internal direct sums

$$\begin{aligned} \Upsilon_{\mathbb{Z}}(G_1, L_{\text{std},1}) &= \text{Crit}(G_1) \oplus \mathbb{Z}x, \\ \Upsilon_{\mathbb{Z}}(G_2, L_{\text{std},2}) &= \text{Crit}(G_2) \oplus \mathbb{Z}f(x). \end{aligned}$$

The map Υf maps the first summand into the first summand and the second summand into the second summand. In particular, f restricts to a map $\text{Crit}(f) : \text{Crit}(G_1) \rightarrow \text{Crit}(G_2)$.

If f is nonconstant, then it is surjective by Proposition 4.5. In this case, f induces a surjection $RV_1 \rightarrow RV_2$, and hence by passing to the quotient, $\Upsilon f : \Upsilon_{\mathbb{Z}}(G_1, L_{\text{std},1}) \rightarrow \Upsilon_{\mathbb{Z}}(G_2, L_{\text{std},2})$ is surjective. It follows that in the direct sum decomposition above, Υf must be surjective on each summand. On the other hand, if f is constant, then f maps every vertex to $f(x)$ and hence the image of Υf is completely contained in $\mathbb{Z}f(x)$. Thus, Υf must map $\text{Crit}(G_1)$ to zero. \square

COROLLARY 4.13. *Suppose $f : G_1 \rightarrow G_2$ is a nonconstant harmonic morphism of graphs. Then the number of spanning trees on G_2 divides the number of spanning trees on G_1 .*

Proof. It is well-known that the number of spanning trees $\tau(G_j)$ equals the size of the critical group $\text{Crit}(G_j)$; see, for instance, [24, section 3]. Since $\text{Crit}(f)$ is surjective, the order of $\text{Crit}(G_2)$ divides the order of $\text{Crit}(G_1)$. \square

PROPOSITION 4.14. *Suppose $f : G_1 \rightarrow G_2$ is a nonconstant harmonic morphism of connected graphs which satisfies $\deg(f, x) = n$ for all $x \in V_1$. Then f induces a $\mathbb{C}[z]$ -network morphism*

$$f_* : (G_1, nzI - L_{\text{std},1}) \rightarrow (G_2, zI - L_{\text{std},2}).$$

Moreover the induced map $\Upsilon_{\mathbb{C}[z]}f_*$ is surjective. In particular, if G_1 and G_2 are finite, then the characteristic polynomials of the standard Laplacians for the two graphs are related by

$$\det(zI - L_{\text{std},2}) \mid \det(nzI - L_{\text{std},1}).$$

Proof. Observe that f defines a $\mathbb{C}[z]$ -network morphism $(G_1, nzI - L_{\text{std},1}) \rightarrow (G_2, zI - L_{\text{std},2})$; the second condition of Definition 4.6 holds because for $L_1 = nzI - L_{\text{std},1}$ and $L_2 = zI - L_{\text{std},2}$, we have

$$d_1(x) = nz = \deg(f, x)z = \deg(f, x) \cdot d_2(f(x)).$$

By Proposition 4.5, f is surjective on the vertex and edge sets. By Lemma 4.9, we have a surjective $\mathbb{C}[z]$ -module morphism

$$\Upsilon_{\mathbb{C}[z]}(G_1, nzI - L_{\text{std},1}) \rightarrow \Upsilon_{\mathbb{C}[z]}(G_2, zI - L_{\text{std},2}),$$

which by application of the Hom functor induces an injective map

$$\mathcal{U}(G_2, zI - L_{\text{std},2}, \mathbb{C}[z]/(z - \lambda)) \rightarrow \mathcal{U}(G_1, nzI - L_{\text{std},1}, \mathbb{C}[z]/(z - \lambda)).$$

By similar reasoning to Example 2.16, this means that λ -eigenvectors of G_2 pull back to $n\lambda$ -eigenvectors of G_1 . Thus, if λ is an eigenvalue of $L_{\text{std},2}$, then $n\lambda$ is also an eigenvalue of $L_{\text{std},1}$ with the same or greater multiplicity. This implies $\det(zI - L_{\text{std},2}) \mid \det(nzI - L_{\text{std},1})$. \square

Remark 4.15. In addition, Lemma 4.11 yields a map

$$\mathcal{U}_0(G_1, nzI - L_{\text{std},1}, \mathbb{C}[z]/(z - \lambda)) \rightarrow \mathcal{U}_0(G_2, zI - L_{\text{std},2}, \mathbb{C}[z]/(z - \lambda)),$$

which implies that $n\lambda$ -eigenvectors of $L_{\text{std},1}$ push forward to λ -eigenvectors of $L_{\text{std},2}$. In particular, if $n\lambda$ is an eigenvalue of G_1 but λ is not an eigenvalue for G_2 , then any $n\lambda$ -eigenvector of G_1 will push forward to zero; in other words, the values on each fiber will add up to zero. These results on the characteristic polynomial and eigenvectors generalize in a straightforward way to weighted graphs.

5. Layer-stripping and harmonic continuation.

5.1. Layerable extensions. Based on the technique of layer-stripping from the electrical inverse problem (see section 1.1 and [15, 18, 13, 26]), we shall now describe three operations (simple layerable extensions), which add a vertex or an edge onto the boundary of a ∂ -graph. Individually, these modifications are simple enough that their effect on Υ , \mathcal{U} , and \mathcal{U}_0 is easy to understand, but when applied in sequence, they provide nontrivial information about our algebraic invariants. We will show, for instance, that if (G', L') is obtained from (G, L) by a sequence of simple layerable extensions, then any harmonic function on (G, L) extends to a harmonic function on (G', L') . For simplicity, we focus on the case of finite ∂ -graphs first and then consider infinite ∂ -graphs in section 5.6.

DEFINITION 5.1. *An isolated boundary vertex x of a ∂ -graph is a boundary vertex with no neighbors incident to it. We say G' is obtained from G by adjoining the isolated boundary vertex x if x is an isolated boundary vertex in G' and*

$$V(G') = V(G) \sqcup \{x\}, \quad E(G') = E(G), \quad V^\circ(G') = V^\circ(G).$$

Equivalently, we say that G is obtained from G' by deleting the isolated boundary vertex x . See Figure 5.1 for an example.

DEFINITION 5.2. *A boundary spike e of a ∂ -graph is an edge e such that one endpoint e_- is an interior vertex and the other e_+ is a boundary vertex with no other edges incident to it. We say G' is obtained from G by adjoining the boundary spike e if e is a boundary spike in G' and we have*

$$V(G') = V(G) \sqcup \{e_+\}, \quad E(G') = E(G) \sqcup \{e, \bar{e}\}, \quad V^\circ(G') = V^\circ(G) \sqcup \{e_-\}.$$

Equivalently, we say that G is obtained from G' by contracting the boundary spike e . See Figure 5.2 for an example.

DEFINITION 5.3. *A boundary edge e of a ∂ -graph is an edge e such that both endpoints are boundary vertices. We say that G' is obtained from G by adjoining a boundary edge e if e is a boundary edge in G' and*

$$V(G') = V(G), \quad E(G') = E(G) \sqcup \{e, \bar{e}\}, \quad V^\circ(G') = V^\circ(G).$$

Equivalently, we say that G is obtained from G' by deleting the boundary edge e . See Figure 5.3 for an example.



FIG. 5.1. Adjoining or deleting an isolated boundary vertex x .



FIG. 5.2. Adjoining or contracting a boundary spike e .



FIG. 5.3. Adjoining or deleting a boundary edge e .

DEFINITION 5.4. If G' is obtained from G by adjoining an isolated boundary vertex, boundary spike, or boundary edge, then we say that G' is a simple layerable extension of G . We will equivalently say that G is obtained from G' by a layer-stripping operation.

OBSERVATION 5.5. If G' is a simple layerable extension of G , then G is a sub- ∂ -graph of G' , as one can check by straightforward casework.

DEFINITION 5.6. We say (G', L') is obtained from (G, L) by adjoining an isolated boundary vertex, boundary spike, or boundary edge if G' is obtained from G by the corresponding operation and in addition $w'|_{E(G)} = w$ and $d'|_{V(G)} = d$. (Note that given the geometric setup, $w'|_{E(G)} = w$ and $d'|_{V(G)} = d$ is a necessary and sufficient condition to make the inclusion $(G, L) \rightarrow (G', L')$ an R -network morphism.)

LEMMA 5.7. Let (G', L') be an R^\times -network. If (G', L') is obtained from (G, L) by adjoining a boundary spike or boundary edge, then the induced maps $\Upsilon(G, L) \rightarrow \Upsilon(G', L')$, $\mathcal{U}(G, L, M) \rightarrow \mathcal{U}(G', L', M)$ are isomorphisms.

If (G', L') is obtained from (G, L) by adjoining an isolated boundary vertex x , then these maps furnish isomorphisms

$$\Upsilon(G', L') \cong \Upsilon(G, L) \oplus Rx$$

and

$$\mathcal{U}(G', L', M) \cong \mathcal{U}(G, L, M) \times M^{\{x\}}.$$

Proof. Suppose G' is obtained from G by adjoining an isolated boundary vertex x . Then we have

$$\begin{aligned} RV(G') &= RV(G) \oplus Rx, \\ L'(RV^\circ(G')) &= L(RV^\circ(G)) \subseteq RV(G). \end{aligned}$$

Taking the quotient of the top row by the bottom row yields $\Upsilon(G', L') \cong \Upsilon(G, L) \oplus Rx$. Then applying $\text{Hom}(-, M)$ yields the desired equation for $\mathcal{U}(-, M)$.

Suppose G' is obtained from G by adjoining a boundary edge. Then $V(G') = V(G)$ and $V^\circ(G') = V^\circ(G)$. Since the endpoints of e are boundary vertices, the two Laplacians L' and L agree on $RV^\circ(G)$. Hence, $\Upsilon(G', L') = \Upsilon(G, L)$, and application of $\text{Hom}(-, M)$ yields the corresponding statement for $\mathcal{U}(-, M)$.

Finally, suppose G' is obtained from G by adjoining a boundary spike e with $x = e_+ \in \partial V(G')$ and $y = e_- \in V^\circ(G')$. Then consider the commutative diagram:

$$\begin{CD} 0 @>>> RV^\circ(G) @>>> RV^\circ(G') @>>> Ry @>>> 0 \\ @. @V L VV @V L' VV @V \psi VV @. \\ 0 @>>> RV(G) @>>> RV(G') @>>> Rx @>>> 0, \end{CD}$$

where the horizontal arrows are given by the direct sum decompositions induced by $V^\circ(G') = V^\circ(G) \sqcup \{y\}$ and $V(G') = V(G) \sqcup \{x\}$, and the vertical arrow $\psi : Ry \rightarrow Rx$ is given by $y \mapsto -w'(e)x$. Commutativity of the left square follows from the fact that $w'|_G = w$ and $d'|_G = d$. To check commutativity of the right square, start with an arbitrary element of $RV^\circ(G')$ written in the form $z + ry$, where $z \in RV^\circ(G)$ and $r \in R$. Going right and then down produces $-rw'(e)x$. In the other direction, going down from $RV^\circ(G')$ to $RV(G')$ yields

$$L'(z + ry) = L'z + r \left(d'(y)y + \sum_{e': e'_+ = y} w'(e')(y - e_-) \right) \in RV(G) - rw'(e)x,$$

and then following the diagram right to Rx yields $-rw'(e)x$.

The rows are clearly exact. Hence, the snake lemma yields an exact sequence

$$\dots \rightarrow \ker \psi \rightarrow \Upsilon(G, L) \rightarrow \Upsilon(G', L') \rightarrow \text{coker } \psi \rightarrow 0.$$

Since ψ is an isomorphism, this shows that $\Upsilon(G, L) \rightarrow \Upsilon(G', L')$ is an isomorphism, and application of $\text{Hom}(-, M)$ yields the corresponding statement for $\mathcal{U}(-, M)$ by Lemma 2.9. \square

LEMMA 5.8. *If an R^\times -network (G', L') is obtained from (G, L) by a simple layerable extension, then the induced map $\mathcal{U}_0(G, L, M) \rightarrow \mathcal{U}_0(G', L', M)$ is an isomorphism.*

Proof. Since (G, L) is a subnetwork of (G', L') , the map $\mathcal{U}_0(G, L, M) \rightarrow \mathcal{U}_0(G', L', M)$ defined by Lemma 4.11 extends a function u on G to a function u' on G' by setting $u'|_{V(G') \setminus V(G)} = 0$. This map is clearly injective. We prove surjectivity by cases.

Suppose G' is obtained from G by adjoining an isolated boundary vertex. If $u' \in \mathcal{U}_0(G', L', M)$, then clearly $u'(x) = 0$. Moreover, $L'u'|_{V(G)} = L(u'|_{V(G)})$. Thus, u' restricts to a function in $u \in \mathcal{U}_0(G, L, M)$, and u is mapped to u' by the extension map.

Suppose G' is obtained from G by adjoining a boundary edge e . Recall $V(G') = V(G)$, so functions on G' and functions on G are equivalent. If $u \in \mathcal{U}_0(G', L', M)$, then $u(e_+) = u(e_-) = 0$ and hence $Lu = L'u$. Thus, $u \in \mathcal{U}_0(G, L, M)$ and u is mapped to itself by the extension map.

Suppose G' is obtained from G by adjoining a boundary spike e with boundary endpoint $e_+ = x$ and interior endpoint $e_- = y$. Suppose that $u' \in \mathcal{U}_0(G', L', M)$ and let $u = u'|_{V(G)}$. Note that

$$0 = L'u'(x) = d(x)u'(x) + w(e)(u'(x) - u'(y)) = 0 + w(e)(0 - u(y)).$$

Since $w(e)$ is a unit in R , this implies $u(y) = 0$. Moreover, we have

$$0 = L'u'(y) = Lu(y) + w(e)(u'(y) - u'(x)) = Lu(y).$$

Thus, $u(y) = Lu(y) = 0$, which shows $u \in \mathcal{U}_0(G, L, M)$. Thus, u' is in the image of the extension map. \square

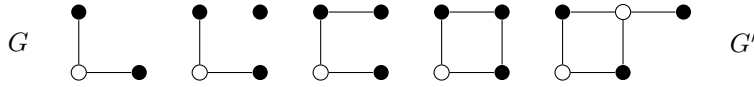


FIG. 5.4. A layerable filtration from a ∂ -graph G to a layerable extension G' of G .

Remark 5.9. In Lemmas 5.7 and 5.8, the only place where we used the fact that $w(e) \in R^\times$ was for the case of a boundary spike. When $w(e)$ is not a unit in R , the snake lemma still yields an exact sequence relating $\Upsilon(G, L)$ and $\Upsilon(G', L')$. Though we will focus on the case of unit edge weights in this paper, applying layerable filtrations in the general case seems like a promising avenue for future research.

Now we will describe layerable extensions formed from a sequence of simple layerable extensions.

DEFINITION 5.10. We say that G' is a finite layerable extension of G if there exists a finite of sequence of sub- ∂ -graphs of G'

$$G = G_0 \subseteq G_1 \subseteq \dots \subseteq G_n = G'$$

such that G_j is a simple layerable extension of G_{j-1} . We call $\{G_j\}$ a layerable filtration from G to G' . We say that G_j is layerable if it is a layerable extension of \emptyset . See Figure 5.4 for an example.

PROPOSITION 5.11. Suppose that a finite R^\times -network (G', L') is a layerable extension of (G, L) through the filtration $\{(G_j, L_j)\}_{j=0}^n$. Let $S \subseteq V(G')$ be the set of vertices which are adjoined as isolated boundary vertices at some step of the filtration. Then the inclusion map $(G, L) \rightarrow (G', L')$ induces isomorphisms

$$\begin{aligned} \Upsilon(G', L') &\cong \Upsilon(G, L) \oplus RS, \\ \mathcal{U}(G', L', M) &\cong \mathcal{U}(G, L, M) \times M^S, \\ \mathcal{U}_0(G', L', M) &\cong \mathcal{U}_0(G, L, M). \end{aligned}$$

Proof. In the first claim, the map $\Upsilon(G, L) \rightarrow \Upsilon(G', L')$ is given by functoriality of Υ and the map $RS \rightarrow \Upsilon(G', L')$ is given by the composition $RS \rightarrow RV(G') \rightarrow \Upsilon(G', L')$. Let $S_j = S \cap V(G_j)$. The three cases of Lemma 5.7 show that the map

$$\Upsilon(G_j, L_j) \oplus R(S_{j+1} \setminus S_j) \rightarrow \Upsilon(G_{j+1}, L_{j+1})$$

is an isomorphism (in the case of adjoining a boundary spike or boundary edge, $S_{j+1} \setminus S_j = \emptyset$). By induction, this implies that

$$\Upsilon(G, L) \oplus RS_n \rightarrow \Upsilon(G_n, L_n) \text{ is an isomorphism,}$$

which completes the proof because $(G_n, L_n) = (G', L')$.

The second claim follows by application of the functor $\text{Hom}_R(-, M)$ to the first claim, or alternatively by inductive application of the second isomorphism in Lemma 5.7. The third claim follows by inductive application of Lemma 5.8. \square

The claim $\mathcal{U}(G', L', M) \cong \mathcal{U}(G, L, M) \times M^S$ in Proposition 5.11 has the following interpretation in terms of harmonic extensions: Any harmonic function u on (G, L) extends to a harmonic function u' on (G', L') . For any $\phi \in M^S$, there is a unique harmonic extension u' of u such that $u'|_S = \phi$. We can see from the inductive

application of Lemma 5.7 that the extension u' can be constructed using step-by-step extensions from (G_j, L_j) to (G_{j+1}, L_{j+1}) . In other words, a layerable filtration provides a geometric model for discrete harmonic continuation, loosely analogous to the sequences of domains used for harmonic continuation in complex analysis.

The special case of Proposition 5.11 where $G = \emptyset$ deserves special comment.

PROPOSITION 5.12. *Suppose that (G', L') is a layerable R^\times -network, and let $\{(G_j, L_j)\}$ be a layerable filtration from \emptyset to (G', L') . Let S be the set of vertices which are adjoined as isolated boundary vertices at some step of the filtration.*

- (1) $\Upsilon(G', L')$ is a free R -module and a basis is given by S .
- (2) For every R -module M , for every $\phi \in M^S$, there is a unique harmonic function u such that $u|_S = \phi$.
- (3) The network (G', L') is nondegenerate and $\mathcal{U}_0(G, L, M) = 0$ for every R -module M .

Remark 5.13. If the network in Proposition 5.12 is finite, then $|S| = |\partial V|$ (one verifies this by induction after observing that adjoining a boundary spike or boundary edge does not change $|\partial V|$). Thus, $\Upsilon(G, L)$ is a free R -module of rank $|\partial V|$.

Remark 5.14. If we unwind the proof leading up to Proposition 5.12(3), it does in fact boil down to the intuitive argument given in section 1.1 that if we assume u is harmonic and has zero boundary data, then we can inductively show that $u = 0$ further and further inside the network using harmonic continuation. Indeed, if $(G_0, L_0) \subseteq (G_1, L_1) \subseteq \dots (G_n, L_n) = (G, L)$ is a layerable filtration, then Lemma 5.8 shows

$$(u|_{\partial V_{j+1}}, L_{j+1}u|_{\partial V_{j+1}}) = 0 \implies (u|_{\partial V_j}, L_j u|_{\partial V_j}) = 0.$$

Inductive application of this statement shows that zero potential and current conditions propagate inward from ∂V_n to ∂V_{n-1} and so forth to ∂V_0 . Since all vertices are contained in some ∂V_j , this shows $u \equiv 0$. By linearity, the same argument shows that a harmonic function is uniquely determined by its boundary data and the values can be deduced using harmonic continuation from the boundary inward.

5.2. Algebraic characterization of layerability. We are now ready to give our algebraic characterization of layerability. We begin by considering networks over a field and then apply our algebraic machinery to state several other equivalent conditions.

LEMMA 5.15. *Let G be a finite ∂ -graph and let F be a field with at least three elements. Then G is layerable if and only if every F^\times -network on the ∂ -graph G is nondegenerate.*

Proof. If G is layerable and (G, L) is an F^\times -network on G , then (G, L) is nondegenerate by Proposition 5.12(3).

To prove the converse, suppose G is not layerable. Let G' be a minimal non-layerable sub- ∂ -graph of G . Since G' is minimal, it cannot have a boundary spike, boundary edge, or isolated boundary vertex, since performing a layer-stripping operation on G' would preserve layerability. Hence, every boundary vertex in G' must have multiple edges incident to it, and all its neighbors are interior vertices. In particular, $\mathcal{E}(x)$ and $\mathcal{E}(y)$ are disjoint for distinct vertices $x, y \in \partial V(G')$. Since F has at least three elements, we can write zero as the sum of n nonzero elements for every $n \in \mathbb{N}$. Thus, we can choose $w' : E(G') \rightarrow F^\times$ such that $\sum_{e \in \mathcal{E}(x)} w'(e) = 0$ for every $x \in \partial V(G')$.

Let $u : V(G') \rightarrow F$ be given by $u = 0$ on $\partial V(G')$ and $u = 1$ on $V^\circ(G')$. Define $d : V(G') \rightarrow F$ such that

$$d'(x) = - \sum_{\substack{e \in \mathcal{E}(x) \\ e_- \in \partial V(G')}} w'(e).$$

Then note that

$$L'u(x) = d'(x) + \sum_{\substack{e \in \mathcal{E}(x) \\ e_- \in \partial V(G')}} w'(e) = 0 \text{ for } x \in V^\circ(G'),$$

while

$$L'u(x) = - \sum_{e \in \mathcal{E}(x)} w'(e) = 0 \text{ for } x \in \partial V(G').$$

Therefore, $u \in \mathcal{U}_0(G', L', F)$ and clearly u is not identically zero.

Let w and d be any extension of w' and d' to G . Then $(G', L') \rightarrow (G, L)$ is an injective F^\times -network morphism and hence extension by zero defines an injective map $\mathcal{U}_0(G', L') \rightarrow \mathcal{U}_0(G, L)$ (as a special case of Lemma 4.11). Thus, $\mathcal{U}_0(G, L)$ is nonzero, so (G, L) is degenerate as desired. \square

Next, we state some other equivalent algebraic conditions. In (4) below, $R^*(G, F)$ and $L^*(G, F)$ are the ring and generalized Laplacian defined in Definition 2.17. The ring R^* is the polynomial algebra $F(t_e^{\pm 1}, e \in E; t_x, x \in V)$, where $t_{\bar{e}} = t_e$, and L^* is given by setting $w(e) = t_e$ and $d(x) = t_x$.

THEOREM 5.16. *Let G be a finite ∂ -graph. The following are equivalent:*

- (1) G is layerable.
- (2) For every ring R , every R^\times -network on G is nondegenerate.
- (3) For every ring R , for every non-degenerate R^\times -network (G, L) on the ∂ -graph G , $\Upsilon(G, L)$ is a free R -module.
- (4) There exists a field F with at least three elements such that $\Upsilon(G, L^*(G, F))$ is a flat $R^*(G, F)$ -module.
- (5) There exists a field F with at least three elements such that every F^\times -network on G is nondegenerate.

Proof. We prove the implications in two cycles. First, (1) \implies (2) by Proposition 5.12; (2) \implies (5) trivially; and (5) \implies (1) by Lemma 5.15.

Next, observe (1) \implies (3) by Proposition 5.12; (3) \implies (4) because (G, L^*) is nondegenerate (Proposition 2.18) and because free modules are automatically flat; (4) \implies (5) was proved in Proposition 2.18; and we already know (5) \implies (1). \square

Remark 5.17. Based on the statement alone, one can come up with several other equivalent conditions: For instance, “ $\Upsilon(G, L)$ is flat for every nondegenerate R^\times -network on G for every ring R ,” or “every F^\times -network on G is nondegenerate for every field F .”

5.3. Transformations of harmonic boundary data. We now describe how adjoining a boundary spike or boundary edge affects the boundary potential and current data of harmonic functions. The simple coordinate system given here will be used later in Theorem 5.23, which gives an algorithm for simplifying computation of $\mathcal{U}_0(G, L, M)$. The discussion will also relate layer-stripping to symplectic matrices as in [30], [27, section 12].

For a harmonic function $u \in \mathcal{U}(G, L, M)$, the term *boundary data* will refer to $(u|_{\partial V}, Lu|_{\partial V})$. If (G', L') is a simple layerable extension of (G, L) and if $u \in \mathcal{U}(G, L, M)$ extends to $u' \in \mathcal{U}(G', L', M')$, then it is easy to explicitly compute the boundary data of u' from that of u .

Adjoining a boundary spike. Suppose G' is obtained from G by adjoining a boundary spike e with $e_+ \in \partial V(G')$ and $e_- \in V^\circ(G')$, and assume $w(e) = w \in R^\times$ and $d(e_+) = d \in R$. For explicitness, we will index the boundary vertices of each ∂ -graph by integers: Let $[m]$ denote $\{1, \dots, m\}$, and let $\ell : [m] \rightarrow \partial V(G)$ and $\ell' : [m] \rightarrow \partial V(G')$ be bijections labeling the vertices. We assume the two labelings are *consistent*, meaning that $e_- \in \partial V(G)$ and $e_+ \in \partial V(G')$ have the same index, and every $x \in \partial V(G) \cap \partial V(G')$ has the same index with respect to the two different labelings.

Let j be the index of $e_- \in \partial V(G)$ and $e_+ \in \partial V(G')$, so that $e_+ = \ell'(j)$ and $e_- = \ell(j)$. Let $E_{p,q}$ denote the matrix with a 1 in the (p, q) entry and zeros elsewhere. Then we claim that

$$\begin{pmatrix} u' \circ \ell' \\ L'u' \circ \ell' \end{pmatrix} = \begin{pmatrix} I & w^{-1}E_{j,j} \\ dE_{j,j} & I + dw^{-1}E_{j,j} \end{pmatrix} \begin{pmatrix} u \circ \ell \\ Lu \circ \ell \end{pmatrix},$$

where the matrix blocks are $m \times m$, and $u \circ \ell$ is viewed as a vector in M^m . Note that the matrix has a simpler form when $d = 0$. To verify the matrix formula, note that

$$0 = L'u'(e_-) = Lu(e_-) + w \cdot (u(e_-) - u(e_+))$$

so that

$$u'(e_+) = u(e_-) + w^{-1}Lu(e_-)$$

and

$$L'u'(e_+) = d \cdot u'(e_+) + w(u'(e_+) - u'(e_-)) = du(e_-) + (dw^{-1} + 1)Lu(e_-)$$

Adjoining a boundary edge. Suppose G' is obtained from G by adjoining a boundary edge e with $w = w(e)$. In this case, $\partial V(G) = \partial V(G')$, so two indexings $\ell : [m] \rightarrow \partial V(G)$ and $\ell' : [m] \rightarrow \partial V(G')$ are called *consistent* if $\ell = \ell'$. Assume that the indices of e_- and e_+ are i and j . Then a similar computation as before shows that

$$\begin{pmatrix} u' \circ \ell' \\ L'u' \circ \ell' \end{pmatrix} = \begin{pmatrix} I & 0 \\ w(E_{i,i} + E_{j,j} - E_{i,j} - E_{j,i}) & I \end{pmatrix} \begin{pmatrix} u \circ \ell \\ Lu \circ \ell \end{pmatrix}.$$

The matrices described above will be called the *boundary data transformations* for adjoining a boundary spike or boundary edge.

Standard form for layerable filtrations. In general a layerable filtration is allowed to mix adjoining boundary spikes, adjoining boundary edges, and adjoining isolated boundary vertices in any order. However, it is sometimes convenient for the sake of computation to assume that all the isolated boundary vertices are adjoined before the other operations. For finite networks, we can always arrange this; simply take all the isolated boundary vertices that are adjoined at any step of the filtration, adjoin them at the beginning, and include them in all subsequent ∂ -graphs in the filtration.

A *standard form layerable filtration* for (G, L) is a sequence of R -networks

$$(G_0, L_0) \subseteq (G_1, L_1) \subseteq \dots \subseteq (G_n, L_n) = (G, L),$$

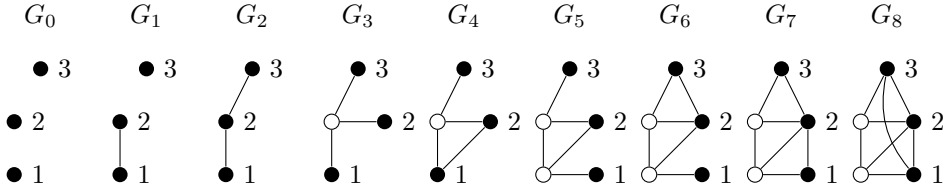


FIG. 5.5. A layerable filtration in standard form with a consistent labeling.

where (G_0, L_0) consists entirely of isolated boundary vertices, and (G_{j+1}, L_{j+1}) is obtained from (G_j, L_j) by adjoining a boundary spike or boundary edge. A *consistent labeling* for such a filtration consists of (bijective) labeling functions $\ell_j : [m] \rightarrow \partial V_j$ such that for each j , ℓ_j and ℓ_{j+1} are consistent in the sense described above for the two cases of adjoining a boundary spike and adjoining a boundary edge. See Figure 5.5 for an example.

LEMMA 5.18. *Let (G, L) be a layerable R^\times -network and let M be an R -module. Let $(G_0, L_0), \dots, (G_n, L_n)$ be a standard form layerable filtration for (G, L) , and let ℓ_0, \dots, ℓ_n be a consistent labeling for it. Let*

$$T_0 = \begin{pmatrix} I & 0 \\ D & I \end{pmatrix},$$

where $D = \text{diag}(d \circ \ell_0(1), \dots, d \circ \ell_0(m))$. For $j \geq 0$, let T_j be the boundary data transformation associated to the operation $(G_{j-1}, L_{j-1}) \mapsto (G_j, L_j)$. For every $\phi \in M^m$, there is a unique harmonic function such that $u|_{\partial V_0} \circ \ell_0 = \phi$, and if $u_j = u|_{V_j}$, we have

$$\begin{pmatrix} u_j \circ \ell_j \\ L_j u_j \circ \ell_j \end{pmatrix} = T_j \dots T_1 T_0 \begin{pmatrix} \phi \\ 0 \end{pmatrix} \text{ for } j = 0, \dots, n.$$

Proof. The existence and uniqueness of u follows from Proposition 5.11. To prove the matrix equation, note that G_0 consists of isolated boundary vertices. Thus, all functions u_0 are harmonic and have $L_0 u(x) = d(x)u(x)$, which establishes the case $j = 0$. Then the equation follows for all j by inductive application of the foregoing computation. \square

Remark 5.19. In the last lemma, computing the boundary data of u_j at every step of the filtration is sufficient to find the values of u on all of G . This is because every vertex of G must be a boundary vertex at some step of the filtration.

Remark 5.20. To simplify the statement of Lemma 5.18, we have assumed that each layerable extension adds only one boundary spike or boundary edge. In general, it may be computationally convenient to adjoin multiple spikes or multiple boundary edges at once. For instance, if $m = 2$ and one adjoins two boundary spikes with parameters w_1, w_2 and d_1, d_2 , then the matrix is

$$\begin{pmatrix} 1 & 0 & w_1^{-1} & 0 \\ 0 & 1 & 0 & w_2^{-1} \\ d_1 & 0 & 1 + d_1 w_1^{-1} & 0 \\ 0 & d_2 & 0 & 1 + d_2 w_2^{-1} \end{pmatrix}.$$

Remark 5.21. Recording boundary potential and current data is not the only feasible bookkeeping method for the harmonic continuation process. In some situations, it could more convenient to record the values of u on the boundary of G_j

and all vertices adjacent to the boundary, rather than the recording u and Lu on the boundary.

Remark 5.22. Direct computation will verify that the boundary data transformations discussed here are all symplectic matrices, that is, they satisfy

$$T^t J T = J,$$

where

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

The relationship between layer-stripping and the symplectic group over \mathbb{R} was studied in [30] from the viewpoint of Lie theory, while our discussion makes the connection in a more elementary and explicit way. The same matrices were written down in [27, p. 48], but without the interpretation as transformations of boundary behavior.

5.4. Using harmonic continuation to compute \mathcal{U}_0 . Harmonic continuation can be applied on nonlayerable networks as well. In section 3.2, we used harmonic continuation to compute $\mathcal{U}_0(\text{CLF}(m, n), L_{\text{std}}, \mathbb{Q}/\mathbb{Z})$. Now we generalize this approach to compute $\mathcal{U}_0(G, L, M)$ for a finite R^\times -network (G, L) , using layerable filtrations to keep track of the harmonic continuation process.

If G is a ∂ -graph and $S \subseteq V^\circ(G)$, then we define $G_{S \rightarrow \partial}$ as the ∂ -graph obtained by changing the vertices in S from interior to boundary, so that

$$V(G_{S \rightarrow \partial}) = V(G), \quad E(G_{S \rightarrow \partial}) = E(G), \quad \partial V(G_{S \rightarrow \partial}) = \partial V(G) \sqcup S.$$

THEOREM 5.23. *Let (G, L) be a finite R^\times -network, let $S \subseteq V^\circ(G)$, and suppose $G_{S \rightarrow \partial}$ is layerable. Then there exists a matrix $A \in M_{|S| \times (|\partial V| + |S|)}(R)$ such that for every R -module M*

$$\mathcal{U}_0(G, L, M) \cong \ker \left(A : M^{|S|} \rightarrow M^{|S| + |\partial V|} \right).$$

The matrix A can be computed explicitly from a given layerable filtration for $G_{S \rightarrow \partial}$, as described by (5.1) and (5.3) below.

Proof. First, let us motivate the proof in light of earlier results. Let $G' = G_{S \rightarrow \partial}$ for short and recall $\partial V(G') = \partial V(G) \sqcup S$. Note that since G' and G differ only in the assignment of boundary vertices, we have

$$\mathcal{U}_0(G, L, M) = \{u \in \mathcal{U}(G', L, M) : u|_{\partial V(G)} = 0, Lu|_{\partial V(G')} = 0\}.$$

In other words, a function $u \in \mathcal{U}_0(G, L, M)$ is equivalent to a harmonic function on (G', L) satisfying the additional boundary conditions that $u|_{\partial V(G)} = 0$ and $Lu|_{\partial V(G')} = 0$.

Now G' is layerable. Thus, as explained in Remark 5.14, a harmonic function u on G' is uniquely determined by its boundary data. Moreover, the values of the function u can be found from the boundary data by harmonically continuing from $\partial V(G')$ inward along a layer-stripping filtration of G' . Thus, to find $\mathcal{U}_0(G, L, M)$, we need to start with boundary data on G' satisfying our extra boundary conditions, harmonically continue inward, and then ensure that the result we get is actually harmonic on G' .

Suppose that we have a standard-form layerable filtration of G' given by

$$\emptyset \subseteq (G_0, L_0) \subseteq \cdots \subseteq (G_n, L_n) = (G', L).$$

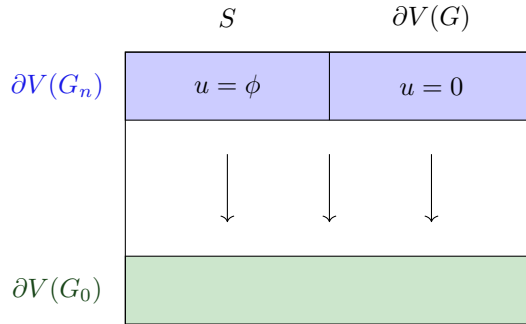


FIG. 5.6. Schematic overview of the proof of Theorem 5.23. Arrows show the direction of harmonic continuation.

Our harmonic continuation will start with a function defined on $\partial V(G_n)$ and then extend it inward to $\partial V(G_{n-1})$ (see Figure 5.6). At step j (counting backward from n), we will have a partially defined harmonic function whose domain is the roughly the complement of G_j .

To make this idea precise, we define a complementary sub- ∂ -graph H_j by

$$(5.1) \quad \begin{aligned} V(H_n) &= V(G') \setminus V^\circ(G_n), \\ E(H_n) &= E(G') \setminus E(G_n), \\ V^\circ(H_n) &= V(G') \setminus V(G_n), \\ \partial V(H_n) &= \partial V(G_n). \end{aligned}$$

See Figure 5.7 for an example. The following facts follow from direct casework:

- If G_{j+1} is obtained from G_j by adjoining a boundary spike, then H_{j+1} is obtained from H_j by contracting a boundary spike.
- If G_{j+1} is obtained from G_j by adjoining a boundary edge, then H_{j+1} is obtained from H_j by deleting a boundary edge.
- The graph H_n consists of isolated boundary vertices.
- The graph H_0 has the same vertex and edge sets as G and \bar{G} , but a different choice of boundary vertices. Specifically, $\partial V(H_0)$ is equal to $\partial V(G_0)$ rather than $\partial V(G_n) = \partial V(G)$.

This shows that $H_n \subseteq H_{n-1} \subseteq \dots \subseteq H_0$ is a standard-form layerable filtration of H_0 .

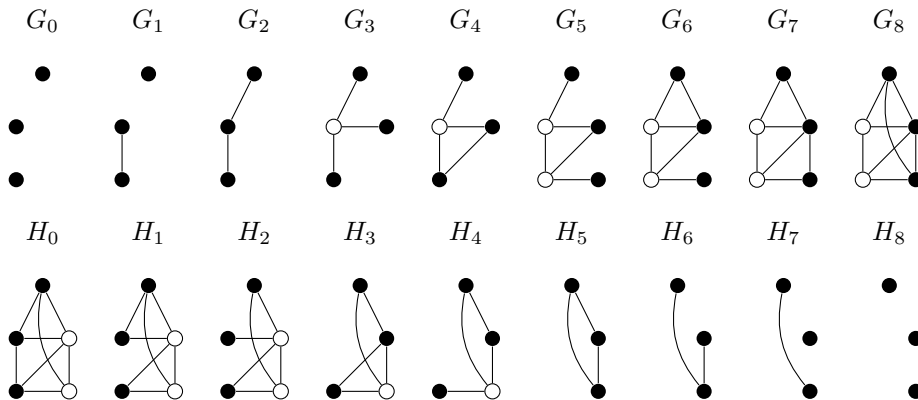


FIG. 5.7. Complementary layerable filtrations as defined by (5.1).

Harmonic continuation inward from the boundary of G toward $\partial V(G_0)$ will correspond to building harmonic extensions through the filtration H_n, H_{n-1}, \dots . But now that H_j has been defined, we no longer need to work directly with the ∂ -graphs G_j . We can express $\mathcal{U}_0(G, L, M)$ in terms of harmonic functions on H_0 rather than harmonic functions on G_n : Note that H_0 and G differ only in the assignment of boundary vertices; expressing the conditions for $u \in \mathcal{U}_0(G, L, M)$ in terms of H_0 yields

$$\mathcal{U}_0(G, L, M) = \{u \in \mathcal{U}(H_0, L, M) : u|_{\partial V(G)} = 0, Lu|_{\partial V(H_0)} = 0\}.$$

As in Lemma 5.18, we can use the filtration $H_n \subseteq \dots \subseteq H_0$ to parametrize the harmonic functions on H_0 in terms of their values on $\partial V(H_n) = \partial V(G) \sqcup S$. The condition $u|_{\partial V(G)} = 0$ simply says that a subset of our initial parameters will be zero, and thus our harmonic function will be parametrized by the values on S with the values on the rest of $\partial V(H_n)$ set to zero.

Thus, we proceed as follows: Given a vector $\phi \in M^S$, we extend ϕ by zero to a vector in $M^{\partial V(H_n)} = \mathcal{U}(H_n, L|_{H_n}, M)$, then apply the sequence of boundary data transformations associated to the filtration H_n, \dots, H_0 as in Lemma 5.18 to compute a harmonic extension u to H_0 . We let A be the transformation $\phi \mapsto Lu|_{\partial V(H_n)}$ that sends $\phi \in M^S$ to the boundary values of Lu for the harmonic extension to H_0 . Then the functions $u \in \mathcal{U}_0(G, L, M)$ correspond to the values of ϕ such that $A\phi = 0$, so that

$$(5.2) \quad \mathcal{U}_0(G, L, M) \cong \ker(A : M^S \rightarrow M^{\partial V(H_n)}).$$

The matrix A is given explicitly as follows. Let $m = |\partial V(H_0)| = |S| + |\partial V(G)|$, and choose a consistent labeling for the filtration $\{H_j\}$. Assume that in the indexing of $\partial V(H_n)$, the vertices in S are indexed first by $1, \dots, s$ and then the vertices of $\partial V(G)$ are indexed by $s + 1, \dots, m$. Let T_n, T_{n-1}, \dots, T_0 be the sequence of boundary data transformations corresponding to the filtration H_n, H_{n-1}, \dots, H_0 (as in Lemma 5.18 except with G_j replaced by H_{n-j}). Here T_n is the transformation for the initial network H_n , and T_j is the transformation from H_{j+1} to H_j . Then set

$$(5.3) \quad A = (0_{m \times m}, I_{m \times m})T_0T_1 \dots T_n \begin{pmatrix} I_{s \times s} \\ 0_{(2m-s) \times s} \end{pmatrix}. \quad \square$$

Example 5.24. Let us apply Theorem 5.23 to the critical group of the graph shown in Figure 5.8, left. By Proposition 2.23, it suffices to compute $\mathcal{U}_0(G, L_{\text{std}}, \mathbb{Q}/\mathbb{Z})$, where G is the middle ∂ -graph in the figure with one boundary vertex. Let $S = \{x, y\}$; then $G' = G_{S \rightarrow \partial}$ is the graph in Figure 5.8, right. We let G_j and H_j be the ∂ -graphs shown earlier in Figure 5.7. For each G_j and H_j , we define ℓ_j by labeling the boundary vertices 1, 2, 3 from bottom to top (thus, 1 corresponds to x or v , 2 corresponds to y or w , and 3 corresponds to z). To compute A by (5.3), we write down the

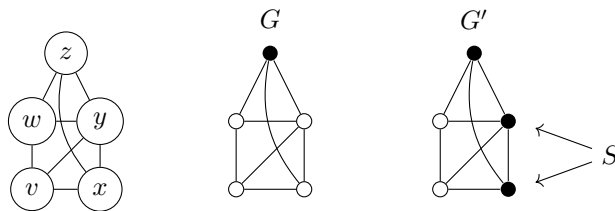


FIG. 5.8. Graphs for Example 5.24.

transformations for H_8, H_7, \dots . Note that because $d = 0$, the initial transformation T_8 is the identity. The next few are

$$\begin{aligned}
 T_7 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, & T_6 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
 T_5 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{pmatrix}, & T_4 &= \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},
 \end{aligned}$$

and the rest of the transformations are computed similarly. After some straightforward computation,

$$A = \begin{pmatrix} 0_{3 \times 3} & I_{3 \times 3} \end{pmatrix} T_0 \dots T_7 T_8 \begin{pmatrix} I_{2 \times 2} \\ 0_{4 \times 2} \end{pmatrix} = \begin{pmatrix} 12 & -9 \\ -15 & 15 \\ 3 & -6 \end{pmatrix}.$$

By performing invertible integer row and column operations, we can transform A into the matrix

$$A' = \begin{pmatrix} 3 & 0 \\ 0 & 15 \\ 0 & 0 \end{pmatrix}.$$

This implies

$$\mathcal{U}_0(G, L_{\text{std}}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/3 \times \mathbb{Z}/15 \cong \mathbb{Z}/3 \times \mathbb{Z}/3 \times \mathbb{Z}/5.$$

Harmonic continuation also allows us to compute the harmonic functions explicitly from the parameters on S . We will demonstrate this by computing $\mathcal{U}_0(G, L_{\text{std}}, \mathbb{Z}/3)$ and $\mathcal{U}_0(G, L_{\text{std}}, \mathbb{Z}/5)$. For a harmonic function u , we have from the last equation of Lemma 5.18 that

$$\begin{pmatrix} u(v) \\ u(w) \\ u(z) \\ 0 \\ 0 \\ 0 \end{pmatrix} = T_0 \dots T_7 T_8 \begin{pmatrix} u(x) \\ u(y) \\ u(z) \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

From explicit computation of the first two columns of $T_0 \dots T_8$, we find that when $u(z) = 0$, we have

$$u(v) = 3u(x) - u(y), \quad u(w) = -4u(x) + 5u(y).$$

To compute $\mathcal{U}_0(G, L_{\text{std}}, \mathbb{Z}/3)$, we observe that since $A = 0 \pmod 3$, every choice of two parameters in $\mathbb{Z}/3$ on S yields a harmonic function, and we obtain two generators

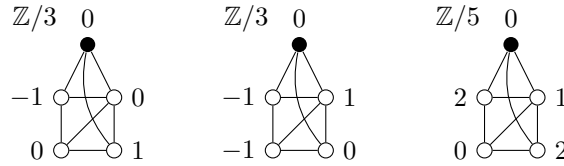


FIG. 5.9. Generators for $\mathcal{U}_0(G, L_{\text{std}}, \mathbb{Z}/3)$ and $\mathcal{U}_0(G, L_{\text{std}}, \mathbb{Z}/5)$ on the network G from Example 5.24.

shown in Figure 5.9. Next, to compute $\mathcal{U}_0(G, L_{\text{std}}, \mathbb{Z}/5)$, we read off from the matrix A that u must satisfy $3u(x) - 6u(y) = 0$, and so a generator is given by taking $u(x) = 2$ and $u(y) = 1$. The resulting harmonic function is shown in Figure 5.9.

Example 5.25. In section 3, we used harmonic continuation to compute $\mathcal{U}_0(\text{CLF}(m, n), L_{\text{std}}, \mathbb{Q}/\mathbb{Z})$ without using Theorem 5.23. But in fact, we *could* have applied Theorem 5.23, and it is instructive to see how our method in section 3 can be derived from the ideas in this section.

Let $G = \text{CLF}(m, n)$. Using the indexing of the vertices from section 3.1, define $S = \{0, 1\} \times \{1, \dots, n\}$ and $S' = \{m-1, 0\} \times \{1, \dots, n\}$. Note $G' = G_{S \rightarrow \partial}$ is layerable, and a filtration is shown in Figure 5.10.

$$\begin{aligned} \partial V(G_n) &= \partial V(G) \cup S, \\ \partial V(G_0) &= \partial V(G) \cup S'. \end{aligned}$$

The layer-stripping filtration strips away the graph column by column; for $j = 0, \dots, m-1$, it removes the edges from $\{j\} \times \{0, \dots, n\}$ and $\{j+1\} \times \{0, \dots, n\}$. In each column, it removes the edges from bottom to top or from top to bottom depending on parity.

The filtration suggests a process of harmonic continuation where the initial parameters are the values of u on $S = \{0, 1\} \times \{0, \dots, n\}$ and the harmonic continuation moves column by column from left to right in the picture. In the notation of section 3.2, this means solving for u in terms of a_0 and a_1 by finding a_j inductively. In section 3.2, we did not use the same method of bookkeeping as in section 5.3 and Theorem 5.23 but rather kept track of potential values on two consecutive columns (see Remark 5.21).

Using the filtration pictured here, a_{j+1} can be found from a_0, \dots, a_j in $2n-1$ steps corresponding to the $2n-1$ edges connecting the j th and $(j+1)$ th columns. Our approach in section 3.2 combined all these operations into one step by writing harmonicity in terms of the matrix E . In fact, executing the $2n-1$ steps for each column amounts to inverting the matrix E (recall $4E^{-1}$ appears in the upper left block of the transformation \mathbf{T} from section 3.2). The reason we did not have to do this was that we avoided dealing directly with E^{-1} in the proof of Lemma 3.5.

After solving for a_2, \dots, a_{m-1} through harmonic continuation, the method of Theorem 5.23 requires us to find the values of the initial parameters a_0 and a_1 that will guarantee $Lu|_{\partial V(G_0)} = 0$. Recall $\partial V(G_0) = \partial V(G) \cup S'$. Because the matrix \mathbf{T} was constructed to check harmonicity on one column of vertices, the condition $L_{\text{std}}u|_{S'} = 0$ is equivalent to

$$\mathbf{T} \begin{pmatrix} a_{m-1} \\ a_{m-2} \end{pmatrix} = \begin{pmatrix} a_0 \\ a_{m-1} \end{pmatrix}, \quad \mathbf{T} \begin{pmatrix} a_0 \\ a_{m-1} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_0 \end{pmatrix}.$$

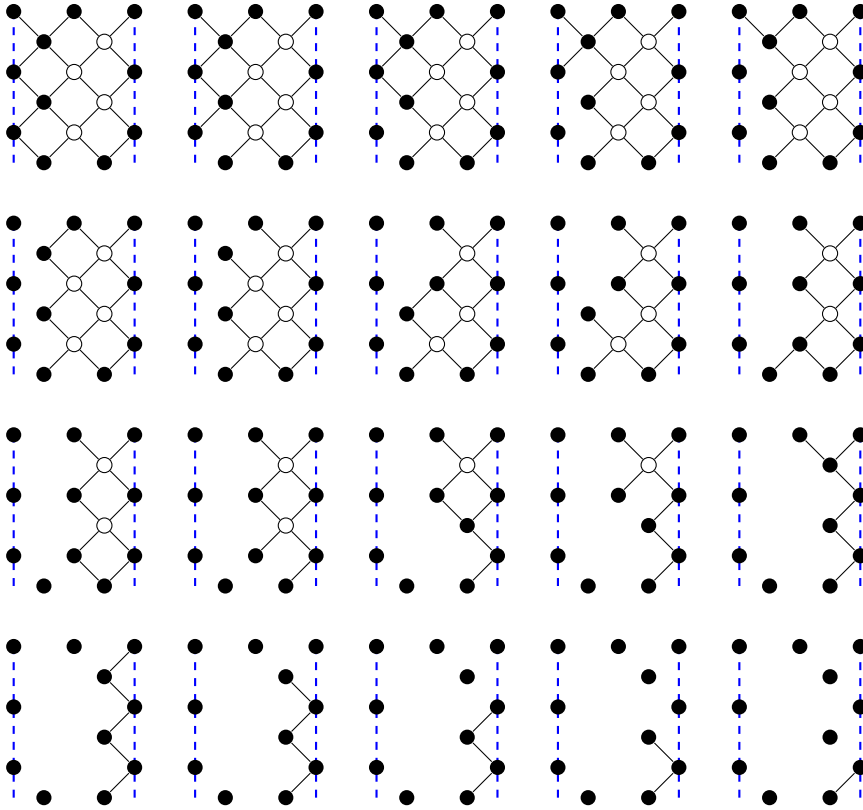


FIG. 5.10. A layerable filtration of $\text{CLF}(4, 2)$ with $S = \{0, 1\} \times \{1, 2\}$ changed to boundary vertices. The vertices on the left and right sides of each picture are identified.

Since the bottom block row of \mathbf{T} is $(I, 0)$, it suffices to check $\mathbf{T}^2(a_{m-1}, a_{m-2})^t = (a_1, a_0)^t$. In short, $Lu|_{S'} = 0$ amounts to the fixed-point condition $\mathbf{T}^m(a_1, a_0)^t = (a_1, a_0)^t$ in section 3.2. Meanwhile, the condition $L_{\text{std}}u|_{\partial V(G)} = 0$ was checked through our computation of M_1 in section 3.3.

5.5. Bounds on the number of invariant factors. If the ring R is a PID, then Theorem 5.23 immediately yields the following bound on the number of invariant factors for the torsion submodule of $\Upsilon(G, L)$, which we will apply below to the critical group and the Laplacian eigenvalues of a graph.

COROLLARY 5.26. *Let R be a PID, and let (G, L) be a finite nondegenerate R -network. Let $S \subseteq V^\circ(G)$, and suppose $G_{S \rightarrow \partial}$ is layerable. Then the torsion submodule of $\Upsilon(G, L)$ has at most $|S|$ invariant factors.*

Proof. By Proposition 2.20, the torsion submodule of $\Upsilon(G, L)$ is isomorphic to $\mathcal{U}_0(G, L, F/R)$, where F is the field of fractions for R . By Theorem 5.23, there is a matrix A with $|S|$ columns such that

$$\mathcal{U}_0(G', L_{\text{std}}, \mathbb{Q}/\mathbb{Z}) \cong \ker_{F/R}(A).$$

Recall that if B is a matrix over a PID R , then B is said to be in *Smith normal form* if the only nonzero entries of B are on the diagonal, and the diagonal entries a_1, a_2, \dots, a_n satisfy $a_j|a_{j+1}$. Moreover, every matrix over R can put into Smith normal

form by multiplying on the left and right by square matrices which are invertible over R (see, for instance, [20, section 12.1, Exercises 16–19]). In particular, if A is the matrix given by Theorem 5.23, then there exist square matrices U and V which are invertible over R such that $A' = UAV$ is in Smith normal form. By interpreting U and V as changes of coordinates, we can see that $\ker_{F/R}(A) \cong \ker_{F/R}(A')$. Thus, the invariant factors for $\text{Crit}(G)$ are found from the diagonal entries of A' . Since A' has $|S|$ columns, there are at most $|S|$ invariant factors. \square

Corollary 5.26 provides a priori upper bounds on the number of invariant factors of the critical group and the multiplicity of Laplacian eigenvalues (Corollaries 5.27 and 5.30 below). Though these bounds are not always sharp, they have the advantage of being computed geometrically without writing down any matrices. Moreover, we will give several families of graphs for which they are sharp.

COROLLARY 5.27. *Suppose that G is a graph without boundary and that $G_{S \rightarrow \partial}$ is layerable. Then $\text{Crit}(G)$ has at most $|S| - 1$ invariant factors.*

Proof. Choose some $x \in S$, and let $G' = G_{x \rightarrow \partial}$. Then by Proposition 2.23, (G', L) is nondegenerate and we have

$$\text{Crit}(G) \cong \mathcal{U}_0(G', L_{\text{std}}, \mathbb{Q}/\mathbb{Z}).$$

Let $S' = S \setminus x$. Then $(G')_{S' \rightarrow \partial}$ is layerable, so by Corollary 5.26, $\mathcal{U}_0(G', L_{\text{std}}, \mathbb{Q}/\mathbb{Z})$ has at most $|S'| = |S| - 1$ invariant factors. \square

Example 5.28. Consider the complete graph K_n . Note that we can make the graph K_n into a layerable ∂ -graph by changing $n - 1$ of the n vertices to boundary vertices. On the other hand, it is shown in [34, Lemma, p. 278] that $\text{Crit}(K_n) \cong (\mathbb{Z}/n)^{n-2}$, which has $n - 2 = (n - 1) - 1$ invariant factors.

Example 5.29. Let Q_n be the 1-skeleton of the k -dimensional cube, described explicitly by $V = \{0, 1\}^n$ with $x \sim y$ if and only if x and y have exactly $n - 1$ coordinates equal to each other.

Let $S = \{0\} \times \{0, 1\}^{n-1}$ and $T = \{1\} \times \{0, 1\}^{n-1}$. Then $(Q_n)_{S \rightarrow \partial}$ is layerable, as we will verify by reducing it to the empty graph through a sequence of layer-stripping operations. All the edges between vertices in S are boundary edges, so we can delete them. After that, all edges from S to T are boundary spikes, so we can contract them, and then the vertices in T become boundary vertices. Finally, we can delete all the edges between vertices in T and we are left with T as a set of isolated boundary vertices.

Since $(Q_n)_{S \rightarrow \partial}$ is layerable, Corollary 5.26 shows that $\text{Crit}(Q_n)$ has at most $|S| - 1 = 2^{n-1} - 1$ invariant factors. This bound is sharp because it was shown in [3, Theorem 1.1] that there are exactly $2^{n-1} - 1$ invariant factors.

COROLLARY 5.30. *Suppose G is a graph without boundary and $G_{S \rightarrow \partial}$ is layerable. Then for any \mathbb{R}^\times -network (G, L) , every eigenvalue of the generalized Laplacian L has multiplicity at most $|S|$. In particular, this holds when L is the standard adjacency matrix or standard Laplacian.*

Proof. Consider the $\mathbb{C}[z]$ -network $(G, zI - L)$ as in Example 2.7. We claim that the number of invariant factors of $\Upsilon(G, zI - L)$ over $\mathbb{C}[z]$ is the maximum multiplicity of an eigenvalue of L . Note that this claim together with Corollary 5.26 will complete the proof.

To prove the claim, note that as in Example 2.7, $\Upsilon(G, zI - L)$ has an elementary divisors decomposition as the direct sum of cyclic modules $\mathbb{C}[z]/(z - \lambda_j)$ over the

eigenvalues $\lambda_1, \dots, \lambda_n$. Because the elements $z - \lambda$ for $\lambda \in \mathbb{C}$ are distinct primes in $\mathbb{C}[z]$, the number of invariant factors decomposition is

$$\Upsilon(G, zI - L) \cong \bigoplus_{k=1}^m \left(\mathbb{C}[z] / \prod_{\lambda: \text{mult}(\lambda) \leq m-k+1} (z - \lambda) \right),$$

where $\text{mult}(\lambda)$ is the multiplicity of λ and $m = \max_j \text{mult}(\lambda_j)$. Thus, the number of invariant factors of $\Upsilon(G, zI - L)$ is the maximum multiplicity of an eigenvalue λ . \square

Example 5.31. We mentioned that for the complete graph K_n , one must assign $n - 1$ boundary vertices. The adjacency matrix of K_n has all off-diagonal entries equal to 1 and thus has the eigenvalue 0 with multiplicity $n - 1$.

Example 5.32. Let C_n by the n -cycle graph. Observe that we can make C_n into a layerable ∂ -graph by changing two adjacent vertices to boundary vertices. This implies that every eigenvalue of the adjacency matrix has multiplicity ≤ 2 . The adjacency matrix is $\Sigma + \Sigma^{-1}$, where Σ is the permutation matrix representing the n -cycle. Since the eigenvalues of Σ are $\{e^{2\pi ik/n}\}_{k=0}^{n-1}$, the eigenvalues of $\Sigma + \Sigma^{-1}$ are $\{2 \cos(2\pi k/n)\}_{k=0}^{n-1}$. In particular, every eigenvalue $\lambda \neq \pm 2$ has multiplicity 2.

5.6. Infinite layerable filtrations. In this section, we generalize Proposition 5.11 to infinite ∂ -graphs, allowing an infinite sequence of simple layerable extensions. We use the following auxiliary definition.

DEFINITION 5.33. Suppose that $\{G_\alpha\}$ is a collection of sub- ∂ -graphs of a given ∂ -graph G . Define $\bigcup_\alpha G_\alpha$ by

$$V\left(\bigcup_\alpha G_\alpha\right) = \bigcup_\alpha V(G_\alpha), \quad V^\circ\left(\bigcup_\alpha G_\alpha\right) = \bigcup_\alpha V^\circ(G_\alpha), \quad E\left(\bigcup_\alpha G_\alpha\right) = \bigcup_\alpha E(G_\alpha).$$

The definition for $\bigcap_\alpha G_\alpha$ is the same with \cup replaced by \cap . One checks straightforwardly from Definition 4.4 that $\bigcup_\alpha G_\alpha$ and $\bigcap_\alpha G_\alpha$ are sub- ∂ -graphs.

DEFINITION 5.34. We say that G' is a layerable extension of G if there is a sequence of sub- ∂ -graphs

$$G = G_0 \subseteq G_1 \subseteq \dots$$

such that $G' = \bigcup_\alpha G_j$.

In order to extend Proposition 5.11 to the infinite case, we need to be able to take limits of our algebraic functors along increasing sequences of sub- ∂ -graphs. We apply the notions of *direct (inductive) limits* and *inverse (projective) limits* of R -modules. For background, see [20, Exercises 7.6.8, 10.3.25, and 10.3.26], [2, Exercises 2.14–19].

LEMMA 5.35. Let $(G_0, L_0) \subseteq (G_1, L_1) \subseteq \dots$ be a sequence of subnetworks of (G', L') and assume that $\bigcup_j G_j = G'$. Then

- (1) $\Upsilon(G', L')$ is (isomorphic to) the direct limit of the sequence

$$\Upsilon(G_0, L_0) \rightarrow \Upsilon(G_1, L_1) \rightarrow \dots,$$

- (2) $\mathcal{U}(G', L', M)$ is (isomorphic to) the inverse limit of the sequence

$$\dots \rightarrow \mathcal{U}(G_1, L_1, M) \rightarrow \mathcal{U}(G_0, L_0, M),$$

- (3) $\mathcal{U}_0(G', L', M)$ is (isomorphic to) the direct limit of the sequence

$$\mathcal{U}_0(G_0, L_0, M) \rightarrow \mathcal{U}_0(G_1, L_1, M) \rightarrow \dots$$

Proof. To prove (1), let Y be the direct limit of the sequence $\Upsilon(G_0, L_0)$. Note that we have a short exact sequence of directed systems

$$\begin{array}{ccccccc}
 0 & \longrightarrow & L_0(RV^\circ(G_0)) & \longrightarrow & RV(G_0) & \longrightarrow & \Upsilon(G_0, L_0) \longrightarrow 0 \\
 & & \downarrow \alpha_{0,1} & & \downarrow \beta_{0,1} & & \downarrow \gamma_{0,1} \\
 0 & \longrightarrow & L_1(RV^\circ(G_1)) & \longrightarrow & RV(G_1) & \longrightarrow & \Upsilon(G_1, L_1) \longrightarrow 0 \\
 & & \downarrow \alpha_{1,2} & & \downarrow \beta_{1,2} & & \downarrow \gamma_{1,2} \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

where the vertical maps are the obvious ones obtained from the inclusions $V(G_n) \rightarrow V(G_{n+1})$ and $V^\circ(G_n) \rightarrow V^\circ(G_{n+1})$. Moreover, we have maps

$$\begin{array}{ccccccc}
 0 & \longrightarrow & L_n(RV^\circ(G_n)) & \longrightarrow & RV(G_n) & \longrightarrow & \Upsilon(G_n, L_n) \longrightarrow 0 \\
 & & \downarrow \alpha_n & & \downarrow \beta_n & & \downarrow \gamma_n \\
 0 & \longrightarrow & L'(RV^\circ(G')) & \longrightarrow & RV(G') & \longrightarrow & \Upsilon(G', L') \longrightarrow 0,
 \end{array}$$

where the vertical arrows are obtained from the inclusion $G_n \rightarrow G$. These maps satisfy $\alpha_n \circ \alpha_{m,n} = \alpha_m$ for $m < n$, and the same holds for β and γ . Therefore, the universal property of direct limits [2, Exercise 2.16] gives us maps

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \varinjlim L_n(RV^\circ(G_n)) & \longrightarrow & \varinjlim RV(G_n) & \longrightarrow & \varinjlim \Upsilon(G_n, L_n) \longrightarrow 0 \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
 0 & \longrightarrow & L'(RV^\circ(G')) & \longrightarrow & RV(G') & \longrightarrow & \Upsilon(G', L') \longrightarrow 0.
 \end{array}$$

Because the direct limit is an exact functor [2, Exercise 2.19], the top row of this diagram is exact. The bottom row is exact by construction of Υ . We easily see that the first two vertical maps are isomorphisms since $V^\circ(G') = \bigcup_n V^\circ(G_n)$ and $V(G') = \bigcup_n V(G_n)$. Therefore, the five-lemma implies that the third vertical map is an isomorphism. This completes the proof of (1).

The statement (2) follows by applying the $\text{Hom}(-, M)$ functor to (1) because whenever $M_0 \rightarrow M_1 \rightarrow \dots$ is a sequence of R -modules, there is a natural isomorphism

$$\text{Hom}(\varinjlim M_n, M) \cong \varprojlim \text{Hom}(M_n, M).$$

The proof of (3) is symmetrical to the proof of (1). Instead of using the short exact sequence

$$0 \rightarrow L(RV^\circ(G)) \rightarrow RV(G) \rightarrow \Upsilon(G, L) \rightarrow 0$$

for each network $G = G_n$ or $G = G'$, we use the short exact sequence

$$0 \rightarrow \mathcal{U}_0(G, L, M) \rightarrow RV^\circ \otimes M \xrightarrow{L \otimes \text{id}} (L \otimes \text{id})(RV^\circ \otimes M) \rightarrow 0.$$

Here $RV^\circ \otimes M$ is viewed as the module of finitely supported functions $V^\circ \rightarrow M$. In the last step, we apply the five-lemma to show that the *first* map out of three is an isomorphism rather than the *last* map as in (1). □

PROPOSITION 5.36. *Suppose that an R^\times -network (G', L') is a layerable extension of (G, L) through the filtration $\{(G_j, L_j)\}_{j=0}^\infty$. Let $S \subseteq V(G')$ be the set of vertices which are adjoined as isolated boundary vertices at some step of the filtration. Then the inclusion map $(G, L) \rightarrow (G', L')$ induces isomorphisms*

$$\begin{aligned} \Upsilon(G', L') &\cong \Upsilon(G, L) \oplus RS, \\ \mathcal{U}(G', L', M) &\cong \mathcal{U}(G, L, M) \times M^S, \\ \mathcal{U}_0(G', L', M) &\cong \mathcal{U}_0(G, L, M). \end{aligned}$$

Proof. Let $S_n = S \cap V(G_n)$. Then by Proposition 5.11, the maps $\Upsilon(G, L) \oplus RS_n \rightarrow \Upsilon(G_n, L_n)$ are isomorphisms. By passing to the direct limit, we see that $\Upsilon(G, L) \oplus RS \rightarrow \Upsilon(G', L')$ is an isomorphism. The arguments for the other two statements are similar. \square

6. Functorial properties of layer-stripping.

6.1. Unramified ∂ -graph morphisms and layer-stripping. In this section, we will show that layer-stripping pulls back through unramified ∂ -graph morphisms. Here it is convenient to take the perspective of removing things from a ∂ -graph rather than adding things, layer-stripping operations rather than layerable extensions, and decreasing rather than increasing filtrations. Recall that the layer-stripping operations are *deleting an isolated boundary vertex*, *contracting a boundary spike*, and *deleting a boundary vertex*. We will also need the following definitions concerning unramified ∂ -graph morphisms and preimages of sub- ∂ -graphs.

DEFINITION 6.1. *We say a ∂ -graph morphism $f: G \rightarrow H$ is an unramified if $\deg(f, x) = 1$ for all $x \in V^\circ(G)$ and $\deg(f, x) \leq 1$ for all $x \in \partial V(G)$. We apply the same terminology to R -network morphisms.*

Lemma 4.2 implies that ∂ -graphs and unramified morphisms form a category, which we will denote ∂ -**graph**_{unrm}. The category of R -networks and unramified morphisms will be denoted R -**net**_{unrm}. The full subcategory of finite ∂ -graphs or networks will be denoted by a superscript “0.” We remark that covering maps and inclusions of sub- ∂ -graphs are unramified. Moreover, restricting an unramified morphism to a sub- ∂ -graph yields another unramified morphism.

DEFINITION 6.2. *If $f: G \rightarrow H$ is ∂ -graph morphism and H' is a sub- ∂ -graph of H , then we can define the pullback or preimage $f^{-1}(H')$ as the ∂ -graph given by*

$$\begin{aligned} V(f^{-1}(H')) \sqcup E(f^{-1}(H')) &= f^{-1}(V(H') \sqcup E(H')), \\ V^\circ(f^{-1}(H')) &= f^{-1}(V^\circ(H')) \cap V^\circ(G). \end{aligned}$$

Straightforward casework verifies that $f^{-1}(H')$ is a sub- ∂ -graph of G .

LEMMA 6.3. *Suppose $f: G \rightarrow H$ is an unramified morphism between finite ∂ -graphs. Suppose $H_2 \subseteq H_1 \subseteq H$ are harmonic sub- ∂ -graphs. If H_2 is obtained from H_1 by a layer-stripping operation, then $f^{-1}(H_2)$ is obtained from $f^{-1}(H_1)$ by a sequence of layer-stripping operations.*

Proof. Suppose that H_2 is obtained from H_1 by deleting a boundary edge e . Since f must map interior vertices to interior vertices, any preimage of a boundary vertex is a boundary vertex. Thus, $f^{-1}(e)$ consists of boundary edges, so $f^{-1}(H_2)$ is obtained from $f^{-1}(H_1)$ by deleting boundary edges.

Suppose that H_2 is obtained from H_1 by deleting an isolated boundary vertex x . Now $f^{-1}(x)$ may contain boundary vertices of $f^{-1}(H_2)$ as well as edges which

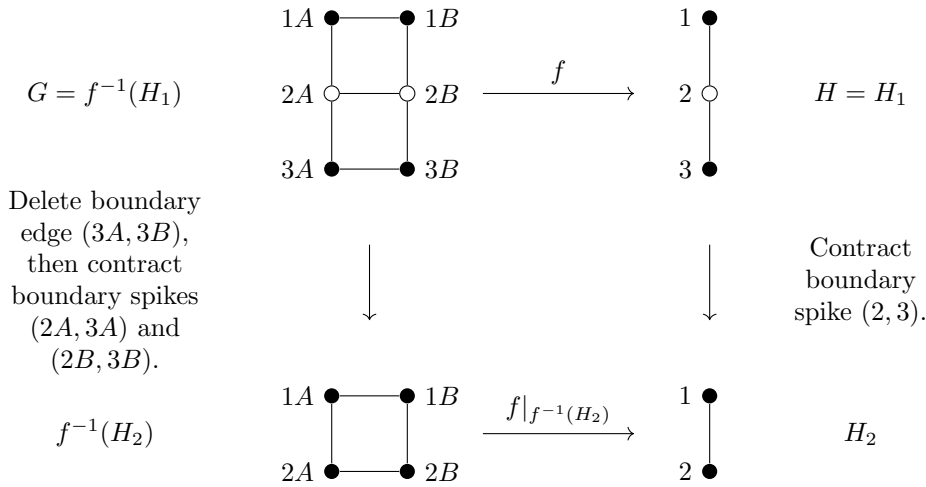


FIG. 6.1. One case in the proof of Lemma 6.3. Here $H_1 = H$, and H_2 is obtained from H_1 by contracting a boundary spike.

are collapsed by the map f into the vertex x . Every edge in $f^{-1}(x)$ is a boundary edge since its endpoints are also in $f^{-1}(x)$ (Definition 4.1(4)) and because f maps interior vertices to interior vertices (Definition 4.1 (2)). Thus we can obtain $f^{-1}(H_2)$ by deleting the boundary edges in $f^{-1}(x)$, and then deleting the vertices in $f^{-1}(x)$, which are now isolated boundary vertices.

Suppose that H_2 is obtained from H_1 by contracting a boundary spike e with boundary endpoint $x = e_+$ and interior endpoint $y = e_-$. We illustrate this case in Figure 6.1. Then we obtain $f^{-1}(H_2)$ from $f^{-1}(H_1)$ in several steps. First, any edges in $f^{-1}(x)$ are boundary edges, and we can delete them. Second, if there are any isolated boundary vertices in $f^{-1}(x)$, we delete them. At this point, any vertex left in $f^{-1}(x)$ is attached to an edge in $f^{-1}(e)$ and no other edges.

Now consider each edge $e' \in f^{-1}(e)$. Note $e'_+ \in f^{-1}(x)$ is a boundary vertex. If e'_- is also a boundary vertex, then we delete e' as a boundary edge and then delete e'_+ as an isolated boundary vertex. If e'_- is an interior vertex, then we can contract e' as a spike, which will automatically remove $e'_+ \in f^{-1}(x)$. These steps will allow us to obtain $f^{-1}(H_2)$ from $f^{-1}(H_1)$. Indeed, all vertices and edges in $f^{-1}(x)$ have been removed and all edges in $f^{-1}(e)$ have been removed. Moreover, all vertices in $f^{-1}(y)$ are now boundary vertices; indeed if $y' \in f^{-1}(y)$ was interior, then because $\mathcal{E}(y') \cap f^{-1}(E(H_1)) \rightarrow \mathcal{E}(y)$ is bijective, y' must have been the endpoint of some e' which mapped to e . This e' was contracted as a boundary spike, which means y' has been changed to a boundary vertex.

These are the only possibilities; thus, by a sequence of layer-stripping operations, we can obtain $f^{-1}(H_2)$ from $f^{-1}(H_1)$. \square

The previous lemma immediately implies the following.

LEMMA 6.4. *Let G and H be finite ∂ -graphs. If H is layerable and there is an unramified morphism $f: G \rightarrow H$, then G is also layerable.*

Before developing our basic ideas further, we must clear up some technical issues pertaining to pulling back layer-stripping operations. Lemma 6.3 shows that each layer-stripping operation pulls back to some sequence of layer-stripping operations,

but this sequence is not unique since the transformation $f^{-1}(H_1) \mapsto f^{-1}(H_2)$ could be broken up into layer-stripping operations in multiple ways. However, with a little cleverness, we can find a type of elementary operation that pulls back uniquely to another operation of the same type.

DEFINITION 6.5. *We define a three-step layer-stripping operation to be a sequence of boundary edge deletion, isolated boundary vertex deletion, and boundary spike contraction (in that order). Multiple edges or vertices are allowed to be removed at each step; however, if multiple boundary spikes are contracted, then they are not allowed to share any endpoints with each other. We allow any one of the three steps to be trivial, and thus a simple layer-stripping operation can be considered a three-step layer-stripping operation. We allow infinitely many vertices or edges to be removed if the ∂ -graph is infinite.*

The following lemma applies even to infinite ∂ -graphs. For an example, see Figure 6.2.

LEMMA 6.6. *Suppose $f: G \rightarrow H$ is an unramified ∂ -graph morphism, and suppose $H_2 \subseteq H_1 \subseteq H$ are sub- ∂ -graphs. If H_2 is obtained from H_1 by a three-step layer-stripping operation, then $f^{-1}(H_2)$ is obtained from $f^{-1}(H_1)$ by a three-step layer-stripping operation.*

Proof. The steps are the same as in Lemma 6.3 but we must order the operations in a nonobvious way:

- Delete all boundary edges in $f^{-1}(H_1)$ that map to the deleted boundary edges in H_1 .
- Delete all boundary edges in $f^{-1}(H_1)$ that map to the deleted isolated boundary vertices in H_1 .
- Delete all boundary edges in $f^{-1}(H_1)$ that map to the boundary endpoints of spikes in H_1 .
- Delete all boundary edges in $f^{-1}(H_1)$ that map to the contracted boundary spikes in H_1 .
- Delete all isolated boundary vertices in $f^{-1}(H_1)$ that map to the deleted isolated boundary vertices in H_1 .
- Delete all isolated boundary vertices in $f^{-1}(H_1)$ that map to the boundary endpoints of contracted spikes in H_1 .
- Contract all boundary spikes in $f^{-1}(H_1)$ that map to the contracted boundary spikes in H_1 .

We leave the reader to verify that this works by adapting the casework in Lemma 6.3. □

We define a three-step layer-stripping filtration of a ∂ -graph G as a sequence of subgraphs $G = G_0 \supseteq G_1 \supseteq \dots$ such that $\bigcap G_j = \emptyset$ and G_{j+1} is obtained from G_j by a three-step layer-stripping operation. The last lemma is convenient because it implies as follows.

LEMMA 6.7. *Suppose that $f: G \rightarrow H$ is an unramified ∂ -graph morphism. If $H_0 \supseteq H_1 \supseteq \dots$ is a three-step layer-stripping filtration of H , then $f^{-1}(H_0) \supseteq f^{-1}(H_1) \supseteq \dots$ is a three-step layer-stripping filtration of G . Therefore, one can define a contravariant functor $\partial\text{-graph}_{\text{unram}} \rightarrow \mathbf{Set}$ by mapping G to the set of three-step layer-stripping filtrations of G .*

Remark 6.8. As in Observation 5.5, if G' is obtained from G by a sequence of layer-stripping operations, then G' is a sub- ∂ -graph of G . Moreover, Lemmas 5.7 and 5.8 generalize to layer-stripping operations, even with infinitely many edges.

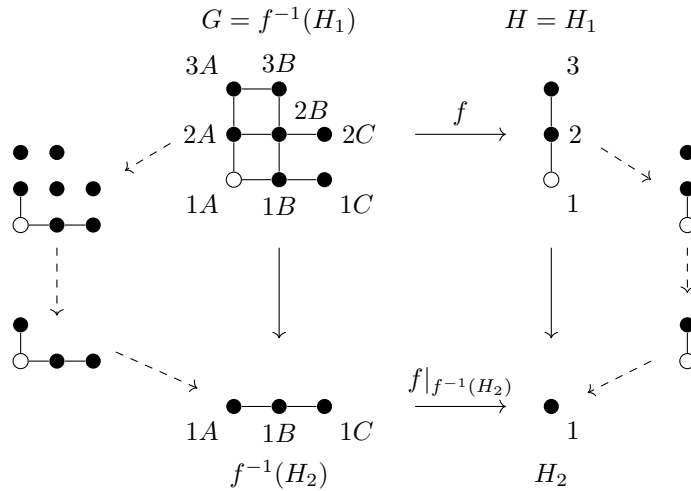


FIG. 6.2. An example of Lemma 6.6.

Remark 6.9. Lemma 6.3 fails for harmonic morphisms in general: If x is the boundary endpoint of a boundary spike e , then a vertex in $f^{-1}(x)$ might not be a boundary spike. It could have degree > 1 since there can be multiple preimages of e attached to it. This problem is illustrated by the ∂ -graph morphism in Figure 4.4.

6.2. The flower functor. Layer-stripping can be viewed as a loose discrete analogue of a deformation retraction; it is a reduction to a smaller space that leaves our algebraic invariant Υ unchanged. This inspires the following analogue of homotopy equivalence.

DEFINITION 6.10. We say two finite ∂ -graphs G and G' are layerably equivalent if there is a finite sequence of ∂ -graphs $G = G_0, G_1, \dots, G_n = G'$ such that for each j , either (1) G_j is obtained from G_{j+1} by a layer-stripping operation or (2) G_{j+1} is obtained from G_j by a layer-stripping operation. As usual, we apply the same terminology to R -networks.

If two finite R^\times -networks (G, L) and (G', L') are layerably equivalent, then by Lemma 5.7 we have

$$\Upsilon(G, L) \oplus R^n \cong \Upsilon(G', L') \oplus R^{n'}$$

for some n and $n' \in \mathbb{N}_0$. Thus, the torsion submodules of $\Upsilon(G, L)$ and $\Upsilon(G', L')$ are isomorphic. Similar reasoning applies the modules \mathcal{U} and \mathcal{U}_0 for the two networks.

Our next goal is to find a canonical representative for each equivalence class. A natural candidate is a ∂ -graph with no boundary spikes, boundary edges, or disconnected boundary vertices, which we will call a *flower*. The name “flower” was coined in 1992 by David Ingberman and James Morrow, who were studying some planar examples which looked like flowers [25], and it was first written down in [38]. Examples of flowers include the ∂ -graph in Figure 6.3, the boundary-interior bipartite ∂ -graphs in Example 2.21 and Figure 2.3, the CLF ∂ -graphs discussed in section 3 (see Figure 3.2), and the ∂ -graph on the left-hand side of Figure 4.4.

Every finite ∂ -graph is layerably equivalent to a flower. Indeed, we can keep removing boundary spikes, boundary edges, and isolated boundary vertices until there are no more left. The result is a sequence of layer-stripping operations that transforms

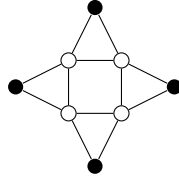


FIG. 6.3. Example of a flower.

G into a flower $\mathfrak{F}(G)$. We claim that in fact this flower is unique and the map $G \mapsto \mathfrak{F}(G)$ is a functor on $\partial\text{-graph}_{\text{unrm}}^0$.

THEOREM 6.11.

- (1) Every finite ∂ -graph G can be layer-stripped to a unique flower $\mathfrak{F}(G)$.
- (2) There is exactly one flower in each layerable equivalence class.
- (3) If $f: G \rightarrow H$ is a UHM, then $\mathfrak{F}(G) \subseteq f^{-1}(\mathfrak{F}(H))$.
- (4) \mathfrak{F} is a functor $\partial\text{-graph}_{\text{unrm}}^0 \rightarrow \partial\text{-graph}_{\text{unrm}}^0$, and the inclusion $\mathfrak{F}(G) \rightarrow G$ is a natural transformation $\mathfrak{F} \rightarrow \text{id}$.

Proof. To prove the uniqueness claim of (1), suppose that we have two flowers H and $H' \subseteq G$, and let

$$G = H_0 \supseteq H_1 \supseteq \dots \supseteq H_n = H, \quad G = H'_0 \supseteq \dots \supseteq H'_m = H'$$

be the corresponding three-step layer-stripping filtrations. Applying Lemma 6.7 to the inclusion map $H \rightarrow G$, we see that

$$H = H \cap H'_0 \supseteq \dots \supseteq H \cap H'_m = H \cap H'$$

is another three-step layer-stripping filtration. Since H is a flower, one cannot perform any nontrivial layer-stripping operations on it, so the filtration must be trivial, so that $H = H \cap H'$. By symmetry, $H' = H \cap H' = H$.

To prove (2), suppose G' is obtained from G by a layer-stripping operation. We just showed $\mathfrak{F}(G)$ is independent of the sequence of layer-stripping operations, so that $\mathfrak{F}(G)$ is obtained by performing the layer-stripping operation $G \mapsto G'$, then reducing G' to a flower. Hence, $\mathfrak{F}(G) = \mathfrak{F}(G')$. In general, if we have a sequence $G = G_0, G_1, \dots, G_n = G'$ witnessing layerable equivalence, then $\mathfrak{F}(G_j)$ and $\mathfrak{F}(G_{j+1})$ are equal (based on the identification of G_j as a subgraph of G_{j+1} or vice versa) and hence $\mathfrak{F}(G) = \mathfrak{F}(G')$.

Claim (3) follows from Lemma 6.7, and (4) follows from (3). □

Remark 6.12. G is layerable if and only if $\mathfrak{F}(G) = \emptyset$.

As a consequence, the study of torsion for finite R^\times -networks can be reduced in a functorial manner to the study of torsion for flowers.

COROLLARY 6.13.

- (1) There is a flower functor $\mathfrak{F}: R^\times\text{-net}_{\text{unrm}}^0 \rightarrow R^\times\text{-net}_{\text{unrm}}^0$.
- (2) $\Upsilon(G, L) = \Upsilon(\mathfrak{F}(G, L)) \oplus R^n$ for some n (depending on (G, L)).
- (3) If F is any functor on R -modules such that $F(M \oplus N) \cong F(M) \oplus F(N)$ and $F(R) = 0$, then the inclusion natural transformation $\mathfrak{F} \rightarrow \text{id}$ on $R^\times\text{-net}_{\text{unrm}}^0$ induces a natural isomorphism $F \circ \Upsilon \circ \mathfrak{F} \rightarrow F \circ \Upsilon$.

In particular, if R is a PID, then we can take F to be the functor which maps an R -module to its torsion submodule, and thus, the torsion part of $\Upsilon \circ \mathfrak{F}(G, L)$ is naturally isomorphic to the torsion part of $\Upsilon(G, L)$.

Proof. Statement (1) follows directly from Theorem 6.11. Statement (2) follows from Lemma 5.7 because $\mathfrak{S}(G, L)$ is obtained from (G, L) by a sequence of layer-stripping operations. Statement (3) follows from (2). \square

7. Complete reducibility. Complete reducibility is a generalization of layer-ability which allows us to applying layer-stripping operations as well as split apart ∂ -graphs that are glued together at one boundary vertex (Definition 7.7). In Theorem 7.13 we will give an algebraic characterization of complete reducibility, which is analogous to Theorem 5.16, except that it uses Laplacians with $d = 0$ rather than generalized Laplacians. In preparation for the proof, we first define a reduced version of Υ suited to Laplacians with $d = 0$. At the end of the section, as an application of Theorem 7.13, we show that boundary-interior bipartite ∂ -graphs of a certain type are not completely reducible.

7.1. The reduced module $\tilde{\Upsilon}$. Recall that a generalized Laplacian L is given by $w : E \rightarrow R$ and $d : V \rightarrow R$. If $d = 0$, we will call L a *Laplacian* or *weighted Laplacian*. A network given by a weighted Laplacian will be called a *normalized R -network*. For weighted Laplacians, constant functions are always harmonic, and dually every element of $L(RV)$ has coordinates which sum to zero. Therefore, it will be convenient to define a reduced version of Υ .

Let L be a weighted Laplacian. Let $\epsilon : RV \rightarrow R$ be the map which sums the coordinates, given by $x \mapsto 1$ for every $x \in V$. Then $\text{im } L \subseteq \ker \epsilon$ because

$$\epsilon(Lx) = \sum_{e \in \mathcal{E}(x)} w(e)(\epsilon x - \epsilon e_-) = 0.$$

Therefore, we can define

$$\tilde{\Upsilon}(G, L) = \ker \epsilon / L(RV^\circ).$$

We remark that since $\ker \epsilon \subseteq RV$, we can regard $\tilde{\Upsilon}(G, L)$ as a submodule of $\Upsilon(G, L)$. Moreover, for each vertex $x \in V(G)$, there is an internal direct sum

$$\Upsilon(G, L) = \tilde{\Upsilon}(G, L) \oplus Rx.$$

We also define

$$\tilde{\mathcal{U}}(G, L, M) = \mathcal{U}(G, L, M) / (\text{constant functions}).$$

Most of the results for Υ and their proofs adapt in a straightforward way to $\tilde{\Upsilon}$. We list the ones we will need for the algebraic characterization. Using similar reasoning as in Lemma 2.9, one can show as follows.

LEMMA 7.1. *If L is a weighted Laplacian on G , then there is a natural R -module isomorphism*

$$\tilde{\mathcal{U}}(G, L, M) \cong \text{Hom}_R(\tilde{\Upsilon}(G, L), M).$$

Moreover, by similar reasoning as in section 2.3, we have the following.

LEMMA 7.2. *If (G, L) is a nondegenerate normalized network, then a free resolution of $\tilde{\Upsilon}(G, L)$ is given by*

$$\dots \rightarrow 0 \rightarrow RV^\circ \xrightarrow{L} \ker \epsilon \rightarrow \tilde{\Upsilon}(G, L) \rightarrow 0.$$

Moreover,

$$\text{Tor}_1^R(\tilde{\Upsilon}(G, L), M) \cong \mathcal{U}_0(G, L, M)$$

and $\text{Tor}_j^R(\tilde{\Upsilon}(G, L), M) = 0$ for $j > 1$.

Lemma 5.7 carries over almost word for word.

LEMMA 7.3. *Let (G', L') be a normalized R^\times -network. If (G', L') is obtained from (G, L) by adjoining a boundary spike or boundary edge, then the induced maps $\tilde{\Upsilon}(G, L) \rightarrow \tilde{\Upsilon}(G', L')$ and $\mathcal{U}(G, L, M) \rightarrow \mathcal{U}(G', L', M)$ are isomorphisms.*

If (G', L') is obtained from (G, L) by adjoining an isolated boundary vertex x and G is nonempty, then we have

$$\tilde{\Upsilon}(G', L') \cong \tilde{\Upsilon}(G, L) \oplus R$$

and

$$\tilde{\mathcal{U}}(G', L', M) \cong \tilde{\mathcal{U}}(G, L, M) \times M.$$

7.2. Completely reducible ∂ -graphs.

DEFINITION 7.4. *Given ∂ -graphs G_1 and G_2 and specified vertices $x_i \in \partial V_i$ for $i = 1, 2$, the boundary wedge-sum*

$$G_1 \vee G_2 = G_1 \vee_{x_1, x_2} G_2$$

is obtained by identifying x_1 with x_2 in the disjoint union $G_1 \sqcup G_2$. Note that G_1 and G_2 are sub- ∂ -graphs of $G_1 \vee G_2$. We apply the same terminology to R -networks as to ∂ -graphs.

Just as with layerable extensions, the behavior of $\tilde{\Upsilon}$ under boundary wedge-sums and disjoint unions is easy to characterize. A similar result for the critical group of graphs without boundary appears in [34, Remark, p. 280].

LEMMA 7.5. *Let (G, L) be a normalized R -network. If $(G, L) = (G_1, L_1) \vee (G_2, L_2)$, then*

$$\begin{aligned} \tilde{\Upsilon}(G, L) &\cong \tilde{\Upsilon}(G_1, L_1) \oplus \tilde{\Upsilon}(G_2, L_2), \\ \tilde{\mathcal{U}}(G, L, M) &\cong \tilde{\mathcal{U}}(G_1, L_1, M) \times \tilde{\mathcal{U}}(G_2, L_2, M), \\ \mathcal{U}_0(G, L, M) &\cong \mathcal{U}_0(G_1, L_1, M) \oplus \mathcal{U}_0(G_2, L_2, M). \end{aligned}$$

If $(G, L) = (G_1, L_1) \sqcup (G_2, L_2)$, then

$$\begin{aligned} \tilde{\Upsilon}(G, L) &\cong \tilde{\Upsilon}(G_1, L_1) \oplus \tilde{\Upsilon}(G_2, L_2) \oplus R, \\ \tilde{\mathcal{U}}(G, L, M) &\cong \tilde{\mathcal{U}}(G_1, L_1, M) \times \tilde{\mathcal{U}}(G_2, L_2, M) \times M, \\ \mathcal{U}_0(G, L, M) &\cong \mathcal{U}_0(G_1, L_1, M) \oplus \mathcal{U}_0(G_2, L_2, M). \end{aligned}$$

Proof. In the case of a boundary wedge-sum, we have an internal direct sum

$$\ker \epsilon = \ker \epsilon_1 \oplus \ker \epsilon_2.$$

Moreover,

$$L(RV^\circ) = L_1(RV_1^\circ) + L_2(RV_2^\circ), \quad L_1(RV_1^\circ) \subseteq \ker \epsilon_1, \quad L_2(RV_2^\circ) \subseteq \ker \epsilon_2.$$

Therefore, taking quotients yields

$$\tilde{\Upsilon}(G, L) \cong \tilde{\Upsilon}(G_1, L_1) \oplus \tilde{\Upsilon}(G_2, L_2).$$

The claim for $\tilde{\mathcal{U}}$ follows by applying $\text{Hom}(-, M)$.

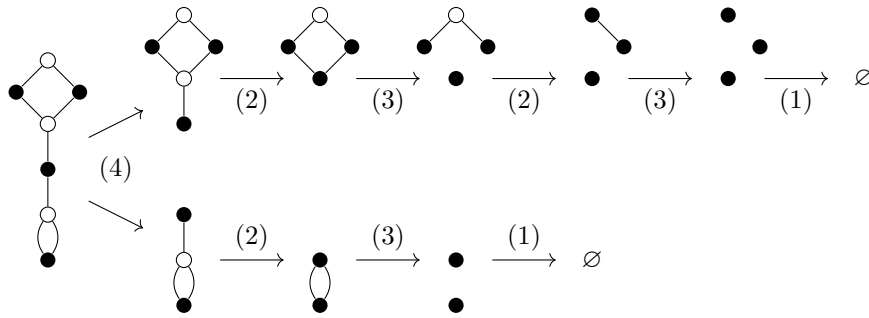


FIG. 7.1. A completely reducible ∂ -graph. The boundary vertices are black and interior vertices are white. The operations are (1) isolated boundary vertex deletion, (2) boundary spike contraction, (3) boundary edge deletion, and (4) splitting a boundary wedge-sum.

To prove the claim for \mathcal{U}_0 , consider the map

$$\Phi : \mathcal{U}_0(G_1, L_1, M) \oplus \mathcal{U}_0(G_2, L_2, M) \rightarrow \mathcal{U}_0(G, L, M)$$

defined by the inclusion maps $(G_j, L_j) \rightarrow (G, L)$. Let x be the common boundary vertex of G_1 and G_2 . To see that Φ is injective, suppose $\Phi(u_1 \oplus u_2) = 0$. Then u_1 and u_2 vanish at x by definition of $\mathcal{U}_0(G_j, L_j, M)$, and they vanish on the rest of G_1 and G_2 , respectively, because G_1 and G_2 intersect only at x . To show surjectivity of Φ , let $u \in \mathcal{U}_0(G, L, M)$, and let $u_j = u|_{V_j}$. Clearly, $u_j = 0$ on ∂V_j since $\partial V_j \subseteq \partial V$. Moreover, $L_j u_j(y) = Lu(y) = 0$ for every $y \in V_j \setminus \{x\}$. Because $\text{im}(L_j \otimes \text{id}_M) \subseteq \ker(\epsilon_j \otimes \text{id}_M)$, this implies that $L_j u(x) = 0$ also. Thus, $u_j \in \mathcal{U}_0(G_j, L_j, M)$ and hence $u = \Phi(u_1 \oplus u_2) \in \text{im } \Phi$. So Φ is an isomorphism as desired.

In the case of a disjoint union, the claim for $\tilde{\Upsilon}$ is proved in a similar way after noting that

$$\ker \epsilon \cong \ker \epsilon_1 \oplus \ker \epsilon_2 \oplus R.$$

The claim for \mathcal{U} follows by applying $\text{Hom}(-, M)$. The argument for \mathcal{U}_0 is similar to the boundary wedge-sum case but easier. \square

DEFINITION 7.6. Completely reducible finite ∂ -graphs are defined to be the smallest class \mathcal{C} of finite ∂ -graphs that contains the empty graph and is closed under layerable extensions, disjoint unions, and boundary wedge-sums. More informally, a graph G is completely reducible if it can be reduced to nothing by layer-stripping and splitting apart boundary wedge-sums and disjoint unions. See Figure 7.1 for an example.

DEFINITION 7.7. A finite ∂ -graph is irreducible if it has no boundary spikes, boundary edges, or isolated boundary vertices, and it is not a boundary wedge-sum or disjoint union. Note that every irreducible ∂ -graph is a flower.

The following is an analogue of Proposition 5.12.

PROPOSITION 7.8. Let G be a finite nonempty completely reducible ∂ -graph. If (G, L) is a normalized R^\times -network, then (G, L) is nondegenerate and $\tilde{\Upsilon}(G, L)$ is a free R -module of rank $|\partial V(G)| - 1$.

Proof. Let \mathcal{C} be the class of ∂ -graphs for which the claims hold, together with the empty ∂ -graph. Lemmas 7.3 and 7.5 imply that \mathcal{C} is closed under layerable extensions, disjoint unions, and boundary wedge-sums. \square

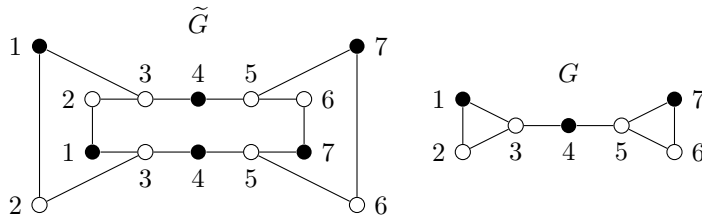


FIG. 7.2. A covering map $f: \tilde{G} \rightarrow G$ such that G decomposes as a boundary wedge-sum and \tilde{G} does not. In fact, G is completely reducible and \tilde{G} is irreducible.

Unlike layer-stripping operations, the operation of splitting apart a boundary wedge-sum does *not* pull back through unramified ∂ -graph morphisms. The problem is illustrated in Figure 7.2. However, we do have the following.

Observation 7.9. Suppose that G is a sub- ∂ -graph of H and that H decomposes as a boundary wedge-sum or disjoint union of H_1 and H_2 . Then G decomposes as a boundary wedge-sum or disjoint union of $G \cap H_1$ and $G \cap H_2$. Together with Lemma 6.3, this implies that a sub- ∂ -graph of a completely reducible ∂ -graph is also completely reducible.

7.3. Algebraic characterization. We shall prove an algebraic characterization of complete reducibility in the same way as we did for layerability (Theorem 5.16). As in the case of layerability, the first and most significant step concerns degenerate networks over fields. Although Lemma 7.10 below is completely analogous to Lemma 5.15, neither one follows from the other. It is more difficult to construct degenerate networks now that we cannot choose the diagonal entries of L arbitrarily, but the condition non-complete-reducibility is also stronger than nonlayerability.

LEMMA 7.10. *Let G be a finite ∂ -graph and let F be an infinite field. Then G is completely reducible if and only if every normalized F^\times -network on G is non-degenerate.*

Proof. The implication \implies follows from Proposition 7.8.

Let G' be a minimal sub- ∂ -graph of G which is not completely reducible. Note that G' must be irreducible. As in the proof of Lemma 5.15, it suffices to construct degenerate edge weights on G' .

Our strategy is to *first* choose a potential function u with $u|_{\partial V} = 0$, and *then* choose an edge-weight function w that will make $Lu \equiv 0$. Let $S \subseteq E(G)$ be the union of all cycles, i.e., S contains every edge that is part of any cycle. Note that every edge in S must have endpoints in distinct components of $G \setminus S$. Define u to be zero on every component of $G \setminus S$ that contains a boundary vertex of G , and assign u a different nonzero value on each component of $G \setminus S$ that does not contain any boundary vertices.

We need to guarantee that u is not identically zero. But in fact, we claim that u is nonzero at every interior vertex. To prove this, it suffices to show that every edge e with endpoints $x \in \partial V$ and $y \in V^\circ$ must be in S , that is, such an edge e must be contained in some cycle. By hypothesis, our edge e is not a boundary spike. Thus, there is some other edge $e' \neq e$ incident to x . Let z be the other endpoint e' . Since G is not a boundary wedge-sum, deleting x leaves G connected. Thus, there is a path $P = \{e_1, \dots, e_k\}$ from y to z which avoids x . Then $P \cup \{e, e'\}$ is a cycle containing e . Consequently, u is nonzero at every interior vertex.

Now we choose the edge weights. Choose oriented cycles C_1, \dots, C_k such that $S = \bigcup_{j=1}^k (C_j \cup \bar{C}_j)$. If $e \in C_j$, then $e \in S$ and hence e_+ and e_- are in distinct components of $G \setminus S$, so $du(e) = u(e_+) - u(e_-) \neq 0$. For each j , define

$$w_j(e) = w_j(\bar{e}) = \begin{cases} 1/du(e) & \text{for } e \in C_j, \\ 0 & \text{for } e \notin C_j \cup \bar{C}_j. \end{cases}$$

Then $w_j(e)du(e)$ is 1 on C_j and -1 on \bar{C}_j and vanishes elsewhere. Therefore, if we let L_j be the Laplacian associated to the edge-weight function w_j , then we have $L_j u = 0$. For each $e \in S$, there is a weight function w_j with $w_j(e) \neq 0$. Since F is infinite and the graph is finite, we may choose $\alpha_j \in F$ such that $\sum_{j=1}^k \alpha_j w_j(e) \neq 0$ for all $e \in S$ simultaneously.

Set $w = 1_{E \setminus S} + \sum_{j=1}^k \alpha_j w_j$ and let L be the associated Laplacian. Then $w(e) \neq 0$ for each e . Because u is constant on each component of $G \setminus S$, we know that $u(e_+) - u(e_-) = 0$ for each $e \in E \setminus S$. Thus, these edges do not contribute to Lu , and so

$$Lu = \sum_{j=1}^k \alpha_j L_j u = 0.$$

Thus, (G, L) is the desired degenerate F^\times -network because $0 \neq u \in \mathcal{U}_0(G, L, F)$. \square

We proved equivalent algebraic characterizations for layerability by assigning indeterminates to the edges (see Proposition 2.18). The analogue for normalized networks is as follows.

DEFINITION 7.11. *Let G be a ∂ -graph and let F be a field. Then $\tilde{R} = \tilde{R}(G, F) = F[t_e^{\pm 1} : e \in E]$ will denote the Laurent polynomial algebra over F with generators indexed by the edges of G . Let $\tilde{L} = \tilde{L}(G, F)$ denote the weighted Laplacian over \tilde{R} given by $\tilde{w}(e) = t_e$.*

PROPOSITION 7.12. *Let G be a finite ∂ -graph such that each component contains at least one boundary vertex, and let F be a field. Then (G, \tilde{L}) is nondegenerate. Moreover, $\tilde{Y}(\tilde{G}, \tilde{L})$ is a flat \tilde{R} -module if and only if every normalized F^\times -network on G is nondegenerate.*

Proof. To prove that (G, \tilde{L}) is nondegenerate, it suffices to prove that each connected component of (G, \tilde{L}) is nondegenerate. Therefore, we may assume without loss of generality that G is connected and has at least one boundary vertex.

Recall that (G, \tilde{L}) is nondegenerate if and only if $\tilde{L}: \tilde{R}V^\circ \rightarrow \tilde{R}V$ is injective (see the proof of Proposition 2.11). For our given boundary vertex x , let $\tilde{L}_x: \tilde{R}(V \setminus x) \rightarrow \tilde{R}(V \setminus x)$ be the Laplacian \tilde{L} , with the domain restricted to chains in $\tilde{R}(V \setminus x) \subseteq \tilde{R}V$, and with the output truncated by applying the canonical projection $\tilde{R}V \rightarrow \tilde{R}(V \setminus x)$. Then injectivity of \tilde{L}_x will imply injectivity of $\tilde{L}: \tilde{R}V^\circ \rightarrow \tilde{R}V$ since $V \setminus x \supseteq V^\circ$ and $V \setminus x \subseteq \partial V$. By the weighted matrix-tree theorem (see [22, Theorem 1] and [28, Theorem 4.2]), we have

$$\det \tilde{L}_x = \sum_{T \in \text{Span}(G)} \prod_{e \in T} t_e \neq 0,$$

where $\text{Span}(G)$ denotes the set of spanning trees of G . Since we assumed G is connected, $\det \tilde{L}_x$ is a nonzero polynomial in $(t_e)_{e \in E}$ and hence is a nonzero element of

the Laurent polynomial algebra \tilde{R} . Since \tilde{R} is an integral domain, it follows that \tilde{L}_x is injective. This completes the proof that (G, \tilde{L}) is nondegenerate.

The rest of the proof is exactly the same as for Proposition 2.18. □

The following theorem is proved the same way as Theorem 5.16.

THEOREM 7.13. *Let G be a finite ∂ -graph such that every component has at least one boundary vertex. The following are equivalent:*

- (1) G is completely reducible.
- (2) For every ring R , every normalized R^\times -network on G is nondegenerate.
- (3) For every ring R , for every nondegenerate normalized R^\times -network (G, L) on the ∂ -graph G , $\tilde{Y}(G, L)$ is a free R -module.
- (4) There exists an infinite field F such that $\tilde{Y}(G, \tilde{L}(G, F))$ is a flat $\tilde{L}(G, F)$ -module.
- (5) There exists an infinite field F such that every normalized F^\times -network on G is nondegenerate.

7.4. Boundary-interior bipartite ∂ -graphs. The correspondence between algebraic and ∂ -graph-theoretic conditions in Theorem 7.13 is illustrated by the following proposition about bipartite graphs. We present both an algebraic proof and an inductive ∂ -graph-theoretic proof for comparison. We say a ∂ -graph is *boundary-interior bipartite* if every edge has one interior endpoint and one boundary endpoint (similar to Example 2.21).

PROPOSITION 7.14. *Suppose that G is a nonempty finite boundary-interior bipartite ∂ -graph, $|V^\circ| \geq |\partial V|$, and every interior vertex has degree ≥ 2 . Then G is not completely reducible.*

Algebraic proof. Let F be any field other than the field F_2 with two elements. We will construct a degenerate F^\times -network on G to conclude that G is not completely reducible by Theorem 7.13. Since each interior vertex has at least two edges incident to it and each edge is incident only to one interior vertex, we can choose $w : E \rightarrow F^\times$ such that $\sum_{e \in \mathcal{E}(x)} w(e) = 0$ for each $x \in V^\circ$. If $u \in 0^{\partial V} \times F^{V^\circ} \subseteq F^V$, then $Lu|_{V^\circ} = 0$ since

$$Lu(x) = \sum_{e:e_+=x} w(e)(u(x) - u(e_-)) = \sum_{e:e_+=x} w(e)u(x) = 0 \text{ for all } x \in V^\circ.$$

Combining this with the fact that $\text{im } L \subseteq \ker \epsilon$ yields

$$L(0^{\partial V} \times F^{V^\circ}) \subseteq \left\{ \phi \in F^{\partial V} : \sum_{x \in \partial V} \phi(x) = 0 \right\} \times 0^{V^\circ}.$$

Therefore, $\dim L(0^{\partial V} \times F^{V^\circ}) \leq |\partial V| - 1 < |V^\circ|$, since we assumed $|\partial V| \leq |V^\circ|$. Therefore, by the rank-nullity theorem,

$$\mathcal{U}_0(G, L, F) = \ker(L : F^{V^\circ} \rightarrow F^V) \neq 0. \quad \square$$

∂ -graph-theoretic proof. By Observation 7.9, it suffices to show that G has a sub- ∂ -graph which is not completely reducible. We proceed by induction on the number of vertices.

Since G is nonempty and $|V^\circ| \geq |\partial V|$, G must have at least one interior vertex x . By assumption x has some neighbor y , and y must be a boundary vertex since the

∂ -graph is boundary-interior bipartite. Therefore, G must have at least one boundary vertex and one interior vertex. If G has only two vertices, it must have exactly one interior vertex and one boundary vertex with at least two parallel edges between them. Then G is irreducible.

Suppose G has $n > 2$ vertices and divide into cases:

- If G is irreducible, we are done.
- Suppose G has a boundary spike (x, y) with $x \in \partial V$ and $y \in V^\circ$. Let G' be the ∂ -graph obtained by contracting the space. Then y is a boundary vertex in G' and by assumption all its neighbors are boundary vertices in G . Thus, we can delete the boundary edges incident to y and then delete the now isolated boundary vertex y to obtain a harmonic sub- ∂ -graph G' which satisfies the original hypotheses. The new ∂ -graph G' is nonempty because $|V(G)| > 2$. By inductive hypothesis, G' is not completely reducible.
- If G can be split apart as a boundary wedge-sum or a disjoint union, then each piece is boundary-interior bipartite with interior vertices that have degree ≥ 2 . Moreover, one of the two subgraphs must have $|\partial V| \leq |V^\circ|$ and hence is not completely reducible by inductive hypothesis.
- G has no boundary edges by assumption. Moreover, if G has an isolated boundary vertex, that can be treated as a special case of disjoint unions. \square

8. Network duality.

8.1. Dual circular planar networks, harmonic conjugates. As shown in [14, Theorem 2], dual planar graphs have isomorphic critical groups. In this section, we generalize this result to circular planar normalized R^\times -networks. The theory here adapts the ideas of duality and discrete complex analysis found in [36, section 2], [16, section 10], [37]. In this section, all the networks will be normalized (that is, they will satisfy $d = 0$).

DEFINITION 8.1. *A circular planar ∂ -graph G is a (finite) ∂ -graph embedded in the closed unit disk \bar{D} in the complex plane such that $V \cap \partial D = \partial V$. The faces of G are the components of $D \setminus G$.*

DEFINITION 8.2. *A connected circular planar ∂ -graph has a circular planar dual G^\dagger defined as follows: The vertices of G^\dagger correspond to the faces of G ; each vertex of G^\dagger is placed in the interior of the corresponding face of G . The edges of G^\dagger correspond to the edges of G . For each oriented edge e of G , there is a dual edge e^\dagger where e^\dagger_+ corresponds to the face on the right of e and e^\dagger_- corresponds to the face on the left of e . A vertex of G^\dagger is considered a boundary vertex if the corresponding face has a side along ∂D . For further explanation and illustration, see [37, Definition 5.1 and Figure 1].*

Remark 8.3. The planar dual is constructed in a similar fashion for a connected planar network without boundary, and the process is well explained in [36, section 2.1 and Figure 2]. To incorporate planar networks without boundary into the circular planar framework, we may designate an arbitrary vertex to be a boundary vertex and embed the ∂ -graph into the disk.

DEFINITION 8.4. *If (G, L) is a circular planar normalized R^\times -network, then the dual network (G^\dagger, L^\dagger) is the network on G^\dagger with $w(e^\dagger) = w(e)^{-1}$. We make the same definition for planar normalized R^\times -networks without boundary.*

THEOREM 8.5. *If (G, L) is a connected circular planar normalized R^\times -network, then*

$$\tilde{\Upsilon}(G^\dagger, L^\dagger) \cong \tilde{\Upsilon}(G, L).$$

The same holds for planar normalized R^\times -networks without boundary.

Theorem 8.5 generalizes [14, Theorem 2] to R^\times -networks. Our proof combines ideas from [8, sections 26–29] and [18, section 7].

Proof. Consider the circular planar case; the proof for planar networks without boundary is the same. The result follows from reformulating $\tilde{\Upsilon}$ in terms of oriented edges rather than vertices. Recall that $C_1(G)$ is the free R -module on the oriented edges E modulo the relations $\bar{e} = -e$ (see section 1.3). Then $\ker \epsilon$ can be identified with the quotient of $C_1(G)$ by the submodule generated by oriented cycles. The cycle submodule is in fact generated by the oriented boundaries of interior faces. Moreover, $L(RV^\circ)$ corresponds to the submodule of $C_1(G)$ generated by $\sum_{e \in \mathcal{E}(x)} w(e)e$. The edges bounding an interior face of G correspond to the edges incident to an interior vertex in G^\dagger . Therefore,

$$\begin{aligned} \tilde{\Upsilon}(G, L) &\cong \frac{C_1(G)}{(\sum_{e^\dagger \in \mathcal{E}(x)} e : x \in V^\circ(G^\dagger)) + (\sum_{e \in \mathcal{E}(x)} w(e)e : x \in V^\circ(G))}, \\ \tilde{\Upsilon}(G^\dagger, L^\dagger) &\cong \frac{C_1(G^\dagger)}{(\sum_{e^\dagger \in \mathcal{E}(x)} w(e^\dagger)e^\dagger : x \in V^\circ(G^\dagger)) + (\sum_{e \in \mathcal{E}(x)} e^\dagger : x \in V^\circ(G))}. \end{aligned}$$

Since $w(e^\dagger) = w(e)^{-1}$, we can define an isomorphism $\tilde{\Upsilon}(G, L) \rightarrow \tilde{\Upsilon}(G^\dagger, L^\dagger)$ by $e \mapsto w(e)^{-1}e^\dagger$. □

Application of $\text{Hom}(-, M)$ yields the following discrete-complex-analytic interpretation of network duality, as in [36, section 2], [37, section 7].

PROPOSITION 8.6. *Let (G, L) be a circular planar normalized R^\times -network. Modulo constant functions, for every M -valued harmonic function u on (G, L) , there is a unique harmonic conjugate v on (G^\dagger, L^\dagger) satisfying the discrete Cauchy–Riemann equation $w(e)du(e) = dv(e^\dagger)$, where $du(e) = u(e_+) - u(e_-)$ and $dv(e^\dagger) = v(e^\dagger_+) - v(e^\dagger_-)$. Moreover, a function $u : V(G) \rightarrow M$ is harmonic if and only if there exists a function v such that $w(e)du(e) = dv(e^\dagger)$. The same holds for planar normalized R^\times -networks without boundary.*

Proof. Given our interpretation of $\tilde{\Upsilon}(G, L)$ in the previous proof, a harmonic function modulo constants is equivalent to a map $\phi : E(G) \rightarrow M$ such that $\phi(e)$ sums to zero around every oriented cycle and $\sum_{e \in \mathcal{E}(x)} w(e)\phi(e) = 0$ for each interior vertex; the correspondence between u and ϕ is given by $\phi(e) = du(e)$. For every such ϕ , we can define a similar function ψ on the dual network by $\psi(e^\dagger) = w(e)\phi(e)$. This proves the existence and uniqueness of harmonic conjugates.

Next, we must prove that if u and v satisfy $w(e)du(e) = dv(e)$, then u is harmonic. But note that for each $x \in V^\circ(G)$, we have

$$Lu(e) = \sum_{e \in \mathcal{E}(x)} w(e)du(e) = \sum_{e \in \mathcal{E}(x)} dv(e^\dagger) = 0$$

because $\{e^\dagger : e \in \mathcal{E}(x)\}$ is a cycle in G^\dagger . The proof for the case without boundary is the same. □

PROPOSITION 8.7. *Let G be a connected circular planar ∂ -graph. Then G is completely reducible if and only if G^\dagger is completely reducible.*

Proof. By Theorem 7.13, G is completely reducible if and only if for every ring R , for every normalized R^\times -network (G, L) on the ∂ -graph, $\tilde{\Upsilon}(G, L)$ is a free R -module.

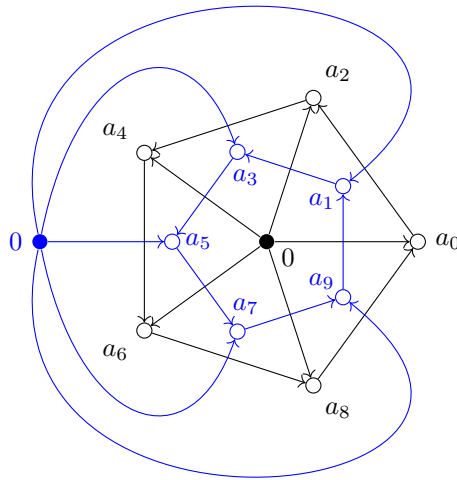


FIG. 8.1. W_5 and its (isomorphic) dual. Arrows indicate the paired dual oriented edges.

Clearly, $(G, L) \mapsto (G^\dagger, L^\dagger)$ defines a bijection between R^\times -networks on G and R^\times -networks on G^\dagger . Thus, Theorem 8.5 implies that G is completely reducible if and only if G^\dagger is completely reducible. \square

Remark 8.8. There is a direct combinatorial proof of Proposition 8.7 as well, which we will merely sketch here. It requires extending the definition of dual to circular planar ∂ -graphs which are disconnected, which is somewhat tricky and tedious since the dual is not unique; this problem is best dealt with by reformulating it using medial graphs as in [27]. One can then show that contracting a boundary spike on G corresponds to deleting a boundary edge in G^\dagger and vice versa. A decomposition of G into a boundary wedge-sum or disjoint union corresponds to a similar decomposition of G^\dagger .

8.2. Wheel graphs. Consider the wheel graph W_n embedded in the complex plane with vertices at $\{e^{2\pi ik/n}\}_{k \in \mathbb{Z}}$ and at 0. Edges connect 0 to $e^{2\pi ik/n}$ and $e^{2\pi ik/n}$ to $e^{2\pi i(k+1)/n}$ for all $k \in \mathbb{Z}$. Figure 8.1 depicts W_5 and its planar dual. Note that the dual of W_n is isomorphic to W_n . We call the vertex 0 the *hub* and the set of vertices $\{e^{2\pi ik/n}\}$ the *rim*, and we apply the same terminology to W_n^\dagger . We denote the hub vertex of W_n^\dagger by 0^\dagger .

The critical group of W_n is computed in [9] using chip-firing, induction, and the symmetry of the graph, and a connection with Lucas sequences is uncovered. We present an alternate approach, computing the critical group using harmonic continuation and planar duality.

PROPOSITION 8.9 (see [9, Theorem 9.2]). *Let W_n be the wheel graph and let $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, \dots$ be the Fibonacci numbers. Then*

$$\text{Crit}(W_n) \cong \begin{cases} \mathbb{Z}/(F_{n-1} + F_{n+1}) \times \mathbb{Z}/(F_{n-1} + F_{n+1}), & n \text{ odd,} \\ \mathbb{Z}/F_n \times \mathbb{Z}/5F_n, & n \text{ even.} \end{cases}$$

Proof. By Proposition 2.23(3), $\text{Crit}(W_n)$ is isomorphic to the \mathbb{Z} -module of \mathbb{Q}/\mathbb{Z} -valued harmonic functions which vanish at the hub vertex. This in turn is isomorphic to the module of \mathbb{Q}/\mathbb{Z} -valued harmonic functions modulo constants, that is,

$$\text{Crit}(W_n) \cong \tilde{\mathcal{U}}(W_n, L_{\text{std}}, \mathbb{Q}/\mathbb{Z}).$$

By Proposition 8.6, it suffices to compute the \mathbb{Z} -module of pairs satisfying the Cauchy–Riemann equation, that is,

$$\{(u, v) \in [(\mathbb{Q}/\mathbb{Z})^{V(W_n)} / (\text{constants}) \times (\mathbb{Q}/\mathbb{Z})^{V(W_n^\dagger)} / (\text{constants})] : w(e)du(e) = dv(e^\dagger)\}.$$

Instead of working modulo constants, we will normalize our functions so that u and v vanish at the hub vertices of W_n and W_n^\dagger , respectively. (The hub vertices are solid in Figure 8.1). Thus, we want to compute

$$\{(u, v) \in [(\mathbb{Q}/\mathbb{Z})^{V(W_n)} \times (\mathbb{Q}/\mathbb{Z})^{V(W_n^\dagger)}] : u(0) = 0, v(0^\dagger) = 0, w(e)du(e) = dv(e^\dagger)\}.$$

Let a_0, a_1, a_2, \dots be the values of u or v on the rim vertices of W_n and W_n^\dagger in counter-clockwise order as shown in Figure 8.1, with indices taken modulo $2n$. The Cauchy–Riemann equation can be rewritten

$$a_{j+1} - a_{j-1} = a_j - 0.$$

In other words, the numbers a_j satisfy the Fibonacci–Lucas recurrence $a_{j+1} = a_j + a_{j-1}$, so that

$$\begin{pmatrix} a_{j+1} \\ a_j \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_j \\ a_{j-1} \end{pmatrix}.$$

This implies that a harmonic pair (u, v) is uniquely determined by (a_1, a_0) . More precisely, if A is the 2×2 matrix of the recursion, then $(a_1, a_0)^t \in (\mathbb{Q}/\mathbb{Z})^2$ will produce a harmonic pair (u, v) through the iteration process if and only if it is a fixed point of A^{2n} . The module of harmonic pairs (u, v) is thus isomorphic to the kernel of $A^{2n} - I$ acting on $(\mathbb{Q}/\mathbb{Z})^2$. So the invariant factors of the critical group are given by the Smith normal form of $A^{2n} - I$, which is the same as the Smith normal form of $A^n - A^{-n}$ because A is invertible over \mathbb{Z} . For $n \geq 1$,

$$A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \quad A^{-n} = (-1)^n \begin{pmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{pmatrix}.$$

If n is odd, then

$$A^n - A^{-n} = (F_{n+1} + F_{n-1})I,$$

and if n is even, then

$$A^n - A^{-n} = \begin{pmatrix} F_{n+1} - F_{n-1} & 2F_n \\ 2F_n & F_{n-1} - F_{n+1} \end{pmatrix} = F_n \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}.$$

From here, the computation of the invariant factors is straightforward. □

Remark 8.10. Johnson [27] in essence developed a system of “discrete analytic continuation” for harmonic conjugate pairs (u, v) . Although we will not do so here, we believe future research should combine his ideas with the algebraic machinery of this paper. Such a theory of discrete analytic continuation would have applications similar to those of Theorem 5.23.

9. Covering maps and symmetry. The ∂ -graphs $\text{CLF}(m, n)$ (section 3) and W_n (section 8.2) had a cyclic structure with a natural action of \mathbb{Z}/m or \mathbb{Z}/n by ∂ -graph automorphisms. In this section, we will sketch potential applications of symmetry in general, showing how symmetry imposes algebraic constraints on the group structure of $\Upsilon(G, L)$. In particular, for \mathbb{Z} -networks, symmetry yields some information about the torsion primes of $\Upsilon(G, L)$. For now we shall be brief and not develop a complete theory. We will merely record a few simple observations for the benefit of future research.

Recall that covering maps of ∂ -graphs were defined in Definition 4.3. We define a *covering map of R -networks* in the obvious way; it is an R -network morphism such that the underlying ∂ -graph morphism is a covering map. We say a covering map is *finite-sheeted* if $|f^{-1}(x)|$ is finite for every $x \in V(G)$ and $|f^{-1}(e)|$ is finite for every $e \in E(G)$. We say f is *n -sheeted* if $|f^{-1}(x)| = n$ for every $x \in V(G)$ and $|f^{-1}(e)| = n$ for every $e \in E(G)$.

We will also use the notation

$$\mathcal{U}_0^+(G, L, M) = \{u \in \mathcal{U}(G, L, M) : u|_{\partial V(G)} = 0, Lu|_{\partial V(G)} = 0\}.$$

This differs from $\mathcal{U}_0(G, L, M)$ in that we no longer require u to be finitely supported; however, for finite networks $\mathcal{U}_0^+(G, L, M) = \mathcal{U}_0(G, L, M)$. Moreover, we assume familiarity with the terminology for the actions of finite groups on sets.

Observation 9.1. Let $f: (\tilde{G}, \tilde{L}) \rightarrow (G, L)$ be a covering map.

- (1) As in Lemma 4.9 f induces a surjection $\Upsilon(\tilde{G}, \tilde{L}) \rightarrow \Upsilon(G, L)$ providing the following isomorphism:

$$\Upsilon(G, L) \cong \Upsilon(\tilde{G}, \tilde{L}) \Big/ \sum_{\substack{x, y \in V(G) \\ f(x) = f(y)}} R(x - y).$$

- (2) As in Lemma 4.10, there is an injective map $f^* : \mathcal{U}(G, L, M) \rightarrow \mathcal{U}(\tilde{G}, \tilde{L}, M)$ given by $u \mapsto u \circ f$ which identifies harmonic functions on (G, L) with harmonic functions on (\tilde{G}, \tilde{L}) that are constant on each fiber of f .
- (3) Moreover, f^* restricts to an injective map $\mathcal{U}_0^+(G, L, M) \rightarrow \mathcal{U}_0^+(\tilde{G}, \tilde{L}, M)$.
- (4) If f is finite-sheeted, then f^* restricts to an injective map $\mathcal{U}_0(G, L, M) \rightarrow \mathcal{U}_0(\tilde{G}, \tilde{L}, M)$.

Observation 9.2. Suppose $f: (\tilde{G}, \tilde{L}) \rightarrow (G, L)$ is a finite-sheeted covering map.

- (1) Proceeding similarly to Lemma 4.11, we can define a map

$$f_* : \mathcal{U}(\tilde{G}, \tilde{L}, M) \rightarrow \mathcal{U}(G, L, M) : (f_*u)(y) = \sum_{x \in f^{-1}(y)} u(x).$$

- (2) Moreover, f_* restricts to define maps $\mathcal{U}_0^+(G, L, M) \rightarrow \mathcal{U}_0^+(G, L, M)$ and $\mathcal{U}_0(G, L, M) \rightarrow \mathcal{U}_0(G, L, M)$.
- (3) If f is n -sheeted, then $f_* \circ f^*u = n \cdot u$.
- (4) Suppose f is n -sheeted and let M be an R -module. Viewing n as an element of R via the ring morphism $\mathbb{Z} \rightarrow R$, we see that multiplication by n defines an R -module morphism $n : M \rightarrow M$. Assume $n : M \rightarrow M$ is an isomorphism and let $n^{-1} : M \rightarrow M$ denote the inverse map. Then $n^{-1}f_* \circ f^* = \text{id}$. Hence, f^* defines a split injection $\mathcal{U}(G, L, M) \rightarrow \mathcal{U}(\tilde{G}, \tilde{L}, M)$ and

$$\mathcal{U}(\tilde{G}, \tilde{L}, M) = f^*\mathcal{U}(G, L, M) \oplus \ker f_*.$$

Similarly,

$$\mathcal{U}_0(\tilde{G}, \tilde{L}, M) = f^*\mathcal{U}_0(G, L, M) \oplus \ker f_*|_{\mathcal{U}_0(\tilde{G}, \tilde{L}, M)}$$

and the same holds for \mathcal{U}_0^+ .

(Compare [6, Lemma 4.1] as well as Maschke’s theorem from representation theory [20, section 18.1, Theorem 1].)

Observation 9.3. Suppose that K is a group which acts by R -network automorphisms on the R -network (\tilde{G}, \tilde{L}) . Assume the action on vertices and edges is free and that $kx \not\sim x$ for every $k \in G \setminus \{\text{id}\}$ and every $x \in V$.

- (1) There exists a quotient network $(G, L) = (\tilde{G}, \tilde{L})/K$ and a covering map $f: (\tilde{G}, \tilde{L}) \rightarrow (G, L)$.
- (2) There is a corresponding action of K on $\mathcal{U}(\tilde{G}, \tilde{L}, M)$ given by $k \cdot u = k_*u$, where k_* is defined as in Observation 9.2. The fixed-point submodule of this action is

$$\mathcal{U}(\tilde{G}, \tilde{L}, M)^K = f^*\mathcal{U}(G, L, M).$$

The same applies with \mathcal{U} replaced by \mathcal{U}_0 or \mathcal{U}_0^+ .

- (3) Suppose K is a finite p -group for some prime p . Then by a standard argument using the orbits of the K -action on $\mathcal{U}(\tilde{G}, \tilde{L}, M)$, we have

$$|\mathcal{U}(\tilde{G}, \tilde{L}, M)| \equiv |\mathcal{U}(G, L, M)| \pmod{p},$$

provided both sides are finite (for instance, assuming \tilde{G} and M are finite). The same holds for \mathcal{U}_0 and \mathcal{U}_0^+ .

While these statements hold in general, the mod p counting formula seems especially useful for the case $R = \mathbb{Z}$. In the following proposition, we make use of the classification of finitely generated \mathbb{Z} -modules (see [20, section 12.1]).

PROPOSITION 9.4. *Suppose (\tilde{G}, \tilde{L}) is a finite nondegenerate \mathbb{Z} -network. Suppose K is a finite p -group which acts by \mathbb{Z} -network automorphisms on (\tilde{G}, \tilde{L}) as in Observation 9.3, let (G, L) be the quotient network, and let $f: (\tilde{G}, \tilde{L}) \rightarrow (G, L)$ be the projection map.*

- (1) *The \mathbb{Z} -network (G, L) is finite and nondegenerate.*
- (2) *The generalized critical group $\Upsilon(\tilde{G}, \tilde{L})$ has nontrivial p -torsion if and only if $\Upsilon(G, L)$ has nontrivial p -torsion.*
- (3) *Let q be a prime distinct from p and let $k \in \mathbb{Z}$. Then*

$$\mathcal{U}_0(\tilde{G}, \tilde{L}, \mathbb{Z}/q^k) = f^*\mathcal{U}_0(G, L, \mathbb{Z}/q^k) \oplus M_{q^k},$$

where $M_{q^k} := \ker f_*|_{\mathcal{U}_0(\tilde{G}, \tilde{L}, \mathbb{Z}/q^k)}$.

- (4) *For $q \neq p$, the action of K on M_{q^k} has no fixed points other than zero and in particular $|M_{q^k}| \equiv 1 \pmod{p}$.*

Proof. (1) G is finite because it is the quotient of a finite network. Moreover, f^* defines an injective map $\mathcal{U}_0(G, L, \mathbb{Z}) \rightarrow \mathcal{U}_0(\tilde{G}, \tilde{L}, \mathbb{Z})$, and $\mathcal{U}_0(\tilde{G}, \tilde{L}, \mathbb{Z}) = 0$ because (\tilde{G}, \tilde{L}) is nondegenerate. Hence, (G, L) is nondegenerate.

- (2) Because both networks are nondegenerate, Proposition 2.11 shows that

$$\text{Tor}_1(\Upsilon(\tilde{G}, \tilde{L}), \mathbb{Z}/p) \cong \mathcal{U}_0(\tilde{G}, \tilde{L}, \mathbb{Z}/p)$$

and the same holds for (G, L) . On the other hand, by Observation 9.3(3), we have

$$|\mathcal{U}_0(\tilde{G}, \tilde{L}, \mathbb{Z}/p)| \equiv |\mathcal{U}_0(G, L, \mathbb{Z}/p)| \pmod{p}.$$

Each of the two \mathbb{Z} -modules in this equation either is zero (hence has cardinality one mod p) or else has cardinality zero mod p . This implies (2).

(3) Note that f is a $|K|$ -sheeted covering map. Since $|K|$ is a power of p , multiplication by $|K|$ acts as an isomorphism on \mathbb{Z}/q^k . Therefore, claim (3) follows from Observation 9.2(4).

(4) It follows from (3) and Observation 9.3(2) that zero is the only fixed point of the K -action on M_{q^k} . Since K is a p -group, we thus have $|M_{q^k}| \equiv 1 \pmod{p}$. \square

Example 9.5. Consider the networks $\text{CLF}(m, n)$ from section 3. There is an obvious translation action of \mathbb{Z}/k on $\text{CLF}(km, n)$ with the quotient $\text{CLF}(m, n)$. The covering map $\text{CLF}(km, n) \rightarrow \text{CLF}(m, n)$ induces an inclusion $\mathcal{U}_0(\text{CLF}(m, n), L_{\text{std}}, \mathbb{Q}/\mathbb{Z}) \rightarrow \mathcal{U}_0(\text{CLF}(km, n), L_{\text{std}}, \mathbb{Q}/\mathbb{Z})$. Note that when k is a power of 2, Proposition 9.4(2) holds because $\mathcal{U}_0(\text{CLF}(m, n), \mathbb{Z}/2^\ell)$ is nontrivial for all $m \geq 2$ and $n \geq 1$ by Theorem 3.1. Moreover, for odd integers k , we have $\mathcal{U}_0(\text{CLF}(m, n), \mathbb{Z}/k) = 0$ for all m , so (2) also holds when k is an odd prime power. One can verify that the other claims in Proposition 9.4 also hold rather vacuously in the case of $\text{CLF}(m, n)$ as well.

Remark 9.6. Though Proposition 9.4 falls far short of computing $\Upsilon(\tilde{G}, \tilde{L})$ from $\Upsilon(G, L)$, it nonetheless gives a significant amount of information, especially in parts (3) and (4). Indeed, one can argue from the classification of finite \mathbb{Z} -modules that the q -torsion component of $\Upsilon(\tilde{G}, \tilde{L})$ is uniquely determined up to isomorphism by the quantities $|\text{Tor}_1(\Upsilon(\tilde{G}, \tilde{L}), \mathbb{Z}/q^k)|$ for $k = 0, 1, \dots$. Moreover, by (3)

$$|\text{Tor}_1(\Upsilon(\tilde{G}, \tilde{L}), \mathbb{Z}/q^k)| = |\mathcal{U}_0(G, L, \mathbb{Z}/q^k)| \cdot |M_{q^k}|.$$

By (4), we know $|M_{q^k}|$ is a power of q which equals 1 mod p and that the group K acts by automorphisms on M_{q^k} with no nontrivial fixed points. This narrows down the possibilities for $|M_{q^k}|$, especially when combined with other information such as bounds on the number of invariant factors for the torsion part of $\Upsilon(G, L)$ from Corollary 5.26 or bounds on the size of $\text{Tor}_1(\Upsilon(\tilde{G}, \tilde{L}), \mathbb{Q}/\mathbb{Z})$ obtained through determinantal computations.

As stated, Proposition 9.4 does not yield optimal information for the case of graphs without boundary and the critical group since it relies on nondegeneracy. The simplest way to handle this problem is by considering ∂ -graphs with one boundary vertex (see Proposition 2.23) and allowing this boundary vertex to be a branching point in our covering map.

DEFINITION 9.7. *Let \tilde{G} and G be ∂ -graphs with exactly one boundary vertex each, called \tilde{x} and x , respectively. A pseudo-covering map $f: \tilde{G} \rightarrow G$ is a ∂ -morphism such that f is surjective on the vertex and edge sets, f maps \tilde{x} to x , f maps interior vertices to interior vertices, f maps edges to edges, and $\deg(f, y) = 1$ for every $y \in V(\tilde{G}) \setminus \{\tilde{x}\}$.*

The foregoing observations all adapt to pseudo-covering maps for *normalized R -networks* (and in particular apply to critical groups). The verifications are straightforward once we make the following observation: Let G be a ∂ -graph with a single vertex x and let L be a weighted Laplacian (recall this means $d = 0$). If $u: V \rightarrow M$ satisfies $Lu(y) = 0$ for all $y \neq x$, then it also satisfies $Lu(x) = 0$ because $\sum_{y \in V(G)} Lu(y) = 0$.

Example 9.8. Let W_n be the wheel graph from section 8.2 where 0 is considered a boundary vertex. For any $k \in \mathbb{N}$, there is a group action of \mathbb{Z}/k on W_{kn} by rotation and a corresponding quotient map $W_{kn} \rightarrow W_n$ which is a pseudo-covering map. By combining the results from section 8.2 with the results from this section, we obtain the following information about the q -torsion components of $\text{Crit}(W_n)$ for each prime q :

- (1) The q -torsion component has at most two invariant factors. Indeed, the harmonic continuation argument in Proposition 8.9 showed that $\mathcal{U}_0(W_n, L_{\text{std}}, \mathbb{Q}/\mathbb{Z})$ is isomorphic to the submodule of $(\mathbb{Q}/\mathbb{Z})^2$ consisting of fixed points of A^{2n} . A submodule of $(\mathbb{Q}/\mathbb{Z})^2$ can have at most two invariant factors. Since $\mathcal{U}_0(W_n, L_{\text{std}}, \mathbb{Q}/\mathbb{Z})$ has at most two invariant factors, so does its q -torsion component.
- (2) For every k , there exists some n such that $\mathcal{U}_0(W_n, L_{\text{std}}, \mathbb{Z}/q^k) \cong (\mathbb{Z}/q^k)^2$. To prove this, it suffices to show that every $\phi \in (\mathbb{Z}/q^k)^2$ will be a fixed point of A^{2n} for some n . Note that A maps $(\mathbb{Z}/q^k)^2$ into itself and $(\mathbb{Z}/q^k)^2$ is finite, so there must exist two distinct integers k and ℓ with $A^{2k}\phi = A^{2\ell}\phi$. Since A is invertible over \mathbb{Z} , we have $A^{2(k-\ell)}\phi = \phi$, so we can take $n = k - \ell$.
- (3) If q is a prime other than 5, then we know from Proposition 8.9 that the q -torsion component of $\text{Crit}(W_n)$ has the form $(\mathbb{Z}/q^k)^2$ for some k . Moreover, the 5-torsion component has the form $(\mathbb{Z}/5^k)^2$ for odd n and $\mathbb{Z}/5^k \times \mathbb{Z}/5^{k+1}$ for even n .
- (4) If $m|n$, then there is a pseudo-covering map $W_m \rightarrow W_n$ and hence by Observation 9.1(4), we can identify the q -torsion component of $\text{Crit}(W_n)$ with a submodule of the q -torsion component for $\text{Crit}(W_m)$.
- (5) Suppose n is such that the q -torsion component $\text{Crit}(W_n)$ has two invariant factors, and let p be a prime other than q . Then $\text{Crit}(W_{pn})$ has the same q -torsion submodule as $\text{Crit}(W_n)$. Indeed, multiplication by p acts as an isomorphism on \mathbb{Z}/q^k . Thus, by Proposition 9.4(3), we have

$$\mathcal{U}_0(W_{pn}, L_{\text{std}}, \mathbb{Z}/q^k) \cong \mathcal{U}_0(W_n, L_{\text{std}}, \mathbb{Z}/q^k) \oplus M_{q^k}.$$

We know that $\mathcal{U}_0(W_n, L_{\text{std}}, \mathbb{Z}/q^k)$ has two invariant factors, while $\mathcal{U}_0(W_{pn}, L_{\text{std}}, \mathbb{Z}/q^k)$ has at most two invariant factors. This implies that $M_{q^k} = 0$ and hence

$$\mathcal{U}_0(W_{pn}, L_{\text{std}}, \mathbb{Z}/q^k) \cong \mathcal{U}_0(W_n, L_{\text{std}}, \mathbb{Z}/q^k).$$

Since this holds for all k , the q -torsion components of $\text{Crit}(W_{pn})$ and $\text{Crit}(W_n)$ are isomorphic.

10. Open problems. Much like the critical group, the fundamental module Υ connects ideas from network theory, combinatorics, algebraic topology, homological algebra, and complex analysis. We have correlated the algebraic properties of Υ with the combinatorial properties of ∂ -graphs, including ∂ -graph morphisms, layer-stripping, boundary wedge-sums, duality, and symmetry and we have given applications to the critical group and Laplacian eigenvalues. Our results lead to the following questions.

QUESTION 1. *Do our algebraic invariants extend to higher-dimensional cell complexes, along the lines of [21]? Do they generalize to directed graphs? What are the analogues of ∂ -graph morphisms and layer-stripping in these settings?*

QUESTION 2. *Can the techniques developed here (particularly Theorem 5.23) aid the computation of previously intractable sandpile groups? What applications do they have for computing eigenvectors and characteristic polynomials?*

Remark 10.1. The layer-stripping process is similar to the inductive construction of threshold graphs by adding isolated vertices and vertices connected to all the preceding ones (see [12, Definition 3]). However, we were not able to find an application of our theory to threshold graphs that would simplify the computation of critical groups.

QUESTION 3. *Corollaries 5.27 and 5.30 used layer-stripping to give a bound on the number of invariant factors for $\text{Crit}(G)$ and the multiplicity of eigenvalues. How sharp is this bound for general graphs? For a graph without boundary, is there an algebraic characterization of the minimal number of boundary vertices one has to assign to achieve layerability? What is the most efficient algorithm for finding a choice of boundary vertices that achieves this minimal number?*

QUESTION 4. *Are there other operations on ∂ -graphs which interact nicely with Υ and with ∂ -graph morphisms? Can such operations be used to compute Υ or at least produce short exact sequences? See Remark 5.9 and [34, Proposition 2], [41, Proposition 21].*

QUESTION 5. *We have studied algebraic invariants which test layerability (Theorem 5.16). Are there algebraic invariants of ∂ -graphs which test whether the electrical inverse problem can be solved by layer-stripping?*

QUESTION 6. *Do Theorems 5.16 and 7.13 extend to infinite ∂ -graphs? In particular, for a fixed infinite graph G , if $\Upsilon(G, L)$ is flat for all unit edge-weight functions w , must $\Upsilon(G, L)$ also be free for all unit edge-weight functions?*

QUESTION 7. *Determine the \mathbb{Z} -module of \mathbb{Q}/\mathbb{Z} -harmonic functions supported in a given subset of the \mathbb{Z}^2 lattice. Applying Lemma 3.6 to $\text{CLF}(\infty, n)$ resolves the case of a diagonal strip with sides parallel to the lines $y = \pm x$. An argument using harmonic continuation shows that these are the only strips with a nonzero answer.*

QUESTION 8. *Can the techniques of section 3 be modified to handle ∂ -graphs built from the triangular or hexagonal lattice rather than the rectangular lattice?*

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