The Big Picture of Basic Linear Algebra

Let A be an $m \times n$ matrix, and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation given by $T(x) = Ax$.

- In each of the columns (P), (Q), (R), all the items are equivalent to each other.
- Column (R) is equivalent to (P) and (Q) combined.

	(P)	(Q)	(R)
1. Linear equations:	$Ax = b$ has at most one solution for every b	$Ax = b$ has at least one solution for every b	$Ax = b$ has exactly one solution for every b
2. Linear maps:	T is injective	T is surjective	T is bijective
3. Subspaces:	$\ker T = \{0\}$	$\operatorname{im} T =$ all of $\mathbb{R}^m$	$\ker T = \{0\}$ and $\operatorname{im} T = \mathbb{R}^m$
4. RREF:	There are no free variables	There are no zero rows	The RREF is the identity matrix.
5. Pivots:	There is a pivot in every column	There is a pivot in every row	There is a pivot in every row and every column.
6. Columns of A:	The columns are linearly indep.	The columns span $\mathbb{R}^m$	The columns are a basis for $\mathbb{R}^m$
7. Rows of A:	The rows span $\mathbb{R}^n$	The rows are linearly indep.	The rows are a basis for $\mathbb{R}^n$
8. Inverses:	A has a left inverse	A has a right inverse	A is invertible

If you understand why the items in each column are equivalent and repeatedly convince yourself it is true to the point where it becomes intuitive, then you will succeed in basic linear algebra. From these equivalences, we can deduce the first major theorem of linear algebra:

Theorem.

a. If (R) is true, then A is a square matrix, that is, $m = n$.

b. If A is square, then (P), (Q), and (R) are equivalent.

Proof. This can be proved in many ways, but it is easiest to look at row 6. (a) If the RREF of A is the identity matrix, then A has to be square because the identity matrix is square. (b) If A is square, then the RREF is square. So if it has a pivot in every column, it must be the identity matrix, hence (P) implies (R). Similarly, if it has a pivot in every row, then it must be the identity matrix. So (Q) implies (R). It is immediate that (R) implies (P) and (R) implies (Q), and this shows (P), (Q), and (R) are equivalent. $\square$