Decimal Expansion of Rational Numbers

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October 5, 2018

Here we describe how to rigorously define the decimal expansion of a rational number using the properties of Euclidean division, and we prove some basic facts about the decimal expansion. This is aimed at math students (probably early undergraduate) who are beginning to learn about proofs.

$\mathbb{N}$ will denote the set of natural numbers $1, 2, 3, \dots$. $\mathbb{N}_0$ will denote the set $\{0, 1, 2, \dots\}$. $\mathbb{Z}$ will denote the integers. We'll take the following result about the division of integers as a given.

Proposition 1 (Euclidean division)**.** *Let $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. Then there exists a unique $q \in \mathbb{Z}$ and $r \in \{0, 1, \dots, b-1\}$ such that $a = qb + r$.*

In elementary school, you learned to compute the quotient q and the remainder r using long division. You also learned how to compute the digits of a decimal expansion of a rational number a/b using long division. Let's rephrase the definition of the decimal digits in terms of Proposition 1. The first step is to find the integer quotient and remainder for a/b. Using Proposition 1, we write

$$a = q_0 b + r_0$$

for some $q_0 \in \mathbb{Z}$ and $r_0 \in \{0, \dots, b-1\}$.

Next, we have to find the decimal expansion of r_0/b. By Proposition 1, we have

$$10r_0 = q_1 b + r_1$$

for some $q_1 \in \mathbb{Z}$ and $r_1 \in \{0, \dots, b-1\}$. Since $0 \leq 10r_0 < 10b$, the quotient q_1 should be in $\{0, 1, \dots, 9\}$. To convince yourself that q_1 corresponds to the first decimal digit, note that

$$\begin{aligned} \frac{a}{b} &= q_0 + \frac{r_0}{b} \\ &= q_0 + \frac{10r_0}{10b} \\ &= q_0 + \frac{q_1}{10} + \frac{r_1}{10b}. \end{aligned}$$

Next, we repeat the process, writing

$$10r_1 = q_2 b + r_2,$$

and then by substituting this into previous equation, we get

$$\frac{a}{b} = q_0 + \frac{q_1}{10} + \frac{q_2}{10^2} + \frac{r_2}{10^2 b}.$$

We can continue this process indefinitely and thus define the digits q_1, q_2, ... of the decimal expansion.

The mathematically rigorous way of saying this is that we will *define* q_1, q_2, ... *by induction.* This depends on the following fact, which we state without proof.

Proposition 2 (Inductive definition)**.** *Suppose that we want to define a mathematical object P_j for $j \in \mathbb{N}_0$. Assume that P_0 is defined. Also, suppose that given P_j, we have a rule which will uniquely define P_{j+1}. Then P_j is defined for all $j \in \mathbb{N}_0$.*

Proposition 3. *Let $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. Then there are unique sequences of integers q_0, q_1, ... and r_0, r_1, ... such that*

$$\begin{aligned} a &= bq_0 + r_0 \\ 10r_j &= bq_{j+1} + r_{j+1} \text{ for } j \geq 0 \\ r_j &\in \{0, \ldots, b-1\} \text{ for } j \geq 0. \end{aligned}$$

Proof. The mathematical object P_j that we want to define is the pair (q_j, r_j). We first check that P_0 is defined (the base case). This follows from Proposition 1 to a and b. Then we check that if P_j is defined, then P_{j+1} is defined. This follows by applying Proposition 1 to $10r_j$ and b. So by the principle of inductive definition, q_j and r_j are defined for all j. □

Now let's give a complete proof that q_j is between 0 and 9 (that is, it is a decimal digit).

Proposition 4. *For $j \geq 1$, we have $0 \leq q_j < 10$.*

Proof. Recall that $q_j b = 10r_{j-1} - r_j$. We know that $0 \leq r_{j-1} < b$ and $0 \leq r_j < b$ Therefore,

$$0 - b < 10r_{j-1} - r_j < 10b - 0.$$

We get strict inequalities on both sides, since in each case at least one of the inequalities we used was strict. Hence,

$$-b < q_j b < 10b.$$

Dividing by b yields $-1 < q_j < 10$, and hence $0 \leq q_j < 10$. □

Earlier, we showed that $a/b = m_0 + m_1/10 + m_2/10^2 + r_2/10^2 b$. The general formula for the first n digits will be

$$\frac{a}{b} = q_0 + \frac{q_1}{10} + \cdots + \frac{q_n}{10^n} + \frac{r_n}{10^n b} = \sum_{j=0}^{n} \frac{q_j}{10^j} + \frac{r_n}{10^n b}.$$

Remark: Here the Σ is a summation notation. It is a compact way of writing the sum of n terms without the .... In general, $\sum_{j=0}^{n} a_j$ means the same thing as $a_0 + a_1 + \cdots + a_n$. Actually, the sum of n terms is another object that is defined inductively. Although you intuitively know what $a_0 + a_1 + \cdots + a_n$ means, the rigorous definition (by induction on n) is that

$$\sum_{j=0}^{0} a_j = a_j$$
$$\sum_{j=0}^{n+1} a_j = \sum_{j=1}^{n} a_j + a_{n+1}.$$

Now let's return to the decimal expansion formula that we want to prove. Since the q_j's and r_j's are *defined* inductively, it would make sense for us to *prove* our claim inductively as well. We use the following fact which we state without proof.

Proposition 5 (Inductive proof)**.** *Suppose that for $n \in \mathbb{N}_0$, P_n is some mathematical statement. Suppose that P_0 is true and that P_n implies P_{n+1}. Then P_n is true for all $n \in \mathbb{N}_0$.*

Proposition 6. *Let $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ and let q_j and r_j be defined as above. Then for $n \geq 0$, we have*

$$\frac{a}{b} = \sum_{j=0}^{n} \frac{q_j}{10^j} + \frac{r_n}{10^n b}.$$

Proof. Let P_n be the equation above that we want to prove. In order to use induction, we'll prove that P_0 (the base case) and prove that P_n implies P_{n+1} (the inductive step). For the base case, the statement P_0 says that

$$\frac{a}{b} = q_0 + r_0,$$

and this is true because that is how we defined q_0 and r_0. For the induction step, assume P_n and then we'll prove P_{n+1}. Now P_n tells us that

$$\frac{a}{b} = \sum_{j=0}^{n} \frac{q_j}{10^j} + \frac{r_n}{10^n b}.$$

By our definition of r_{n+1} and q_{n+1}, we have $10r_n = q_{n+1}b + r_{n+1}$. Thus,

$$\frac{r_n}{10^n b} = \frac{10r_n}{10^{n+1}b} = \frac{q_{n+1}b + r_{n+1}}{10^{n+1}} = \frac{q_{n+1}}{10^{n+1}} + \frac{r_{n+1}}{10^{n+1}b}.$$

Therefore,

$$\frac{a}{b} = \sum_{j=0}^{n} \frac{q_j}{10^j} + \frac{q_{n+1}}{10^{n+1}} + \frac{r_{n+1}}{10^{n+1}b} = \sum_{j=0}^{n+1} \frac{q_j}{10^j} + \frac{r_{n+1}}{10^{n+1}b}. \qquad \square$$

The next statement we want to prove is that a/b is "approximately" $\sum_{j=0}^{n} m_j/10^j$. Rigorous statements about approximation are often written as estimates for how small the error is, such as the following proposition.

Proposition 7. *Let a, b and q_j, r_j be as above. Then*

$$\left| \frac{a}{b} - \sum_{j=0}^{n} \frac{q_j}{10^j} \right| < \frac{1}{10^n}.$$

Proof. The difference between the two terms is $r_n/10^n b$. Because $0 \leq r_n < b$, we have $r_n/10^n b < 1/10^n$. □

Finally, we'll show that a/b equals the infinite sum $\sum_{j=0}^{\infty} m_j/10^j$. But first, we have to explain what the infinite sum means. What we want to prove is really a statement about limits of real numbers. We recall the following definitions from basic analysis.

Definition 1. If $\{x_n\}$ is a sequence of real numbers, then we say $x_n \to x$ if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - x| < \epsilon$.

Definition 2. If $\{y_n\}$ is a sequence of real numbers, then we say $y = \sum_{j=0}^{\infty} y_j$ if $\sum_{j=0}^{n} y_j \to y$.

Proposition 8. *With the setup from above, we have*

$$\frac{a}{b} = \sum_{j=0}^{\infty} \frac{m_j}{10^j}.$$

Proof. Let $y_j = m_j/10^j$. Let $x_n = \sum_{j=0}^{n} m_j/10^j$. We want to show that $\sum_{j=0}^{\infty} y_j = a/b$. By definition, this means we must show that $x_n \to a/b$. And to prove this, we need to prove that for every $\epsilon > 0$, there exists N such that $n \geq N$ implies $|x_n - a/b| < \epsilon$. In order to check this claim for every ϵ, we'll start out by saying "Let $\epsilon > 0$" and then write an argument that works for every possible value of ϵ.

Let $\epsilon > 0$. Recall that $1/10^n \to 0$ (prove this as an exercise or look up why it is true). Because $1/10^n \to 0$, there exists N such that $n \geq N$ implies that $1/10^n < \epsilon$. We showed earlier that $|x_n - a/b| < 1/10^n$. Therefore, if $n \geq N$, then $|x_n - a/b| < \epsilon$. Now we're finished because for an arbitrary $\epsilon > 0$, we exhibited an N such that $n \geq N$ implies $|x_n - a/b| < \epsilon$. □