Orthogonal Stuff

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This worksheet was made for UCLA Math 33A Winter 2016 with Omer Ben Neria; it covers material related to Sections 5.1 and 5.2 of Otto Bretscher's *Linear Algebra with Applications*.

Part 1: Gram-Schmidt Computation

Define

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$

a. Compute $\vec{u}_1 = \vec{v}_1 / \|\vec{v}_1\|$.

b. Compute $\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$.

c. Make sure $\vec{w}_2$ is orthogonal to $\vec{u}_1$. Why does this happen?

d. Compute $\vec{u}_2 = \vec{w}_2 / \|\vec{w}_2\|$.

e. Compute $\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$.

f. Verify that $\vec{w}_2$ is orthogonal to $\vec{u}_1$ and $\vec{u}_2$.

g. Compute $\vec{u}_3 = \vec{w}_3 / \|\vec{w}_3\|$.

h. Verify directly that $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ are orthonormal (that is, they are unit vectors that are perpendicular to each other). If they are not orthonormal, then check your computations for fraction and square-root errors!

Part 2: Thinking about Gram-Schmidt

a. Show that

$$\begin{aligned} \operatorname{span}(\vec{u}_1) &= \operatorname{span}(\vec{v}_1) \\ \operatorname{span}(\vec{u}_1, \vec{u}_2) &= \operatorname{span}(\vec{v}_1, \vec{v}_2) \\ \operatorname{span}(\vec{u}_1, \vec{u}_2, \vec{u}_3) &= \operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3). \end{aligned}$$

b. What orthonormal basis would you get if you applied Gram-Schmidt to $\vec{v}_3, \vec{v}_2, \vec{v}_1$ (in that order) instead of $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$?

c. Why are there square roots in the $\vec{u}_j$'s, but not the $\vec{w}_j$'s?

d. If $\vec{x}$ is a nonzero vector and c is a positive number, then $(c\vec{x})/\|c\vec{x}\| = \vec{x}/\|\vec{x}\|$. How might this save you time in the Gram-Schmidt computations?

Part 3: QR Factorization

a. Verify that

$$\vec{v}_1 = \|\vec{v}_1\|\,\vec{u}_1, \qquad \vec{v}_2 = \vec{w}_2 + (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 + \|\vec{w}_2\|\,\vec{u}_2.$$

b. Find a similar formula for $\vec{v}_3$, directly from the definition of $\vec{w}_3$ and $\vec{u}_3$.

c. Show that the equations

$$\begin{aligned} \vec{v}_1 &= r_1\vec{u}_1 \\ \vec{v}_2 &= s_1\vec{u}_1 + s_2\vec{u}_2 \\ \vec{v}_3 &= t_1\vec{u}_1 + t_2\vec{u}_2 + t_3\vec{u}_3 \end{aligned}$$

can be rewritten as

$$\begin{pmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3\end{pmatrix} = \begin{pmatrix}\vec{u}_1 & \vec{u}_2 & \vec{u}_3\end{pmatrix}\begin{pmatrix} r_1 & s_1 & t_1 \\ 0 & s_2 & t_2 \\ 0 & 0 & t_3 \end{pmatrix}.$$

d. Plug in the specific vectors and numbers from the example in part 1 to get a QR factorization of the matrix $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$.

Part 4: Projection

a. Let $V = \text{span}(\vec{v}_1, \vec{v}_2) = \text{span}(\vec{u}_1, \vec{u}_2)$. What is the orthogonal projection of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ onto V? (See page 206) Compute it for this specific example.

b. If $\vec{x}$ is any vector, show that

$$(\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2 = \vec{u}_1\vec{u}_1^T\vec{x} + \vec{u}_2\vec{u}_2^T\vec{x}.$$

c. Let P be the matrix of projection onto V. Show that

$$P = \vec{u}_1\vec{u}_1^T + \vec{u}_2\vec{u}_2^T = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 \end{pmatrix} \begin{pmatrix} \vec{u}_1 & \vec{u}_2 \end{pmatrix}^T.$$

d. Compute P for this specific example. If you want, verify that the formulas in part (b) and (c) are true for this specific P.

e. Show directly from part (c) that $P\vec{u}_1 = \vec{u}_1$, $P\vec{u}_2 = \vec{u}_2$, $P\vec{u}_3 = 0$.

f. Directly from the equations in part (e), show that

$$P \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{pmatrix} = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and hence

$$P = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{pmatrix}^{-1}.$$

g. Deduce that all projections onto planes in $\mathbb{R}^3$ are similar to each other.

h. Suppose that some matrix A satisfies $A\vec{u}_1 = \vec{u}_1$, $A\vec{u}_2 = \vec{u}_2$, $A\vec{u}_3 = 0$. Show that A also satisfies the equations in part (e) and conclude that $A = P$.

i. Conclude that P is the *unique* matrix that satisfies

$$P\vec{w} = \vec{w} \text{ for } \vec{w} \in V$$
$$P\vec{w} = \vec{0} \text{ for } \vec{w} \in V^\perp.$$

j. How would you change the answers if V were a line instead of a plane? Is the projection onto a line similar to the projection onto a plane?