(Uniform) Continuity, (Uniform) Convergence

David Jekel

February 10, 2018

The distinctions between continuity, uniform continuity, convergence, and pointwise convergence deserve repeated explanation, since they are important but easily confused.

Let's compare the definitions. In the following, X and Y are metric spaces, and $f : X \to Y$ and $f_n : X \to Y$ are functions.

Continuity

- In continuity, you are only considering one function f (not a sequence of functions).[1]
- Continuity describes how $f(x)$ changes *when you change* x.
- It says that for each x_0, if x is close to x_0, then $f(x)$ is close to $f(x_0)$.
- The definition reads: $\forall x_0 \in X, \forall \epsilon > 0, \exists \delta > 0$ such that $\forall x \in X$, $d(x, x_0) < \delta$ implies $d(f(x), f(x_0)) < \epsilon$.

Uniform Continuity

- Uniform continuity is a stronger version of continuity. As before, you are only considering one function f (not a sequence of functions).
- Uniform continuity describes how $f(x)$ changes *when you change* x.
- If f is uniformly continuous, that means that if x is close to x_0, then $f(x)$ is close to $f(x_0)$. Importantly, it requires that how close $f(x)$ and $f(x_0)$ are only depends on how close x and x_0 are. The same estimate works for all possible values of x and x_0.
- The definition reads: $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x_0 \in X, \forall x \in X$, $d(x, x_0) < \delta$ implies $d(f(x), f(x_0)) < \epsilon$.

[1] Well, maybe you have a sequence $\{f_n\}$ of continuous functions, but in that case, the definition of continuity applies to each function f_n independently. It only considers one function at a time.

- Note that the only thing that changed relative to the definition of continuity was that "$\forall x_0$" moved later, but this makes all the difference. In the statement of continuity, putting x_0 first allows the value of δ to depend on both ϵ and x_0, but for uniform continuity the value of δ only depends on ϵ. Thus, you can make *one* choice of δ such that $f(x)$ and $f(x_0)$ will be *uniformly* close together for all values of x and x_0 within a distance of δ from each other.

Pointwise Convergence

- To discuss pointwise convergence $f_n \to f$, you need to have a sequence of functions $\{f_n\}$, not just one function.
- Convergence describes how $f_n(x)$ changes *when you change* n (but don't change x).
- $f_n \to f$ pointwise means that for each $x \in X$, if n is large enough, then $f_n(x)$ is close to $f(x)$.
- The definition reads: $\forall x \in X, \forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n \geq N$ implies $d(f_n(x), f(x)) < \epsilon$.

Uniform Convergence

- Uniform convergence is a stronger version of convergence. To discuss pointwise convergence $f_n \to f$, you need to have a sequence of functions $\{f_n\}$, not just one function.
- Uniform convergence describes how $f_n(x)$ changes *when you change* n (but don't change x).
- $f_n \to f$ means tha if n is large enough, then $f_n(x)$ is close to $f(x)$ uniformly for all values of x.
- The definition reads: $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall x \in X$, $n \geq N$ implies $d(f_n(x), f(x)) < \epsilon$.
- Note that the only thing that changed relative to the definition of pointwise convergence was that "$\forall x$" moved later, but this makes all the difference. In the statement of convergence, putting x first allows the value of N to depend on both ϵ and x, but for uniform converence the value of N only depends on ϵ. Thus, you can make *one* choice of N such that $f_n(x)$ and $f(x)$ will be *uniformly* close together for all values of x whenever $n \geq N$.

Examples and Theorems

- The following functions $\mathbb{R} \to \mathbb{R}$ are uniformly continuous: x, $\sin x$, $1/(1+x^2)$, $\arctan x$.

- The following functions $\mathbb{R} \to \mathbb{R}$ are *not* uniformly continuous: x^2, $\sin x^2$, any polynomial of degree at least 2, e^x.
- If f_n is continuous for each n and $f_n \to f$ uniformly, then f is continuous.
- If f_n is continuous for each n and $f_n \to f$ pointwise, then f might not be continuous. For example, consider $f_n : \mathbb{R} \to \mathbb{R}$ given by

  $$f_n(x) = \begin{cases} 0, & x \le 0 \\ nx, & 0 \le x \le 1/n \\ 1, & x \ge 1. \end{cases} \qquad f(x) = \begin{cases} 0, & x \le 0, \\ 1, & x > 0. \end{cases}$$

  Then $f_n \to f$ pointwise but not uniformly.
- If f_n is uniformly continuous and $f_n \to f$ uniformly, then f is uniformly continuous.
- If f_n is uniformly continuous and $f_n \to f$ pointwise, then f might not be continuous, or f might be continuous and not uniformly continuous.