Operator-Valued Chordal Loewner Chains and Non-Commutative Probability

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Introduction

Non-Commutative Probability

- In classical probability, a bounded real random variable X can be thought of as a bounded self-adjoint operator, namely as a multiplication operator on $L^2(\Omega, P)$.

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- In non-commutative probability, the algebra $L^\infty(\Omega, P)$ of bounded random variables is replaced by a possibly non-commutative operator algebra $\mathcal{A}$, and the expectation is positive, unital map $E : \mathcal{A} \to \mathbb{C}$.

Non-Commutative Probability

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- Muraki (2001), building on work of Speicher, showed that for NC variables, there are five types of independence satisfying certain axioms; they are tensor, free, boolean, monotone, and anti-monotone independence.

Non-Commutative Probability

- The Cauchy transform $G_X(z) = E[(z - X)^{-1}]$ as well as its reciprocal $F_X(z) = 1/G_X(z)$ play an important role in non-commutative probability like the Fourier transform in classical probability (e.g. R-transform, analytic subordination).

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- Given a process (X_t) with independent increments, we want to understand the evolution of the F_{X_t} through a differential equation.
- Loewner chains from complex analysis are relevant here.

Chordal Loewner Chains

Definition

A *normalized chordal Loewner chain* on $[0, T]$ is a family of analytic functions $F_t : \mathbb{H} \to \mathbb{H}$ such that

- $F_0(z) = z$.
- The F_t's are analytic in a neighborhood of ∞.
- If $F_t(z) = z + t/z + O(1/z^2)$.
- For $s < t$, we have $F_t = F_s \circ F_{s,t}$ for some $F_{s,t} : \mathbb{H} \to \mathbb{H}$.

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Fact

The F_t's are conformal maps from $\mathbb{H}$ onto $\mathbb{H} \setminus K_t$, where K_t is a growing compact region touching the real line, e.g. a growing slit.

Loewner Chains and Monotone Probability

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Observation (Schleißinger 2017)

If X_t is a process with monotone independent increments, and if $E(X_t) = 0$ and $E(X_t^2) = t$, then $F_t(z) = 1/G_{X_t}(z)$ is a normalized chordal Loewner chain. Every normalized Loewner chain arises in this way.

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Hasebe (2010) studied the evolution equation for processes with monotone independent and stationary increments, but as Schleißinger realized, this was a special case of the older chordal Loewner equation ...

The Loewner Equation

Theorem (Bauer 2005)

Every normalized Loewner chain satisfies the generalized Loewner equation

$$\partial_t F_t(z) = D_z F_t(z) \cdot V(z,t)$$

where $V(z,t)$ is some vector field of the form $V(z,t) = -G_{\nu_t}(z)$. Conversely, given such a vector field, the Loewner equation has a unique solution.

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History

Loewner chains in the disk were studied by Loewner in 1923 in the case F_t maps $\mathbb{D}$ onto $\mathbb{D}$ minus a slit. Kufarev and Pommerenke considered more general Loewner chains in the disk. Loewner chains in the half-plane were studied by Schramm in the case $V(z,t) = -1/(z - B_t)$ where B_t is a Brownian motion (SLE, 1980's).

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Goal

Study the operator-valued version of Loewner theory, and prove operator-valued analogues of the above results of Bauer and Schleißinger.

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Talk Overview:

1. Background on operator-valued probability.
2. Operator-valued chordal Loewner equation for $F_t = F_{X_t}$.
3. Realization of monotone increment processes on a Fock space and application to CLT (if time).

Operator-valued Non-commutative Probability

$\mathcal{B}$-valued Probability Spaces

Definition

Let $\mathcal{B}$ be a C^*-algebra. An *$\mathcal{B}$-valued probability space* $(\mathcal{A}, E)$ is a C^* algebra $\mathcal{A} \supseteq \mathcal{B}$ together with a completely positive, unital, $\mathcal{B}$-bimodule map $E : \mathcal{A} \to \mathcal{B}$, called the *expectation*.

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Definition

The *$\mathcal{B}$-valued law* of a self-adjoint random variable X in $\mathcal{A}$ is the induced map from $\mathcal{B}\langle X \rangle$ (non-commutative polynomials over $\mathcal{B}$) to $\mathcal{B}$ given by $p(X) \mapsto E(p(X))$.

Notation / Facts

Fact

There are axioms to characterize the maps $\mu : \mathcal{B}\langle X \rangle \to \mathcal{B}$ which can be realized as the law of some random variable. Specifically, it is a completely positive map with exponentially bounded moments which satisfies $\mu|_{\mathcal{B}} = \text{id}$.

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"Definition"

We denote by $\mathrm{rad}(\sigma)$ the norm of the associated operator, which is analogous to the radius of the support of a measure.

Matricial Upper Half-Plane

The $\mathcal{B}$-valued Cauchy transform $G_X(z) = E[(z - X)^{-1}]$ should be understood as a fully matricial (non-commutative) function on the matricial upper half-plane (J.L. Taylor, D. Voiculescu, M. Popa, V. Vinnikov, J. Williams, ...).

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Definition

The matricial upper half-plane is defined by

$$\mathbb{H}^{(n)}(\mathcal{B}) = \bigcup_{\epsilon > 0} \{z \in M_n(\mathcal{B}) : \operatorname{Im} z \geq \epsilon\}$$

$$\mathbb{H}(\mathcal{B}) = \{\mathbb{H}^{(n)}(\mathcal{B})\}_{n \geq 1}.$$

Cauchy Transforms

Definition

A *fully matricial function* on $\mathbb{H}(\mathcal{B})$ is a sequence of functions $F^{(n)} : \mathbb{H}^{(n)}(\mathcal{B}) \to M_n(\mathcal{B})$ such that F preserves direct sums of matrices and conjugation by scalar matrices, together with a local boundedness condition which is uniform in n.

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Definition (Voiculescu)

The Cauchy transform of a generalized law μ is defined by $G_\mu^{(n)}(z) = \mu \otimes \mathrm{id}_{M_n(\mathbb{C})}[(z - X \otimes 1_{M_n(\mathbb{C})})^{-1}]$. This is a fully matricial function on $\mathbb{H}(\mathcal{B})$.

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Fact

There is an analytic characterization of $\mathcal{B}$-valued Cauchy transforms due to [Williams 2013, Williams-Anshelevich 2015].

$\mathcal{B}$-valued Chordal Loewner Chains

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A *Lipschitz, normalized $\mathcal{B}$-valued chordal Loewner chain* on $[0, T]$ is a family of matricial analytic functions $F_t(z) = F(z, t)$ on $\mathbb{H}(\mathcal{B})$ such that

- $F_0 = \text{id}$
- F_t is the recriprocal Cauchy transform of an $\mathcal{B}$-valued law μ_t.
- If $s < t$, then $F_t = F_s \circ F_{s,t}$ for some matricial analytic $F_{s,t} : \mathbb{H}(\mathcal{B}) \to \mathbb{H}(\mathcal{B})$.
- $\mu_t(X) = 0$ and $\mu_t(X^2)$ is Lipschitz in t.

Basic Properties

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Loewner chains relate to monotone independence over $\mathcal{B}$ just as in the scalar case.

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Lemma

- $F_{s,t}$ *is unique.*
- $F_{0,t} = F_t$.
- $F_{s,t} \circ F_{t,u} = F_{s,u}$.
- $F_{s,t}$ *is the* F*-transform of a law* $\mu_{s,t}$.
- $\sup_{s,t} \operatorname{rad}(\mu_{s,t}) \leq C \operatorname{rad}(\mu_T)$.

Basic Properties

Lemma

There exists a generalized law $\sigma_{s,t}$ *such that*

$$F_{s,t}(z) = z - G_{\sigma_{s,t}}(z).$$

We have $\mathrm{rad}(\sigma_{s,t}) \leq \mathrm{rad}(\mu_{s,t})$ *and* $\sigma_{s,t}(1) = \mu_{s,t}(X^2) = \mu_t(X^2) - \mu_s(X^2)$.

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Proposition

Each F_t is a biholomorphic map onto a fully matricial domain and the inverse is fully matricial.

The Loewner Equation?

- The operator-valued version of the Loewner equation is

$$\partial_t F(z,t) = DF(z,t)[V(z,t)],$$

where $DF(z,t)$ is the Fréchet derivative with respect to z (also known as $\Delta F_t(z,z)$ in the NC function theory),

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- We want to show that the Loewner equation defines a bijection between Loewner chains $F(z,t)$ and Herglotz vector fields $V(z,t)$ on $[0,T]$.

Problems with Pointwise Differentiation

- We should allow Loewner chains which are merely Lipschitz in t, so we need to differentiate Lipschitz functions $[0, T] \to M_n(\mathcal{B})$.

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- Pointwise differentiation won't work because a C^*-algebra $\mathcal{B}$ is a bad Banach space for differentiation (often not reflexive or separable).
- So consider $\partial_t F(z, \cdot)$ as an $M_n(\mathcal{B})$-valued distribution on $[0, T]$.

Distributional Differentiation

- But we need to manipulate $\partial_t F(z, \cdot)$ like a pointwise defined function, e.g. we want to have the chain rule:

$$\partial_t[F(G(z,t),t)] = \partial_t F(G(z,t),t) + DF(G(z,t),t)[\partial_t G(z,t)].$$

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- Thus, $\partial_t F^{(n)}(z,t)$ is an element of $\mathcal{L}(L^1[0,T], M_n(\mathcal{B}))$, which is "almost as nice" as an L^∞ function $[0,T] \to M_n(\mathcal{B})$.

Distributional Differentiation

- A family of Banach-valued analytic functions $F(z,t)$ for $t \in [0,T]$ is a called a *locally Lipschitz family* if it is Lipschitz in t with uniform Lipschitz constants for z in a neighborhood of each z_0 in the domain.

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- If $F(z, t)$ and $G(z, t)$ are locally Lipschitz families, then we can define the composition

$$\partial_t F(G(z, t), t) \in \mathcal{L}(L^1[0, T], \mathcal{X})$$

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- We can define $DF(G(z,t),t)[\partial_t G(z,t)]$ similarly.
- The chain rule computation above is correct in $\mathcal{L}(L^1[0,T], \mathcal{X})$.

The Loewner Equation: Setup

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Definition

A *distributional Herglotz vector field* $V(z, t)$ to be a matricial analytic function $\mathbb{H}(\mathcal{B}) \to \mathcal{L}(L^1[0, T], M_n(\mathcal{B}))$ such that for each nonnegative $\phi \in L^1[0, T]$, the function $-\int V(z, t)\phi(t)\, dt$ is the Cauchy transform of a generalized law ν_ϕ with $\sup_\phi \operatorname{rad}(\nu_\phi) < +\infty$.

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Definition

In this case, we call the map $\nu : \mathcal{B}\langle X \rangle \times L^1[0,T] \to \mathcal{B}$ a *distributional family of generalized laws* and denote $\operatorname{rad}(\nu) = \sup_{\phi \geq 0} \operatorname{rad}(\nu_\phi)$. We also denote $\nu_\phi = \int_0^T \nu(\cdot, t)\phi(t)\,dt$.

The Loewner Equation: Main Theorem

Theorem

On an interval $[0, T]$, the Loewner equation $\partial_t F(z,t) = DF(z,t)[V(z,t)]$ defines a bijection between Lipschitz, normalized $\mathcal{B}$-valued Loewner chains and distributional Herglotz vector fields (and hence distributional generalized laws).

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Now that all the machinery has been set up, the proof proceeds exactly as Bauer did in the scalar case. Specifically,

- To construct the Herglotz vector field from the Loewner chain, we show that the distributional time derivative has the correct form using some step function approximation arguments.
- To construct the Loewner chain from the Herglotz vector field, we can reverse time to convert the problem to an ODE, then solve it with Picard iteration and make explicit estimates for convergence.

Fock Space Construction

Fock Space Construction

Let $\mathcal{C} = C([0, T], \mathcal{B})$. For a distributional family of generalized laws ν on $[0, T]$, define $I = I_\nu : \mathcal{C}\langle X \rangle \to \mathcal{C}$ by

$$I_\nu[f(X, t)](t) = \int_t^T \nu(f(X, s), s)\, ds.$$

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We define a Fock space $\mathcal{H}_\nu = \bigoplus_{n=0}^\infty \mathcal{H}_n$, where

$$\mathcal{H}_n = \mathcal{C}\langle X \rangle \otimes \cdots \otimes \mathcal{C}\langle X \rangle \otimes \mathcal{C}$$

with the $\mathcal{C}$-valued inner product

$$\langle f_n \otimes \cdots \otimes f_0, g_n \otimes \cdots \otimes g_0 \rangle = f_0^* I_\nu(f_1^* \dots I_\nu(f_n^* g_n) \dots g_1) g_0.$$

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- For $f(X,t) \in \mathcal{C}\langle X \rangle$, define the creation operator $\ell(f)$ by

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- The annihilation operator $\ell(f)^*$ is given by $\ell(f)^* f_0 = 0$ for $f_0 \in \mathcal{C}$ and otherwise

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$$\ell(f)^*[f_n \otimes \cdots \otimes f_0] = I(f^* f_n) f_{n-1} \otimes \cdots \otimes f_0.$$

- Every $f(X,t) \in \mathcal{C}\langle X \rangle$ defines a multiplication operator acting on the left-most coordinate, where the action on $\mathcal{H}_0 = \mathcal{C}$ is defined to be multiplication by $f(0,t)$.

Fock Space Construction

Theorem

Let $Y_{t_1,t_2} = \ell(\chi_{[t_1,t_2)}) + \ell(\chi_{[t_1,t_2)})^* + \chi_{[t_1,t_2)}(t)X$. *Define a* $\mathcal{B}$*-valued expectation by*

$$E(T) = \langle \Omega, T\Omega \rangle_{\mathcal{H}_\nu}|_{t=0}.$$

Then

1. $Y_{t_1,t_3} = Y_{t_1,t_2} + Y_{t_2,t_3}$.
2. Y_{t_1,t_2} *and* Y_{t_2,t_3} *are monotone independent over* $\mathcal{B}$ *with respect to* E.
3. Y_{t_1,t_2} *has the law* μ_{t_1,t_2} *with respect to* E.

Central Limit Theorem for Loewner Chains

Background for CLT

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Definition

Let $\eta : \mathcal{B} \times L^1[0,T] \to \mathcal{B}$ be a distributional family of completely positive maps. We define the corresponding $\mathcal{B}$-valued generalized arcsine law μ_η as the law obtained by running the Loewner equation up to time T with $V(z,t) = -\eta_t(z^{-1})$.

CLT via Coupling

Let ν be a distributional generalized law and let $\eta_t = \nu_t|_{\mathcal{B}}$. Using the Fock space $\mathcal{H}_\nu$, define

- $Y_{t_1,t_2} = \ell(\chi_{[t_1,t_2)}) + \ell(\chi_{[t_1,t_2)})^* + \chi_{[t_1,t_2)}(t)X$.
- $Z_{t_1,t_2} = \ell(\chi_{[t_1,t_2)}) + \ell(\chi_{[t_1,t_2)})^*$.

Let $F_t = F_{\mu_t}$ be the solution to the Loewner equation for $-G_{\nu_t}(z)$.

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Theorem

Y_{t_1,t_2} has the law μ_{t_1,t_2} and Z_{t_1,t_2} has the generalized arcsine law for $\eta|_{[t_1,t_2]}$. Moreover, we have

$$\|Y_{t_1,t_2} - Z_{t_1,t_2}\| \leq \operatorname{rad}(\nu).$$

As a consequence, for $\operatorname{Im} z \geq \epsilon$,

$$\|T^{1/2} G_{Y_{0,T}}(T^{1/2} z) - T^{1/2} G_{Z_{0,T}}(T^{1/2} z)\| \leq T^{-1/2} \epsilon^{-2} \operatorname{rad}(\nu).$$

CLT via Loewner Equation

Another proof for the CLT is a "continuous-time Lindeberg exchange" where we interpolate between $Y_{0,T}$ and $Z_{0,T}$ using $Y_{0,t} + Z_{t,T}$. In other words, we write

$$G_{Y_{0,T}} - G_{Z_{0,T}} = \int_0^T \partial_t [G_{Y_{0,t}} \circ F_{Z_{t,T}}] \, dt.$$

Evaluate this using the chain rule and the Loewner equation and make some straightforward estimates ...

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Theorem

For $\operatorname{Im} z \geq \epsilon$, *we have*

$$\| T^{1/2} G_{Y_{0,T}}(T^{1/2} z) - T^{1/2} G_{Z_{0,T}}(T^{1/2} z) \| \\ \leq T^{-1/2} \epsilon^{-4} \operatorname{rad}(\nu) \|\nu(1)\|_{\mathcal{L}(L^1[0,T],\mathcal{B})}.$$

Concluding Remarks

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For the other types of independence, you also get differential equations for (sufficiently regular) processes with independent increments:

- Free: $\partial_t F(z,t) = DF(z,t)[V(F(z,t),t)]$.
- Monotone: $\partial_t F(z,t) = DF(z,t)[V(z,t)]$.
- Anti-monotone: $\partial_t F(z,t) = V(F(z,t),t)$.
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This results in a "Bercovici-Pata bijection" for processes with independent (non-stationary) increments, where the processes with the same $V(z,t)$ correspond to each other.

This extends the usual BP bijection for infinitely divisible laws ($\cong$ processes with independent and stationary increments).

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The Fock space constructions, and their application to CLT, adapt to free and Boolean independence also, with similar but easier proofs.

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Actually, the coupling argument for non-commutative CLT works in much greater generality and doesn't require a continuous-time process. See joint work with W. Liu on "An Operad of Non-commutative Independences Defined by Trees."

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Question

Is there a version of the Riemann mapping theorem for matricial domains?

Selected References

This is based on arXiv:1711.02611, which contains complete citations. Some of the most important references:

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