Review of Complex Numbers

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This note reviews the definition and basic properties of complex numbers. Basic manipulations of complex numbers are an essential prerequisite for much of analysis. In doing the exercises, feel free to skip proving properties you already know, but make sure you read over each one. The most interesting exercises are 1.f, 1.g, 2.f, and those of section 3.

1 The Field of Complex Numbers

A **complex number** is an expression of the form $x + iy$, where $x, y \in \mathbb{R}$. Logically, a complex number is no different than a pair (x, y) of real numbers; however, the notation $x+iy$ is convenient for multiplication of complex numbers. Addition and multiplication are defined by

$$\begin{aligned}(a + bi) + (c + di) &= (a + b) + (c + d)i \\ (a + bi)(c + di) &= (ac - bd) + (ad + bc)i.\end{aligned}$$

To remember the definition of multiplication, just expand using the distributive property and the rule $i^2 = -1$.

$\mathbb{C}$ denotes the set of complex numbers. We regard $\mathbb{R}$ as a subset of $\mathbb{C}$ by identifying the real number x with the complex number $x + 0i$. We also write iy for $0 + iy$ and write i for $0 + i1$. Then $i^2 = -1$.

Exercise 1. *Verify that the complex numbers are a* **field***, that is, they satisfy the following properties. Here $z = x+iy$ and $z_j = x_j +iy_j$ is a complex number.*

a. Addition is commutative: $z_1 + z_2 = z_2 + z_1$.

b. Addition is associative: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.

c. $0 = 0 + 0i$ is an additive identity: $z + 0 = 0 + z = z$ for all z.

d. A complex number $z = x + iy$ has an additive inverse given by

$$-z = -x + i(-y).$$

e. Multiplication is commutative: $z_1 z_2 = z_2 z_1$.

f. *Multiplication is associative:* $(z_1 z_2)z_3 = z_1(z_2 z_3)$.

g. $1 = 1 + 0i$ *is a multiplicative identity:* $z \cdot 1 = 1 \cdot z$ *for all* z.

h. *A nonzero complex number* $z = x + iy$ *has a multiplicative inverse given by*

$$\frac{1}{z} = \frac{x}{x^2 + y^2} + \frac{-y}{x^2 + y^2} i.$$

i. *Multiplication distributes over addition:* $z(z_1 + z_2) = zz_1 + zz_2$.

2 The Norm, Real and Imaginary Parts

For a complex number $z = x + iy$, we define

$$\begin{aligned} \operatorname{Re} z &= x \\ \operatorname{Im} z &= y \\ |z| &= \sqrt{x^2 + y^2}. \end{aligned}$$

Exercise 2. *Verify the following properties:*

a. $z = (\operatorname{Re} z) + i(\operatorname{Im} z)$.

b. $|\operatorname{Re} z| \leq |z|$.

c. $|\operatorname{Im} z| \leq |z|$.

d. $|z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|$.

e. $|z_1 + z_2| \leq |z_1| + |z_2|$.

f. $|z_1 z_2| = |z_1||z_2|$.

g. $|z^n| = |z|^n$.

3 Sequences and Series of Complex Numbers

A sequence $\{z_n\}$ of complex number is said to **converge to** z if we have $|z_n - z| \to 0$ as $n \to \infty$. In this case, we write $z_n \to z$ and $\lim_{n\to\infty} z_n = z$.

Exercise 3. *Show that* $z_n \to z$ *if and only if* $\operatorname{Re} z_n \to \operatorname{Re} z$ *and* $\operatorname{Im} z_n \to \operatorname{Im} z$. *Moreover,* $\{z_n\}$ *is Cauchy if and only if* $\{\operatorname{Re} z_n\}$ *and* $\{\operatorname{Im} z_n\}$ *are Cauchy.*

Exercise 4. *Show that the complex numbers are complete, that is, any Cauchy sequence converges.*

A **series** of complex numbers is an expression of the form $\sum_{n=0}^{\infty} a_n$, where $a_n \in \mathbb{C}$. The series **converges** if

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n \text{ exists.}$$

The series **converges absolutely** if

$$\sum_{n=0}^{\infty} |a_n| < \infty.$$

Exercise 5. *If $\sum_{n=0}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$.*

Exercise 6. *If a series converges absolutely, then it is converges. That is, if $\sum_{n=0}^{\infty} |a_n| < \infty$, then $\lim_{N\to\infty} \sum_{n=0}^{N} a_n$ exists.*

Exercise 7. *$\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} \operatorname{Re} a_n$ and $\sum_{n=0}^{\infty} \operatorname{Im} a_n$ both converge.*

Exercise 8. *$\sum_{n=0}^{\infty} a_n$ converges absolutely if and only if $\sum_{n=0}^{\infty} \operatorname{Re} a_n$ and $\sum_{n=0}^{\infty} \operatorname{Im} a_n$ both converge absolutely.*

Exercise 9. *For $|z| < 1$, show that $\sum_{n=0}^{\infty} z^n$ converges absolutely to $1/(1-z)$.*