Notes on Darboux Integration

David Jekel

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Definitions

Definition 1. A *partition* of $[a, b]$ is a sequence of points $a = t_0 < t_1 < \cdots < t_n = b$. We can also view $\mathcal{P}$ as the set $\{t_0, t_1, \dots, t_n\}$.

Definition 2. Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function and $\mathcal{P} = \{t_0, t_1, \dots, t_n\}$ is a partition. Denote

$$M_j(f) = \sup_{t \in [t_{j-1}, t_j]} f(t),$$
$$m_j(f) = \inf_{t \in [t_{j-1}, t_j]} f(t).$$

We define the *upper and lower sums* of f with respect to $\mathcal{P}$ by

$$U_{\mathcal{P}}(f) = \sum_{j=1}^{n} \left(\sup_{t \in [t_{j-1}, t_j]} f(t) \right) (t_j - t_{j-1})$$
$$L_{\mathcal{P}}(f) = \sum_{j=1}^{n} \left(\inf_{t \in [t_{j-1}, t_j]} f(t) \right) (t_j - t_{j-1}).$$

Definition 3. We define the *upper and lower Darboux integrals* of f by

$$\overline{\int_a^b} f = \inf_{\mathcal{P}} U_{\mathcal{P}}(f)$$
$$\underline{\int_a^b} f = \sup_{\mathcal{P}} L_{\mathcal{P}}(f).$$

Definition 4. We say that f is *Darboux integrable* if the upper and lower Darboux integrals agree. In this case, we define $\int_a^b f$ to be the common value of $\overline{\int_a^b} f$ and $\underline{\int_a^b} f$.

Basic Properties

Lemma 5. *If $\mathcal{P}$ is a partition of $[a,b]$ and $f : [a,b] \to \mathbb{R}$ is bounded, then $L_{\mathcal{P}}(f) \leq U_{\mathcal{P}}(f)$.*

Proof. Let $\mathcal{P} = \{t_0, \ldots, t_n\}$ and let $M_j(f)$ and $m_j(f)$ be as above. It is immediate from the definition that $m_j(f) \leq M_j(f)$. Thus,

$$\sum_{j=1}^{n} m_j(f)(t_j - t_{j-1}) \leq \sum_{j=1}^{n} M_j(f)(t_j - t_{j-1}).$$

□

Definition 6. We say that a partition $\mathcal{P}'$ is a *refinement* of $\mathcal{P}$ if $\mathcal{P} \subseteq \mathcal{P}'$.

Observation 7. *If $\mathcal{P}_1$ and $\mathcal{P}_2$ are partitions of $[a,b]$, then $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ is also a partition of $[a,b]$. Moreover, $\mathcal{P}_1 \cup \mathcal{P}_2$ is a refinement of both $\mathcal{P}_1$ and $\mathcal{P}_2$.*

Lemma 8. *Let $f : [a,b] \to \mathbb{R}$ be bounded. Suppose $\mathcal{P}'$ is a refinement of $\mathcal{P}$. Then*

$$L_{\mathcal{P}}(f) \leq L_{\mathcal{P}'}(f) \leq U_{\mathcal{P}'}(f) \leq U_{\mathcal{P}}(f).$$

Proof. Let us denote by $\mathcal{P}' = \{t_0, \ldots, t_n\}$. Since $\mathcal{P} \subseteq \mathcal{P}'$, we must have $\mathcal{P} = \{t_{i_0}, \ldots, t_{i_m}\}$ where $m \leq n$ and $0 = i_0 < \cdots < i_m = n$ are a subset of the indices $1, \ldots, n$. Then we have

$$\begin{aligned}
\sum_{j=i_{k-1}+1}^{i_k} \left(\sup_{t \in [t_{j-1}, t_j]} f(t) \right) (t_j - t_{j-1}) &\leq \sum_{j=i_{k-1}+1}^{i_k} \left(\sup_{t \in [t_{i_{k-1}}, t_{i_k}]} f(t) \right) (t_j - t_{j-1}) \\
&= \left(\sup_{t \in [t_{i_{k-1}}, t_{i_k}]} f(t) \right) \sum_{j=i_{k-1}+1}^{i_k} (t_j - t_{j-1}) \\
&= \left(\sup_{t \in [t_{i_{k-1}}, t_{i_k}]} f(t) \right) (t_{i_{k-1}} - t_{i_k})
\end{aligned}$$

In the first step above, we have used the fact that $[t_{i_{k-1}}, t_{i_k}]$ contains $[t_{j-1}, t_j]$ and thus will have a larger sup for the values of f. Now summing the previous inequality from $k = 1, \ldots, m$, we have

$$\begin{aligned}
\sum_{j=1}^{n} \left(\sup_{t \in [t_{j-1}, t_j]} f(t) \right) (t_j - t_{j-1}) &= \sum_{k=1}^{m} \left(\sum_{j=i_{k-1}+1}^{i_k} \left(\sup_{t \in [t_{j-1}, t_j]} f(t) \right) (t_j - t_{j-1}) \right) \\
&\leq \sum_{k=1}^{m} \left(\sup_{t \in [t_{i_{k-1}}, t_{i_k}]} f(t) \right) (t_{i_{k-1}} - t_{i_k}).
\end{aligned}$$

This means exactly that

$$U_{\mathcal{P}'}(f) \leq U_{\mathcal{P}}(f).$$

The proof that $L_{\mathcal{P}'}(f) \geq L_{\mathcal{P}}(f)$ is symmetrical. And we already know from the previous lemma that $L_{\mathcal{P}'}(f) \leq U_{\mathcal{P}'}(f)$. □

Lemma 9. *Let $f : [a,b] \to \mathbb{R}$ be bounded. Then $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.*

Proof. Given partitions $\mathcal{P}_1$ and $\mathcal{P}_2$, we can let $\mathcal{P}$ be the common refinement. Then we have

$$L_{\mathcal{P}_1}(f) \leq L_{\mathcal{P}}(f) \leq U_{\mathcal{P}}(f) \leq U_{\mathcal{P}_2}(f).$$

This means that $U_{\mathcal{P}_2}(f)$ is an upper bound for the set $\{L_{\mathcal{P}_1}(f) : \mathcal{P}_1 \text{ a partition}\}$. Therefore,

$$\underline{\int_a^b} f = \sup_{\mathcal{P}_1} L_{\mathcal{P}_1}(f) \leq U_{\mathcal{P}_2}(f).$$

Similarly, $\underline{\int_a^b} f$ is a lower bound for the set of upper sums, and hence

$$\overline{\int_a^b} f = \inf_{\mathcal{P}_2} U_{\mathcal{P}_2}(f) \geq \underline{\int_a^b} f.$$

□

Lemma 10. *Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is Darboux integrable if and only if for every $\epsilon > 0$ there exists a partition $\mathcal{P}$ such that $U_{\mathcal{P}}(f) - L_{\mathcal{P}}(f) < \epsilon$.*

Proof. Suppose that f is Darboux integrable. Then $\underline{\int_a^b} f = \overline{\int_a^b} f$. Let $I = \int_a^b f$ be their common value. Because $\underline{\int_a^b} f$ is the supremum of all lower sums, there exists are partition $\mathcal{P}_1$ such that

$$L_{\mathcal{P}_1}(f) > I - \epsilon/2.$$

Similarly, there exists a partition $\mathcal{P}_2$ such that

$$U_{\mathcal{P}_2}(f) < I + \epsilon/2.$$

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then $L_{\mathcal{P}}(f) \geq L_{\mathcal{P}_1}(f)$ and $U_{\mathcal{P}}(f) \leq U_{\mathcal{P}_2}(f)$. Therefore,

$$U_{\mathcal{P}}(f) - L_{\mathcal{P}}(f) \leq U_{\mathcal{P}_2}(f) - L_{\mathcal{P}_1}(f) < (I + \epsilon/2) - (I - \epsilon/2) = \epsilon.$$

Conversely, suppose that for every $\epsilon > 0$, there exists a partition $\mathcal{P}$ such that $U_{\mathcal{P}}(f) - L_{\mathcal{P}}(f) < \epsilon$. If ϵ is given and we consider such a partition $\mathcal{P}$, then we have $\overline{\int_a^b} f \leq U_{\mathcal{P}}(f)$ and $\underline{\int_a^b} f \geq L_{\mathcal{P}}(f)$. Hence,

$$\overline{\int_a^b} f - \underline{\int_a^b} f \leq U_{\mathcal{P}}(f) - L_{\mathcal{P}}(f) < \epsilon.$$

Since this holds for every ϵ and since $\underline{\int_a^b} f \leq \overline{\int_a^b} f$ automatically, we conclude that $\underline{\int_a^b} f = \overline{\int_a^b} f$, and thus f is Darboux integrable. □

Monotonicity and Additivity Properties

Lemma 11. *Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions and suppose that $f \leq g$. Then*

$$\overline{\int_a^b} f \leq \overline{\int_a^b} g, \qquad \underline{\int_a^b} f \leq \underline{\int_a^b} g.$$

In particular, if f and g are integrable, then $\int_a^b f \leq \int_a^b g$.

Proof. Choose a partition $\mathcal{P} = \{t_0, \dots, t_n\}$ and let $M_j(f)$ and $m_j(f)$ be the corresponding suprema and infima on subintervals. It is clear that $M_j(f) \leq M_j(g)$. It follows that $U_{\mathcal{P}}(f) \leq U_{\mathcal{P}}(g)$. Then by taking the infimum over all $\mathcal{P}$, we obtain $\overline{\int_a^b} f \leq \overline{\int_a^b} g$. The argument for the lower integral is symmetrical. ☐

Lemma 12. *Let $f, g : [a, b] \to \mathbb{R}$ be bounded. Then*

$$\overline{\int_a^b} [f + g] \leq \overline{\int_a^b} f + \overline{\int_a^b} g$$

and

$$\underline{\int_a^b} [f + g] \geq \underline{\int_a^b} f + \underline{\int_a^b} g.$$

Moreover, if f and g are Darboux integrable, then $f + g$ is Darboux integrable and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

Proof. Let $\mathcal{P}$ be a partition and let M_j and m_j be the associated suprema and infima. For all $t \in [t_{j-1}, t_j]$, we have

$$f(t) + g(t) \leq M_j(f) + M_j(g).$$

Therefore, taking the supremum on the left hand side, we have $M_j(f + g) \leq M_j(f) + M_j(g)$. It follows that for every partition $\mathcal{P}$,

$$U_{\mathcal{P}}(f + g) \leq U_{\mathcal{P}}(f) + U_{\mathcal{P}}(g).$$

Moreover, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are partitions, then we have

$$\overline{\int_a^b} (f + g) \leq U_{\mathcal{P}_1 \cup \mathcal{P}_2}(f + g) \leq U_{\mathcal{P}_1 \cup \mathcal{P}_2}(f) + U_{\mathcal{P}_1 + \mathcal{P}_2}(g) \leq U_{\mathcal{P}_1}(f) + U_{\mathcal{P}_2}(g).$$

Thus, we have for all $\mathcal{P}_1$ and $\mathcal{P}_2$ that

$$\overline{\int_a^b} (f + g) \leq U_{\mathcal{P}_1}(f) + U_{\mathcal{P}_2}(g).$$

By taking the infimum over $\mathcal{P}_1$ and then the infimum over $\mathcal{P}_2$, we obtain

$$\overline{\int_a^b} (f + g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g.$$

A symmetrical argument proves our second claim that

$$\underline{\int_a^b}(f+g) \geq \underline{\int_a^b} f + \underline{\int_a^b} g.$$

Combining and rearranging these inequalities shows that

$$\begin{aligned}\overline{\int_a^b}(f+g) - \underline{\int_a^b}(f+g) &\leq \left(\overline{\int_a^b} f + \overline{\int_a^b} g\right) - \left(\underline{\int_a^b} f - \underline{\int_a^b} g\right) \\ &= \left(\overline{\int_a^b} f - \underline{\int_a^b} f\right) + \left(\overline{\int_a^b} g - \underline{\int_a^b} g\right)\end{aligned}$$

If we assume that f and g are Darboux integrable, then the right hand side is zero. This implies that the left hand side is zero (since it is nonnegative by Lemma 9). Thus, $f+g$ is Darboux integrable. $\square$

Lemma 13. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function, and let $c \in \mathbb{R}$ be a constant. If $c \geq 0$, then*

$$\overline{\int_a^b} cf = c\overline{\int_a^b} f, \qquad \underline{\int_a^b} cf = c\underline{\int_a^b} f.$$

If $c \leq 0$, then we have

$$\overline{\int_a^b} cf = c\underline{\int_a^b} f, \qquad \underline{\int_a^b} cf = c\overline{\int_a^b} f.$$

If f is Darboux integrable, then so is cf, and we have $\int_a^b cf = c\int_a^b f$.

Proof. If $c \leq 0$, and if $\mathcal{P}$ is a given partition, then we have $M_j(cf) = cM_j(f)$ and $m_j(cf) = cm_j(f)$. It follows that $U_{\mathcal{P}}(cf) = cU_{\mathcal{P}}(f)$ and $L_{\mathcal{P}}(cf) = cL_{\mathcal{P}}(f)$, and then the first claim follows by taking the infimum over upper partitions and the supremum over lower partitions of these equalities.

In the case where $c \leq 0$, we have $M_j(cf) = cm_j(f)$ and $m_j(cf) = cM_j(f)$, and the rest of the argument proceeds in the same way.

The final claim about Darboux integrability of f implying Darboux integrability of cf is immediate. $\square$

Lemma 14. *Suppose that $a < b < c$. Let $f : [a,c] \to \mathbb{R}$ be a bounded function. Then we have*

$$\overline{\int_a^c} f = \overline{\int_a^b} f + \overline{\int_b^c} f.$$

and

$$\underline{\int_a^c} f = \underline{\int_a^b} f + \underline{\int_b^c} f.$$

The function f is Darboux integrable on $[a,c]$ if and only if $f|_{[a,b]}$ and $f|_{[b,c]}$ are both Darboux integrable. In this case

$$\int_a^c f = \int_a^b f + \int_b^c f.$$

Proof. Suppose that $\mathcal{P}$ is a partition of $[a,c]$. Then $\mathcal{P}' = \mathcal{P} \cup \{b\}$ is also a partition of $[a,c]$. Moreover, $\mathcal{P}_1 = \mathcal{P}' \cap [a,b]$ is a partition of $[a,b]$ and $\mathcal{P}_2 = \mathcal{P}' \cap [b,c]$ is a partition of $[b,c]$. It follows from direct computation that

$$U_{\mathcal{P}'}(f) = U_{\mathcal{P}_1}(f|_{[a,b]}) + U_{\mathcal{P}_2}(f|_{[b,c]}).$$

By Lemma 8, we have $U_{\mathcal{P}}(f) \geq U_{\mathcal{P}'}(f)$. Also, by definition of the upper integral, $U_{\mathcal{P}_1}(f|_{[a,b]}) \geq \overline{\int}_a^b f$ and $U_{\mathcal{P}_2}(f|_{[b,c]}) \geq \overline{\int}_b^c f$. Altogether,

$$U_{\mathcal{P}}(f) \geq \overline{\int}_a^b f + \overline{\int}_b^c f.$$

Since $\mathcal{P}$ was arbitrary, we can take the infimum on the left hand side to get $\overline{\int}_a^c f \geq \overline{\int}_a^b f + \overline{\int}_b^c f$.

To prove the opposite inequality, consider partitions $\mathcal{P}_1$ of $[a,b]$ and $\mathcal{P}_2$ of $[b,c]$. Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ and note that this is a partition of $[a,c]$. Then we have

$$U_{\mathcal{P}}(f) = U_{\mathcal{P}_1}(f|_{[a,b]}) + U_{\mathcal{P}_2}(f|_{[b,c]}),$$

and in particular

$$\overline{\int}_a^c f \leq U_{\mathcal{P}_1}(f|_{[a,b]}) + U_{\mathcal{P}_2}(f|_{[b,c]}).$$

Since $\mathcal{P}_1$ and $\mathcal{P}_2$ were arbitrary, we obtain

$$\overline{\int}_a^c f \leq \overline{\int}_a^b f + \overline{\int}_b^c f.$$

Thus, we have shown that $\overline{\int}_a^c f = \overline{\int}_a^b f + \overline{\int}_b^c f$. A symmetrical argument shows that $\underline{\int}_a^c f = \underline{\int}_a^b f + \underline{\int}_b^c f$.

It follows that

$$\overline{\int}_a^c f - \underline{\int}_a^c f = \left(\overline{\int}_a^b f - \underline{\int}_a^b f\right) + \left(\overline{\int}_b^c f - \underline{\int}_b^c f\right).$$

Each of the three numbers $\overline{\int}_a^c f - \underline{\int}_a^c f$ and $\overline{\int}_a^b f - \underline{\int}_a^b f$ and $\overline{\int}_b^c f - \underline{\int}_b^c f$ is nonnegative. Thus, the number on the left hand side is zero if and only if *both* of the numbers on the right hand side are zero. It follows that f is Darboux integrable on $[a,c]$ if and only if $f|_{[a,b]}$ and $f|_{[b,c]}$ are both Darboux integrable. Moreover, in this case it is immediate that $\int_a^c f = \int_a^b f + \int_b^c f$. □

Integration and Continuous Functions

Lemma 15. *If $f : [a, b] \to \mathbb{R}$ is continuous, then f is Darboux integrable.*

Proof. Because f is a continuous function and $[a, b]$ is compact, we know that f achieves a maximum and a minimum, and hence it is bounded. Moreover, because $[a, b]$ is compact, f is uniformly continuous. Thus, given $\epsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon/(b - a)$ for all $x, y \in [a, b]$. Choose an integer n such that $(b - a)/n < \delta$. Let $\mathcal{P}$ be the partition $\{t_0, \dots, t_n\}$ where $t_j = a + (b - a)j/n$.

For $t, t' \in [t_{j-1}, t_j]$, we have $|t - t'| \le (b - a)/n < \delta$ and hence $|f(t) - f(t')| < \epsilon$. In particular,

$$f(t) \le f(t') + \frac{\epsilon}{b - a}.$$

Now we take the supremum over $t \in [t_{j-1}, t_j]$ on the left hand side and the infimum over $t' \in [t_{j-1}, t_j]$ on the right hand side. This shows that

$$M_j(f) \le m_j(f) + \frac{\epsilon}{b - a}.$$

Therefore, we have

$$\sum_{j=1}^{n} M_j(f)(t_j - t_{j-1}) \le \sum_{j=1}^{n} m_j(f)(t_j - t_{j-1}) + \sum_{j=1}^{n} \frac{\epsilon}{b - a}(t_j - t_{j-1})$$

which means that

$$U_{\mathcal{P}}(f) \le L_{\mathcal{P}}(f) + \frac{\epsilon}{b - a}(b - a).$$

Hence, $U_{\mathcal{P}}(f) - L_{\mathcal{P}}(f) \le \epsilon$. Since ϵ was arbitrary, f is Darboux integrable by Lemma 10. $\square$