Eigenvalues and Eigenvectors

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Note: You may need a piece of scratch paper.

Review: Let A be a square matrix. If $A\vec{v} = \lambda\vec{v}$ for some nonzero vector $\vec{v}$, then we say that $\vec{v}$ is an *eigenvector* and λ is an *eigenvalue*.

Exercise 1. Verify that the following are equivalent:

1. $A\vec{v} = \lambda\vec{v}$.
2. $(A - \lambda I)\vec{v} = 0$.
3. $\vec{v} \in \ker(A - \lambda I)$.

Exercise 2. Verify that the following are equivalent:

1. λ is an eigenvalue of A (that is, it has a nonzero eigenvector).
2. $\ker(A - \lambda I)$ is nonzero.
3. $A - \lambda I$ is not invertible.
4. $\det(A - \lambda I) = 0$.

Exercise 3. Compute the eigenvalues and eigenvectors of $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$ using the following steps:

1. Compute $\det(A - \lambda I)$.
2. For which values of λ does $\det(A - \lambda I) = 0$? In other words, find the roots λ_1 and λ_2 of the polynomial $p(\lambda) = \det(A - \lambda I)$.
3. Note by the previous problem that λ_1 and λ_2 are the eigenvalues of A.
4. Find a basis for $\ker(A - \lambda_1 I)$.
5. Let $\vec{v}_1$ be a nonzero vector in $\ker(A - \lambda_1 I)$. Verify that $A\vec{v}_1 = \lambda_1\vec{v}_1$, and thus $\vec{v}_1$ is an eigenvector.
6. Find a basis $\vec{v}_2$ for $\ker(A - \lambda_2 I)$.

Exercise 4. Use the same matrices and vectors from the previous problem.

1. Show that $\vec{v}_1$ and $\vec{v}_2$ are a basis for $\mathbb{R}^2$.
2. Find the matrix of A with respect to the basis $(\vec{v}_1, \vec{v}_2)$.
3. Conclude that A is similar to a diagonal matrix.

Exercise 5. Suppose that $\vec{v}$ is an eigenvector of A with eigenvalue λ and also an eigenvector of B with eigenvalue μ. Then

1. $\vec{v}$ is an eigenvector of $A + B$. What is its eigenvalue?
2. $\vec{v}$ is an eigenvector of AB. What is its eigenvalue?
3. If $k \geq 0$, then $\vec{v}$ is an eigenvector of A^k. What is its eigenvalue?
4. If A is invertible, then $\vec{v}$ is an eigenvector of A^{-1}. What is its eigenvalue?
5. If $p(x)$ is any polynomial, then we can plug the matrix A into p. Then $\vec{v}$ is an eigenvector of $p(A)$ with eigenvalue $p(\lambda)$.
6. The same is true if we take a two variable polynomial $p(x, y)$ and consider the matrix $p(A, B)$.