Basis for Kernel and Image

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[Relevant to midterm 1 question 2]
Consider the matrix

$$A = \begin{pmatrix} 1 & 3 & 0 & 1 & 3 \\ 2 & 6 & 1 & 1 & 4 \\ 1 & 3 & 1 & 0 & 5 \end{pmatrix}.$$

1. Find the RREF of A.
2. Which columns have pivots? Which columns represent free variables?
3. Why is the kernel of A the same as the kernel of RREF(A)?
4. The kernel of RREF(A) contains two vectors of the form

   $$\begin{pmatrix} * \\ 1 \\ * \\ 0 \\ * \end{pmatrix} \text{ and } \begin{pmatrix} * \\ 0 \\ * \\ 1 \\ * \end{pmatrix}.$$

   Find these two vectors. Call them $\vec{v}_1$ and $\vec{v}_2$.
5. Suppose that $\vec{x}$ is in the kernel of RREF(A). Show that $\vec{x} = x_2\vec{v}_1 + x_4\vec{v}_2$.
6. Show that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent (you want to show that if $a\vec{v}_1 + b\vec{v}_2 = \vec{0}$, then a and b must both be zero).
7. You have just proved that $\vec{v}_1$ and $\vec{v}_2$ are a basis for the kernel of A. Explain.
8. What is the dimension of the kernel of A? What is the dimension of the image of A? Verify that the rank-nullity theorem is true in this case.
9. Show that columns 1, 3, and 5 of A form a basis for the image of A.
10. What is the image of A (besides being the span of these three vectors)?
11. *Challenge:* Write a description in your own words of how to find a basis for the kernel and image of matrix A in general. Why does it work?