Compactness in Metric Spaces

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Abstract

This note, developed for Math 334 at UW and Math 131B at UCLA, explains some well-known and fundamental results about compactness in metric spaces. First, we prove that a subset of a metric space is compact if and only if it is sequentially compact if and only if it is complete and totally bounded. Second, we describe several consequences of compactness, providing two parallel proofs for each result, one with open covers and one with sequences. Third, we prove the Arzela-Ascoli theorem, which characterizes when a subset of $C(X;\mathbb{R})$ is compact.

Contents

1 Three Equivalent Definitions of Compactness

Our goal in this section is to show that three different definitions of compactness are equivalent to each other.

Definition 1.1. Let (X, d) be a metric space and $E \subseteq X$. An *open cover* of E is a collection of open sets $(U_\alpha)_{\alpha \in I}$ (indexed by some set I) such that $E \subseteq \bigcup_{\alpha \in I} U_\alpha$. Given an open cover $(U_\alpha)_{\alpha \in I}$, a *subcover* is a subcollection $(U_\alpha)_{\alpha \in J}$ given by some $J \subseteq I$, such that $E \subseteq \bigcup_{\alpha \in J} U_\alpha$. We say that an open cover is *finite* if the index set is finite.

Definition 1.2. We say that $E \subseteq X$ is *compact* if every open cover has a finite subcover. In other words, if $(U_\alpha)_{\alpha \in E}$ is a collection of open sets with $E \subseteq \bigcup_{\alpha \in I} U_\alpha$, then there exists a finite set $F \subseteq I$ such that $E \subseteq \bigcup_{\alpha \in F} U_\alpha$.

Definition 1.3. We say that $E \subseteq X$ is *sequentially compact* if for every sequence $(x_n)_{n \in \mathbb{N}}$ in E, there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ that converges to some point $x_0 \in E$.

Definition 1.4. We say that $E \subseteq X$ is *complete* if every Cauchy sequence in E converges to some point in E.

Definition 1.5. We say that $E \subseteq X$ is *totally bounded* if for every $r > 0$, there exist a finite set F of points in E such that $E \subseteq \bigcup_{x \in F} B(x, r)$.

Theorem 1.6. *Let (X, d) be a metric space and let $E \subseteq X$. Then the following are equivalent:*

(a) E is compact.

(b) E is sequentially compact.

(c) E is complete and totally bounded.

In the next three subsections, we will show (a) $\implies$ (c), (c) $\implies$ (b), and (b) $\implies$ (a), which will prove the theorem.

1.1 Compact implies Complete and Totally Bounded

Lemma 1.7. *Let (X, d) be a metric space and $E \subseteq X$. The following are equivalent:*

(a) E is compact.

(b) Suppose that $(K_\alpha)_{\alpha \in I}$ is a collection of closed subsets of X. Suppose that for every finite $F \subseteq I$, we have

$$E \cap \bigcap_{\alpha \in F} K_\alpha \neq \varnothing.$$

Then we have

$$E \cap \bigcap_{\alpha \in I} K_\alpha \neq \varnothing.$$

Proof. By taking the contrapositive, condition (b) is equivalent to the following: If $(K_\alpha)_{\alpha\in I}$ is a collection of closed subsets of X and if $E \cap \bigcap_{\alpha\in I} K_\alpha = \varnothing$, then there exists a finite $F \subseteq I$ such that $E \cap \bigcap_{\alpha\in F} K_\alpha = \varnothing$.

Recall that a set is open if and only if the complement is closed. Note that there is a one-to-one correspondence between collections of open sets $\{U_\alpha\}_{\alpha\in I}$ and collections of closed sets $(K_\alpha)_{\alpha\in I}$, given by $K_\alpha = U_\alpha^c$. Moreover, using DeMorgan's laws,

$$\left(\bigcup_{\alpha\in I} U_\alpha\right)^c = \bigcap_{\alpha\in I} K_\alpha.$$

It follows that

$$E \subseteq \bigcup_{\alpha\in I} U_\alpha \iff E \cap \bigcap_{\alpha\in I} K_\alpha = \varnothing.$$

Similarly, if F is a finite subset of I, then

$$E \subseteq \bigcup_{\alpha\in F} U_\alpha \iff E \cap \bigcap_{\alpha\in F} K_\alpha = \varnothing.$$

Therefore, if we rewrite condition (b) in terms of the collection of open sets $(U_\alpha)_{\alpha\in I}$ rather than the closed sets $(K_\alpha)_{\alpha\in I}$, it means that if $(U_\alpha)_{\alpha\in I}$ is a collection of open sets in X and if $E \subseteq \bigcup_{\alpha\in I} U_\alpha$, then there exists a finite $F \subseteq I$ such that $E \subseteq \bigcup_{\alpha\in F} U_\alpha$. This is exactly the definition of compactness. □

Lemma 1.8. *If $E \subseteq X$ is compact, then E is complete.*

Proof. Assume that E is a compact. To prove completeness, suppose that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E. Define

$$\gamma_n = \sup_{m\geq n} d(x_m, x_n).$$

Note that $\gamma_n \to 0$ as $n \to \infty$. Indeed, because $(x_n)_{n\in\mathbb{N}}$ is Cauchy, for every $\epsilon > 0$, there exists N such that

$$m, n \geq N \implies d(x_m, x_n) < \epsilon.$$

Then if $n \geq N$, we have

$$\sup_{m\geq n} d(x_m, x_n) \leq \epsilon.$$

Thus, $n \geq N \implies \gamma_n \leq \epsilon$. This implies that $\gamma_n \to 0$ (since $\gamma_n \geq 0$ obviously).

Now define $K_n = \{x \in X : d(x, x_n) \leq \gamma_n\}$, that is, K_n is the closed ball of radius of γ_n. We want to apply the previous lemma to the collection $(K_n)_{n\in\mathbb{N}}$. Note that K_n is a closed set (exercise). Suppose that $F \subseteq \mathbb{N}$ and let $m = \max F$. Then for each $n \in F$, we have $m \geq n$ and hence by definition of γ_n, we have $d(x_m, x_n) \leq \gamma_n$, hence $x_m \in K_n$. It follows that $x_m \in E \cap \bigcap_{n\in F} K_n$. Therefore, we have $E \cap \bigcap_{n\in F} K_n \neq \varnothing$ for every finite $F \subseteq \mathbb{N}$.

By the previous lemma, since E is compact, we know that $E \cap \bigcap_{n\in\mathbb{N}} K_n \neq \varnothing$. Let $x_0 \in E \cap \bigcap_{n\in\mathbb{N}} K_n$. Then by definition of K_n, we have $d(x_n, x_0) \leq \gamma_n$. Since $\gamma_n \to 0$, we have $x_n \to x_0$. Therefore, every Cauchy sequence in E converges to a point in E as desired. □

Lemma 1.9. *If $E \subseteq X$ is compact, then E is totally bounded.*

Proof. Suppose that E is compact. Let $r > 0$. Note that $(B(y, r))_{y \in E}$ is an open cover of E. Indeed, we know that an open ball $B(x, r)$ is an open set (exercise), and the collection $\{B(x, r)\}_{x \in E}$ covers E because each x is contained in the corresponding ball $B(x, r)$. By compactness, there exists a finite $F \subseteq E$ such that $E \subseteq \bigcup_{x \in F} B(x, r)$. This means precisely that E is totally bounded. □

1.2 Complete and Totally Bounded Implies Sequentially Compact

Lemma 1.10. *Suppose that $E \subseteq X$ is complete and totally bounded. Then E is sequentially compact.*

Proof. Let $(x_n)_{n \in \mathbb{N}}$ be an arbitrary sequence in E, and we will show that there is a subsequence converging to some point $x_0 \in E$.

We define infinite sets $S_k \subseteq \mathbb{N}$ by induction on k as follows. For the base case, let $S_0 = \mathbb{N}$. For the inductive step, suppose that S_{k-1} has been chosen. Because E is a totally bounded, there exists a finite set F_k such that $E \subseteq \bigcup_{y \in F_k} B(y, 1/k)$. For each $n \in S_{k-1}$, the point x_n must be in one of the balls $B(y, 1/k)$. Because S_{k-1} is infinite but there are only finitely many balls, there must be one ball that contains x_n for infinitely many values of $n \in S_{k-1}$. Let us call this ball $B(y_k, 1/k)$ and let $S_k = \{n \in S_{k-1} : x_n \in B(y_k, 1/k)\}$. Note that S_k is infinite by construction. We also have $S_k \subseteq S_{k-1}$.

Now we choose the indices n_k for our subsequence inductively. For the base case, let $n_0 = 1$. For the inductive step, once n_{k-1} has been chosen, we may select $n_k \in S_k$ such that $n_k > n_{k-1}$ (because S_k is infinite). If $j, k \geq K$, then we have $S_j \subseteq S_K$ and $S_k \subseteq S_K$, and hence $n_j, n_k \in S_K$, which implies that $x_{n_j}, x_{n_k} \in B(y_K, 1/K)$. But if $x_{n_j}, x_{n_k} \in B(y_K, 1/K)$, then

$$d(x_{n_j}, x_{n_k}) \leq d(x_{n_j}, y_K) + d(y_K, x_{n_k}) \leq \frac{1}{K} + \frac{1}{K} = \frac{2}{K}.$$

Overall, we have for natural numbers j, k, and K that

$$j, k \geq K \implies d(x_{n_j}, x_{n_k}) \leq \frac{2}{K}.$$

Given $\epsilon > 0$, we may choose K such that $2/K < \epsilon$. Then

$$j, k \geq K \implies d(x_{n_j}, x_{n_k}) \leq \frac{2}{K} < \epsilon.$$

This means that $(x_{n_k})_{k \in \mathbb{N}}$ is a Cauchy sequence. Therefore, by completeness of E, $(x_{n_k})_{k \in \mathbb{N}}$ converges to some $x_0 \in E$. Therefore, $(x_n)_{n \in \mathbb{N}}$ has a convergent subsequence as desired. □

1.3 Sequentially Compact Implies Compact

Lemma 1.11. *Suppose that $E \subseteq X$ is sequentially compact. Then E is totally bounded.*

Proof. We proceed by contrapositive. Suppose that E is not totally bounded. Then there exists an $r > 0$ such that no finite collection of balls of radius r will cover E. Now we construct a sequence $(x_n)_{n\in\mathbb{N}}$ by induction. Note that E must be nonempty, so we can choose $x_1 \in E$. For the induction step, suppose that $x_1, \ldots, x_{n-1}$ have been chosen. Then the balls $B(x_1, r), \ldots, B(x_{n-1}, r)$ do not cover E, and therefore, we may choose some x_n in $E \setminus \bigcup_{j=1}^{n-1} B(x_j, r)$.

By construction, if $m > n$, then $x_m \notin B(x_n, r)$ and hence $d(x_m, x_n) \geq r$. It follows that if $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$, then $d(x_{n_j}, x_{n_k}) \geq r$ for $j \neq k$. Therefore, any subsequence of $(x_n)_{n\in\mathbb{N}}$ cannot be Cauchy and hence it cannot converge. Thus, $(x_n)_{n\in\mathbb{N}}$ has no convergent subsequence, so E is not sequentially compact. □

Lemma 1.12. *Suppose that $E \subseteq X$ is sequentially compact, and let $(U_\alpha)_{\alpha\in I}$ is an open cover of E. Then there exists some $r > 0$, such that for every $x \in E$, the ball $B(x, r)$ is contained in one of the sets U_α.*

Proof. We proceed by contradiction. Assume that X is sequentially compact but that the conclusion fails. Then for every $r > 0$, there exists some $x \in E$ such that $B(x, r)$ is not contained in one of the sets U_α. In particular, for each $n \in \mathbb{N}$, there exists $x_n \in E$ such that $B(x_n, 1/n)$ is not contained in one of the sets U_α.

Because E is sequentially compact, the sequence $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which converges to some $x_0 \in E$. Because $(U_\alpha)_{\alpha\in I}$ is an open cover, there exists some index α such that $x_0 \in U_\alpha$. Because U_α is open, there exists some $r > 0$ such that $B(x_0, r) \subseteq U_\alpha$. Since $x_{n_k} \to x_0$ and $1/n_k \to 0$, we have

$$\lim_{k\to\infty} \left(d(x_{n_k}, x_0) + \frac{1}{n_k} \right) = 0.$$

In particular, there exists some k such that

$$d(x_{n_k}, x_0) + \frac{1}{n_k} < r.$$

Now if $y \in B(x_{n_k}, 1/n_k)$, then we have

$$d(y, x_0) \leq d(y, x_{n_k}) + d(x_{n_k}, x_0) \leq \frac{1}{n_k} + d(x_{n_k}, x_0) < r,$$

and hence $y \in B(x_0, r)$; this implies that

$$B(x_{n_k}, 1/n_k) \subseteq B(x_0, r) \subseteq U_\alpha.$$

However, by our choice of x_n, we know that $B(x_{n_k}, 1/n_k)$ cannot be contained in any set U_α. This is a contradiction, so the proof is complete. □

Lemma 1.13. *If $E \subseteq X$ is sequentially compact, then E is compact.*

Proof. Assume that E is sequentially compact. To prove compactness, suppose that $(U_\alpha)_{\alpha \in I}$ is an open cover of E. By Lemma 1.12, there exists an $r > 0$ such that for every $x \in E$, the ball $B(x, r)$ is contained in one of the sets U_α. By Lemma 1.11, E is totally bounded, and hence there exists a finite set $F \subseteq E$ such that $E \subseteq \bigcup_{x \in F} B(x, r)$. For each $x \in F$, there exists an index $\alpha_x \in I$ such that $B(x, r) \subseteq U_{\alpha_x}$, by our choice of r. Therefore, $E \subseteq \bigcup_{x \in F} U_{\alpha_x}$, and hence $(U_{\alpha_x})_{x \in F}$ is our desired finite subcover. □

1.4 Corollaries; the Heine-Borel Theorem

In the special case $X = \mathbb{R}^n$, the general Theorem 1.6 reduces to the Heine-Borel theorem, which says that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. The point is that the conditions of completeness and total boundedness can be expressed in a simpler way if E is a subset of $\mathbb{R}^n$. We now explain how to deduce this special case, as well as making other general observations.

Lemma 1.14. *Let X be a metric space and $E \subseteq X$. If E is complete, then E is closed. The converse holds if X is complete.*

Proof. For the first claim, suppose that E is complete. To show that E is closed, suppose that $(x_n)_{n \in \mathbb{N}}$ is a sequence in E that converges in X to some point $x_0 \in X$, and we will show $x_0 \in E$. Since $(x_n)_{n \in \mathbb{N}}$ is convergent in X, it is a Cauchy sequence. By assumption $(x_n)_{n \in \mathbb{N}}$ must converge to some point x in E. The limit of a sequence in X is unique and hence $x = x_0$. Thus, $x_0 = x \in E$ as desired.

For the second claim, suppose that X is complete and E is closed. To show that E is complete, suppose that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in E and we will show that x_n is convergent in E. Then $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in X. Therefore, it converges to some $x_0 \in X$. Because E is closed and $x_n \to x_0$, we know $x_0 \in E$. Therefore, $x_n \to x_0$ in E. □

Lemma 1.15. *Let X be a metric space and $E \subseteq X$. If E is totally bounded, then E is bounded. The converse holds if $X = \mathbb{R}^d$.*

Proof. Since E is totally bounded, there exist finitely many balls of radius 1 that cover E; call them $B(x_1, 1), \ldots, B(x_n, 1)$. Pick some point $x_0 \in E$ and let $R = \max(d(x_0, x_j)) + 1$. Then for each $x \in E$, we have $x \in B(x_j, 1)$ for some j and by the triangle inequality $d(x, x_0) < R$. Therefore, $E \subseteq B(x_0, R)$, so E is bounded.

For the second claim, suppose that E is a bounded subset of $\mathbb{R}^d$. Let $r > 0$. Because E is bounded, there exists some $M > 0$ such that $E \subseteq [-M, M]^d$. For each $n \in \mathbb{N}$, we may subdivide $[-M, M]^d$ in a grid-like fashion into $(2n)^d$ (d-dimensional) closed cubes of side length M/n. Each such cube is contained in an open ball of radius $d^{1/2}(M/n)$ about the center point of the cube. Thus, if we choose n large enough, that $Md^{1/2}/n < r$, then we have covered E by finitely many balls of radius of r. □

Theorem 1.16 (Heine-Borel)**.** *A subset of $\mathbb{R}^d$ is compact if and only if it is closed and bounded.*

Proof. Let $E \subseteq \mathbb{R}^d$. We know that E is compact if and only if it is complete and totally bounded. Because $\mathbb{R}^n$ is complete, we know that E is complete if and only if it is closed. Moreover, by the previous lemma, E is totally bounded if and only if it is bounded. Hence, E is compact if and only if it is closed and bounded. □

Remark. For a general metric space, a compact set must be closed and bounded, but the converse is not true.

The next results allow us to test when $\overline{E}$ is compact.

Lemma 1.17. *Let X be a metric space and $E \subseteq X$. If E is totally bounded, then $\overline{E}$ is totally bounded.*

Proof. Let $r > 0$. Since E is totally bounded, E can be covered by finitely many balls $(B(x, r/2))_{x \in F}$. Let $\overline{B}(x, r/2)$ be the closed ball of radius $r/2$, and recall that this is a closed set. Since a finite union of closed sets is closed, we know $\bigcup_{x\in F} \overline{B}(x, r/2)$ is closed. Now since $E \subseteq \bigcup_{x\in F} \overline{B}(x, r/2)$, which is closed, we know that $\overline{E} \subseteq \bigcup_{x\in F} \overline{B}(x, r/2)$. Clearly, $\overline{B}(x, r/2) \subseteq B(x, r)$ and hence $\overline{E} \subseteq \bigcup_{x\in F} B(x, r)$. Thus, $\overline{E}$ is totally bounded. □

Lemma 1.18. *Suppose that X is a complete metric space. If $E \subseteq X$ is totally bounded, then $\overline{E}$ is compact.*

Proof. By the previous lemma, $\overline{E}$ is totally bounded. Also, since X is complete and $\overline{E}$ is closed, we know $\overline{E}$ is complete. Thus, $\overline{E}$ is complete and totally bounded, hence compact. □

2 Consequences of Compactness

In this section, we prove various well-known and useful consequences of compactness (see for instance [**?**, chapter 2] and [**?**]). We present two proofs of each result, one using open covers and one using sequences. By examining these parallel proofs, we hope the reader will get a better intuitive grasp on why compactness and sequential compactness are equivalent.

2.1 Closed Subsets of Compact Sets

Proposition 2.1. *Suppose that X is a compact metric space and $E \subseteq X$ is closed. Then E is compact.*

Covering proof. Suppose that $(U_\alpha)_{\alpha\in I}$ is an open cover of E. Since E is closed $E^c = X \setminus E$ is open. Thus, $(U_\alpha)_{\alpha\in I} \cup \{E^c\}$ is an open cover of X, since $X = E \cup E^c \subseteq \bigcup_{\alpha\in I} U_\alpha \cup E^c$. Since X is compact, there exists a finite subcover of X. This finite subcover will certainly cover E. If this finite subcover includes the E^c, we may delete E^c from it and the remaining sets will still cover E. Thus, $(U_\alpha)_{\alpha\in I}$ contains a finite subcover of E. □

Sequential proof. Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in E. Then $(x_n)_{n\in\mathbb{N}}$ is also a sequence in X, and by compactness of X, there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges to some $x_0 \in E$. Because E is closed, we know that $x_0 \in E$. Therefore, $(x_n)_{n\in\mathbb{N}}$ has a subsequence that is convergent in E. Because $(x_n)_{n\in\mathbb{N}}$ was an arbitrary sequence in E, we know that E is compact. □

2.2 Images Under Continuous Functions

Proposition 2.2. *If X and Y are metric spaces, $f : X \to Y$ is continuous, and $E \subseteq X$ is compact, then $f(E)$ is compact.*

Covering proof. Suppose that $(U_\alpha)_{\alpha\in I}$ is an open cover of $f(E)$. Because f is continuous, we know that $f^{-1}(U_\alpha)$ is open in X. Moreover, $(U_\alpha)_{\alpha\in E}$ covers E, because if $x \in E$, then $f(x)$ is in one of the U_α's, and hence $x \in f^{-1}(U_\alpha)$.

Because E is compact, there exists a finite set $F \subseteq I$ such that $E \subseteq \bigcup_{\alpha\in F} f^{-1}(U_\alpha)$. Then we claim that $(U_\alpha)_{\alpha\in F}$ cover $f(E)$. Indeed, if $y \in f(E)$, then $y = f(x)$ for some $x \in E$. By assumption x is contained in $f^{-1}(U_\alpha)$ for some $\alpha \in F$. But that means that $f(x) \in U_\alpha$, that is, $y \in U_\alpha$. Therefore, $(U_\alpha)_{\alpha\in F}$ is the desired finite subcover. □

Sequential proof. Suppose that $(y_n)_{n\in\mathbb{N}}$ is a sequence in $f(E)$. Then by definition of $f(E)$, we have $y_n = f(x_n)$ for some $x_n \in E$. Because E is compact, the sequence $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges to some point $x_0 \in E$. By continuity of f, we have $f(x_{n_k}) \to f(x_0)$, which means that $y_{n_k} \to f(x_0) \in f(E)$. Thus, $(y_n)_{n\in\mathbb{N}}$ has a convergent subsequence as desired. □

Proposition 2.3. *Let X be a metric space and $f : X \to \mathbb{R}$ be continuous. Then f achieves a maximum and a minimum on X and hence is a bounded function.*

Proof from the previous result. Because f is continuous and X is compact, the previous proposition shows that $f(X)$ is a compact subset of $\mathbb{R}$. So $f(X)$ is closed and bounded. Since $f(X)$ is bounded, it has a finite supremum and infimum. Because $f(X)$ is closed, the supremum and infimum must be in the set $f(X)$, and hence they are achieved by the function f. □

Direct sequential proof. By basic properties of the supremum, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $f(x_n) \to \sup_{x\in X} f(x)$. By compactness, the sequence $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges to some $x_0 \in X$. Then $f(x_{n_k}) \to f(x_0)$. Hence, $f(x_0) = \sup_{x\in X} f(x)$, so f achieves a maximum. Similarly, f achieves a minimum. □

2.3 Continuity Implies Uniform Continuity

Proposition 2.4. *Suppose that X and Y are metric spaces, X is compact, and $f : X \to Y$ is continuous. Then f is uniformly continuous.*

Covering Proof. To prove uniform continuity, choose some $\epsilon > 0$. Then for each $\xi \in X$, there exists $\delta_\xi > 0$ such that

$$d(x, \xi) < \delta \implies d(f(x), f(\xi)) < \epsilon/2.$$

Note that the balls $(B_X(\xi, \delta_\xi/2))_{\xi \in X}$ are an open cover of X. By compactness, there exists a finite set $F \subseteq X$ such that $X = \bigcup_{\xi \in F} B_X(\xi, \delta_\xi)$. Let $\delta = \min_{\xi \in F} \delta_\xi/2$.

Suppose that $x, x' \in X$ with $d(x, x') < \delta$. Then x must be in $B_X(\xi, \delta_\xi/2)$ for some $\xi \in F$. Hence,

$$d(x, \xi) < \delta_\xi/2 < \delta_\xi$$

and

$$d(x', \xi) \leq d(x', x) + d(x, \xi) < \delta + \delta_\xi/2 \leq \delta_\xi.$$

Therefore, by our choice of δ_ξ, we have $d(f(x), f(\xi)) < \epsilon/2$ and $d(f(x'), f(\xi)) < \epsilon/2$. So by the triangle inequality, $d(f(x'), f(x)) < \epsilon$.

Overall, we have obtained a δ such that $d(x, x') < \delta$ implies $d(f(x), f(x')) < \epsilon$. Thus, f is uniformly continuous. □

Sequential proof. Suppose for the sake of contradiction that f is not uniformly continuous. That means that there exists an $\epsilon > 0$ such that for every $\delta > 0$, there exist x and x' such that $d(x, x') < \delta$ but $d(f(x), f(x')) \geq \epsilon$. In particular, for each $n \in \mathbb{N}$, we may take $\delta = 1/n$, and thus there exist x_n and x'_n such that $d(x_n, x'_n) < 1/n$ but $d(f(x_n), f(x'_n)) \geq \epsilon$.

By compactness, the sequence $(x_n)_{n \in \mathbb{N}}$ has a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ that converges to a point $x_0 \in X$. Note that

$$d(x'_{n_k}, x_0) \leq d(x'_{n_k}, x_{n_k}) + d(x_{n_k}, x_0) \leq \frac{1}{n_k} + d(x_{n_k}, x_0) \to 0.$$

Therefore, $(x'_{n_k})_{k \in \mathbb{N}}$ also converges to x_0. Because f is continuous, we have

$$f(x_{n_k}) \to f(x_0), \qquad f(x'_{n_k}) \to f(x_0).$$

Since the distance function is continuous, it follows that

$$d(f(x_{n_k}), f(x'_{n_k})) \to d(f(x_0), f(x_0)) = 0.$$

But this contradicts the fact that $d(f(x_{n_k}), f(x'_{n_k})) \geq \epsilon$ for all k. □

2.4 Uniform Convergence of Monotone Sequences

Proposition 2.5. *Let X be a compact metric space. Suppose that $f_n : X \to [0, +\infty)$ is continuous, $f_{n+1} \leq f_n$, and $f_n \to 0$ pointwise. Then $f_n \to 0$ uniformly.*

Covering proof. Fix $\epsilon > 0$. Let $U_n = \{x \in X : f_n(x) < \epsilon\}$. Then $U_n = f_n^{-1}((-\infty, \epsilon))$ and hence U_n is open. Note that $(U_n)_{n\in\mathbb{N}}$ is an open cover of X; indeed, if $x \in X$, then $f_n(x) \to 0$ and hence $f_n(x) < \epsilon$ for sufficiently large n, which means that $x \in U_n$.

By compactness $(U_n)_{n\in\mathbb{N}}$ has a finite subcover, so there exists $F \subseteq \mathbb{N}$ finite with $X = \bigcup_{n\in F} U_n$. Now because $f_{n+1} \leq f_n$, we have $U_{n+1} \subseteq U_n$. Thus, if we let $N = \max F$, then $X = \bigcup_{n\in F} U_n = U_N$. Now if $n \geq N$, then $f_n(x) \leq f_N(x) < \epsilon$ for all x. Thus, $f_n \to 0$ uniformly. □

Sequential proof. Suppose for contradiction that f_n does not converge uniformly to zero. Then there exists an $\epsilon > 0$ such that for every N, there exists $n \geq N$ and $x \in X$ such that $f_n(x) \geq \epsilon$. Now given N, we know there exists $n \geq N$ and $x \in X$ such that $f_n(x) \geq \epsilon$, but we also have $f_N(x) \geq f_n(x)$. Thus, for every N, there exists x_N such that $f_N(x_N) \geq \epsilon$.

By compactness, the sequence $(x_n)_{n\in\mathbb{N}}$ must have a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges to some $x_0 \in X$. Now since $\lim_{n\to\infty} f_n(x_0) = 0$, there exists N such that $f_n(x_0) < \epsilon$ for $n \geq N$ and in particular, $f_N(x_0) < \epsilon$. Because f_N is continuous, we have

$$\lim_{k\to\infty} f_N(x_{n_k}) = f_N(x_0) < \epsilon.$$

In particular, for sufficiently large k, we have $f_N(x_{n_k}) < \epsilon$. But if k is sufficiently large, we also have $n_k \geq N$ and hence

$$f_{n_k}(x_{n_k}) \leq f_N(x_{n_k}) < \epsilon,$$

which contradicts our choice of $(x_n)_{n\in\mathbb{N}}$. □

3 The Arzela-Ascoli Theorem

3.1 The Space of Continuous Functions

Let X be a compact metric space. Let $C(X;\mathbb{R})$ denote the space of continuous functions $X \to \mathbb{R}$. For $f \in C(X;\mathbb{R})$, denote

$$\|f\|_\infty = \sup_{x\in X} |f(x)|.$$

Because X is compact, a continuous function must achieve a maximum and a minimum and therefore $\|f\|_\infty$ is always finite. One can check that $\|\cdot\|_\infty$ satisfies the following axioms:

1. $\|f\|_\infty = 0$ if and only if $f = 0$.
2. $\|cf\|_\infty = |c| \|f\|_\infty$ for $c \in \mathbb{R}$.
3. $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

In other words, $(C(X;\mathbb{R}), \|\cdot\|_\infty)$ is a *normed vector space*. It follows from these axioms that

$$d_\infty(f,g) := \|f-g\|_\infty$$

defines a metric on $C(X;\mathbb{R})$.

Lemma 3.1. *$C(X;\mathbb{R})$ is complete with respect to the metric d_∞.*

Proof. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. For each $x \in X$ and for each $m, n \in \mathbb{N}$, we have

$$|f_m(x) - f_n(x)| \le d_\infty(f,g).$$

It follows that for each $x \in X$, the sequence $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence of real numbers. Because $\mathbb{R}$ is complete, this sequence converges. We define $f(x) = \lim_{n\to\infty} f_n(x)$.

By construction f_n converges pointwise to f, but we claim that in fact f_n converges *uniformly* to f. Suppose that $\epsilon > 0$. Then there exists N such that

$$m, n \ge N \implies d_\infty(f_m, f_n) \le \epsilon.$$

In particular, if $m, n \ge N$, then for each $x \in X$, we have

$$|f_m(x) - f_n(x)| \le \epsilon.$$

We know that $f_m(x) \to f(x)$ as $m \to \infty$ and hence for $n \ge N$ and $x \in X$,

$$|f(x) - f_n(x)| \le \epsilon.$$

Since this holds for all x, we may take the supremum over X and hence

$$n \ge N \implies \|f_n - f\|_\infty \le \epsilon.$$

Since ϵ was arbitrary, we have shown that $f_n \to f$ uniformly.

Next, we show that f is continuous. Suppose that $x_0 \in X$ and $\epsilon > 0$. By uniform convergence, there exists N such that $n \ge N$ implies that $\|f_n - f\|_\infty \le \epsilon/3$. In particular, $\|f_N - f\|_\infty \le \epsilon/3$. Because f_N is continuous, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f_N(x) - f_N(x_0)| < \epsilon/3.$$

Therefore, if $|x - x_0| < \delta$, we have

$$|f(x)-f(x_0)| \le |f(x)-f_N(x)|+|f_N(x)-f_N(x_0)|+|f_N(x_0)-f(x_0)| \le \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon.$$

Therefore, f is continuous. We have shown that $d_\infty(f_n, f) \to 0$ and hence our Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ converges in $(C(X;\mathbb{R}), d_\infty)$. □

3.2 Proof of Arzela-Ascoli

Let X be a compact metric space. When is a set $\mathcal{E} \subseteq C(X;\mathbb{R})$ compact? Recall that a set is compact if and only if it is complete and totally bounded. We also know that $C(X;\mathbb{R})$ is complete, and hence $\mathcal{E}$ is complete if and only if it is closed in d_∞. Thus, we are left with the question of when a set $\mathcal{E} \subseteq C(X;\mathbb{R})$ is totally bounded. The next theorem will answer this question.

Definition 3.2. Let $\mathcal{E} \subseteq C(X;\mathbb{R})$. Then we say $\mathcal{E}$ is *pointwise bounded* if for every $x \in X$, the set $\{f(x) : f \in \mathcal{E}\}$ is a bounded subset of $\mathbb{R}$, or equivalently,

$$\sup_{f \in \mathcal{F}} |f(x)| < +\infty.$$

Definition 3.3. Let $\mathcal{E} \subseteq C(X;\mathbb{R})$. Then we say that $\mathcal{E}$ is *equicontinuous* if for every $x_0 \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x, x_0) < \delta \implies |f(x) - f(x_0)| \text{ for all } x \in X \text{ and } f \in \mathcal{E}.$$

Remark. The point of equicontinuity is that the *same* δ will work for *all* the functions in the set $\mathcal{E}$.

Theorem 3.4. *Let X be a compact metric space, and let $\mathcal{E} \subseteq C(X;\mathbb{R})$. Then $\mathcal{E}$ is totally bounded in d_∞ if and only if it is equicontinuous and pointwise bounded.*

This theorem and its proof standard and can be found for instance in [**?**, §4.6] and [**?**, §11.4].

Proof. ($\implies$) Suppose $\mathcal{E}$ is totally bounded. Then it must be bounded with respect to d_∞. Therefore, there exists some $f_0 \in \mathcal{E}$ and $R > 0$ such that $\mathcal{E} \subseteq B_{d_\infty}(f_0, R)$. Let $M = \|f_0\|_\infty + R$. Then we have $\|f\|_\infty \leq M$ for all $f \in \mathcal{E}$. In particular, $|f(x)| \leq M$ for all $f \in \mathcal{E}$ and $x \in X$, and hence the set $\{f(x) : f \in \mathcal{E}\} \subseteq \mathbb{R}$ is bounded for each $x \in X$. (This proof in fact shows $\mathcal{E}$ is *uniformly bounded* since the M does not depend on x.)

Next, let us show that $\mathcal{E}$ is equicontinuous. Let $x_0 \in X$ and $\epsilon > 0$. By assumption, $\mathcal{E}$ can be covered by finitely many balls of radius $\epsilon/3$; call them $B_{d_\infty}(f_1, \epsilon/3), \ldots, B_{d_\infty}(f_n, \epsilon/3)$. For each f_k, there is a δ_k such that

$$d(x, x_0) < \delta_k \text{ implies } |f_k(x) - f_k(x_0)| < \epsilon/3 \text{ for all } x \in X.$$

Let $\delta = \min(\delta_1, \ldots, \delta_n)$. Every $f \in \mathcal{E}$ is contained in some ball $B_{d_\infty}(f_k, \epsilon/3)$, which implies $\|f_k - f\| < \epsilon/3$. If $d(x, x_0) < \delta \leq \delta_k$, then we have

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(x_0)| + |f_k(x_0) - f(x_0)| \\ &\leq \|f - f_k\|_\infty + |f_k(x) - f_k(x_0)| + \|f - f_k\|_\infty \\ &< \epsilon. \end{aligned}$$

Since the same δ works for all f, we have equicontinuity.

($\Longleftarrow$) Assume that $\mathcal{E}$ is equicontinuous and pointwise bounded. To prove it is totally bounded, suppose that $r > 0$. By equicontinuity, for each $x \in X$, there is a δ_x such that $d(y, x) < \delta_x$ implies $|f(y) - f(x)| < r/4$ for all $f \in \mathcal{E}$. Now the open balls $\{B_X(x, \delta_x)\}_{x \in X}$ are an open cover of the space X. By compactness of X, there exists a finite subcover, call it $(B_X(x_j, \delta_{x_j}))_{j=1}^m$. By pointwise boundedness, for each $j = 1, \ldots, m$ there is an M_j such that $|f(x_j)| \leq M_j$ for all $f \in \mathcal{E}$. Let $M = \max(M_1, \ldots, M_m)$. Now $[-M, M]$ is a bounded set in $\mathbb{R}$ and hence it is totally bounded. Thus, it can be covered by finitely many balls $B_{\mathbb{R}}(y_1, r/4)$, ..., $B_{\mathbb{R}}(y_n, r/4)$.

Let $A = \{x_1, \ldots, x_m\}$ and $B = \{y_1, \ldots, y_n\}$. Let B^A be the set of functions $\phi : A \to B$, which is a finite set because A and B are finite. Let

$$\mathcal{E}_\phi = \{f \in \mathcal{E} : f(x_j) \in B_{\mathbb{R}}(\phi(x_j), r/4) \text{ for } j = 1, \ldots, m\}.$$

Here $\phi(x_j)$ is one of the y_k's. Note that every function $f \in \mathcal{E}$ must be in one of the sets $\mathcal{E}_\phi$. Indeed, for each j, $f(x_j)$ must be in one of the balls $B_{\mathbb{R}}(y_k, r/4)$ because these balls were chosen to cover $[-M, M]$, and we may define $\phi(x_j)$ to be this y_k. Thus, we have

$$\mathcal{E} = \bigcup_{\phi \in B^A} \mathcal{E}_\phi.$$

Of course, if we let $\Phi = \{\phi \in B^A : \mathcal{E}_\phi \neq \varnothing\}$, then we still have

$$\mathcal{E} = \bigcup_{\phi \in \Phi} \mathcal{E}_\phi.$$

For each $\phi \in \Phi$, pick one function $f_\phi \in \mathcal{E}_\phi$. We claim that $\mathcal{E}_\phi \subseteq B_{d_\infty}(f_\phi, r)$. Indeed, suppose that $f \in \mathcal{E}_\phi$ and $x \in X$. Then x is contained in one of the balls $B(x_j, \delta_{x_j})$. By our choice of δ_{x_j}, this implies that

$$|f(x) - f(x_j)| < r/4, \qquad |f_\phi(x) - f_\phi(x_j)| < r/4.$$

By the definition of $\mathcal{E}_\phi$, we know that $f(x_j)$ and $f_\phi(x_j)$ are in $B_{\mathbb{R}}(\phi(x_j), r/4)$ and hence

$$|f(x_j) - \phi(x_j)| < r/4, \qquad |f_\phi(x_j) - \phi(x_j)| < r/4.$$

By the triangle inequality,

$$|f(x) - f_\phi(x)| \leq |f(x) - f(x_j)| + |f(x_j) - \phi(x_j)| + |\phi(x_j) - f_\phi(x_j)| + |f_\phi(x_j) - f_\phi(x)| < r.$$

Hence, for all x, we have $|f(x) - f_\phi(x)| < r$. Now $|f(x) - f_\phi(x)|$ achieves a maximum on X and this maximum must be less than r, and hence $\|f - f_\phi\|_\infty < r$. This shows that $f \in B_{d_\infty}(f_\phi, r)$, and therefore, $\mathcal{E}_\phi \subseteq B_{d_\infty}(f_\phi, r)$.

We know that $\mathcal{E}$ is the union of the $\mathcal{E}_\phi$'s and each $\mathcal{E}_\phi$ is contained in $B_{d_\infty}(f_\phi, r)$. Therefore,

$$\mathcal{E} \subseteq \bigcup_{\phi \in B^A} B_{d_\infty}(f_\phi, r).$$

Since r was arbitrary, we have shown that $\mathcal{E}$ is totally bounded. □

3.3 Variants and Corollaries

Corollary 3.5. *Suppose that X is a compact metric space and $\mathcal{E} \subseteq C(X;\mathbb{R})$. Then $\mathcal{E}$ is compact if and only if it is closed (in d_∞), equicontinuous, and pointwise bounded.*

Proof. As remarked earlier, $C(X;\mathbb{R})$ is complete. Hence, $\mathcal{E}$ is complete if and only if it is closed. Theorem 3.4 shows that $\mathcal{E}$ is totally bounded if and only if it is equicontinuous and pointwise bounded. □

Corollary 3.6. *Suppose that X is a compact metric space and $\mathcal{E} \subseteq C(X;\mathbb{R})$. If $\mathcal{E}$ is equicontinuous and pointwise bounded, then $\overline{\mathcal{E}}$ is compact.*

Proof. By Theorem 3.4, we know that $\overline{\mathcal{E}}$ is totally bounded. Since $C(X;\mathbb{R})$ is complete, Lemma 1.18 shows that $\overline{\mathcal{E}}$ is compact. □

Corollary 3.7. *Let X be a compact metric space. If $(f_n)_{n\in\mathbb{N}}$ is a sequence in $C(X;\mathbb{R})$ which is equicontinuous and pointwise bounded, then there exists a uniformly convergent subsequence.*

Proof. Let $\mathcal{E} = \{f_n : n \in \mathbb{N}\} \subseteq C(X;\mathbb{R})$. The sequence being equicontinuous and pointwise bounded means exactly that the set $\mathcal{E}$ is equicontinuous and pointwise bounded. By the previous Corollary, $\overline{\mathcal{E}}$ is compact in d_∞. Since $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\overline{\mathcal{E}}$, it must have a convergent subsequence with respect to d_∞. □

Corollary 3.8. *Let X be a compact metric space. If $(f_n)_{n\in\mathbb{N}}$ is a sequence in $C(X;\mathbb{R})$ that converges uniformly, then $(f_n)_{n\in\mathbb{N}}$ is equicontinuous and pointwise bounded.*

Proof. Let f be the limit of the sequence f_n. Let $\mathcal{E} = \{f_n : n \in \mathbb{N}\} \cup \{f\}$. Then $\mathcal{E}$ is totally bounded in d_∞. Indeed, given $r > 0$, there exists N such that $n \geq N$ implies $d_\infty(f_n, f) < r$. Therefore,

$$\mathcal{E} \subseteq B_{d_\infty}(f, r) \cup \bigcup_{j=1}^{N-1} B_{d_\infty}(f_j, r).$$

Thus, $\overline{\mathcal{E}}$ is totally bounded. Hence, by Theorem 3.4, $\mathcal{E}$ is equicontinuous and pointwise bounded. □

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