Results of Aleksandrov and Peller: Operator Bernstein's Inequality and Modulus of Continuity

David Jekel

October 31, 2016

1 Introduction

This script will explain results from several papers of A.B. Aleksandrov and V.V. Peller. The proofs for §1 are drawn from [1] and the proofs for §2 are drawn from [2].

Recall the spectral theorem and functional calculus: If A is a self-adjoint operator on a Hilbert space $\mathcal{H}$, there is a projection-valued spectral measure E_A (Borel measure on $\mathbb{R}$). Let $\mathcal{B}(\mathbb{R})$ be the space of bounded Borel functions on $\mathbb{R}$ with the sup norm. For $f \in \mathcal{B}(\mathbb{R})$, we can define $f(A)$ by

$$f(A) = \int_{\mathbb{R}} f(\lambda)\, dE_A(\lambda).$$

Our goal is to understand how smoothly $f(A)$ depends on A. It's a fact that if $f : \mathbb{R} \to \mathbb{C}$ is Lipschitz, then it is not necessarily operator Lipschitz, that is,

$$|f(s) - f(t)| \leq C|s - t| \text{ for } s, t \in \mathbb{R}$$

does NOT imply

$$\|f(A) - f(B)\| \leq C' \|A - B\|$$

for bounded self-adjoint operators on a Hilbert space. However, we will prove that if $f : \mathbb{R} \to \mathbb{C}$ is Hölder continuous, then f is operator Hölder continuous, and

$$\|f(A) - f(B)\| \leq C[f]_\alpha \|A - B\|^\alpha,$$

where $[f]_\alpha$ is the α Hölder seminorm. We will also see what happens to uniformly continuous f with an arbitrary modulus of continuity. Similar questions can be asked about operator derivatives, higher order differences, and so forth.

2 Bernstein's Inequality

Though the authors originally proved these results using tensor product estimates, they later found a shorter proof using Bernstein's inequality from complex analysis. Let $\mathcal{E}_\sigma$ be the class of bounded functions $f : \mathbb{R} \to \mathbb{C}$ such that f

extends to an entire function $f : \mathbb{C} \to \mathbb{C}$ satisfying $|f(z)| \le C_\delta e^{(\sigma+\delta)|z|}$ for any $\delta > 0$. Then

Theorem 1 (Bernstein's Inequality)**.** *If $f \in \mathcal{E}_\sigma$, then*

$$\|f'\|_{L^\infty(\mathbb{R})} \le \sigma \|f\|_{L^\infty(\mathbb{R})}.$$

In order to handle $f(A)$ for a self-adjoint operator A, we extend Bernstein's inequality to X-valued functions for a Banach space X. If X is a Banach space, then let $\mathcal{E}_\sigma(X)$ be the class of bounded functions $f : \mathbb{R} \to X$ such that $\phi(f(X)) \in \mathcal{E}_\sigma$ for all $\phi \in X^*$. Then we have

Theorem 2. *If $f \in \mathcal{E}_\sigma(X)$, then*

$$\|f(s) - f(t)\|_X \le \sigma|s-t|\|f\|_{L^\infty(\mathbb{R},X)} \text{ for } s, t \in \mathbb{R}.$$

We need some lemmas from complex analysis. The next two lemmas are taken from [3, Lecture 6].

Lemma 3 (Phragmen-Lindelöf)**.** *Suppose that f is analytic on $D = \{re^{i\theta} : \theta_0 < \theta < \theta_0 + \lambda\}$ and extends continuously to $\overline{D}$. If $|f(z)| \le Ce^{c|z|^\rho}$ with $\rho < \pi/\lambda$ and $|f(z)| \le M$ on ∂D, then $|f(z)| \le M$ on D.*

Proof. By rotation, assume that $D = \{re^{i\theta} : -\lambda/2 < \theta < \lambda/2\}$. Let $\rho < \alpha < \pi/\lambda$. Note that the canonical choice of z^α maps D analytically onto $\{re^{i\theta} : |\theta| < \alpha\lambda/2 < \pi/2\}$ and hence $\operatorname{Re} z^\alpha \ge \delta|z|^\alpha$ for some $\delta > 0$. This implies that

$$|f(z)e^{-\epsilon z^\alpha}| \le Ce^{c|z|^\rho - \epsilon\delta|z|^\alpha},$$

which approaches zero as $z \to \infty$. Since we also have $|f(z)e^{-\epsilon z^\alpha}| \le |f(z)|$ on ∂D, the maximum principle implies that $|f(z)e^{-\epsilon z^\alpha}| \le M$. Taking $\epsilon \to 0$ completes the proof. □

Lemma 4. *If $f \in \mathcal{E}_\sigma$, then $|f(z)| \le e^{\sigma|\operatorname{im} z|}\|f\|_{L^\infty(\mathbb{R})}$.*

Proof. Let us prove the inequality for the upper half-plane. The lower half-plane is symmetrical. For $\delta > 0$, note that $e^{i(\sigma+\delta)z}f(z)$ is bounded on $\mathbb{R}$ by $\|f\|_\infty$. We also have $e^{i(\sigma+\delta)z}f(z) \le C_\delta$ for z on the imaginary axis. We apply Phragmen-Lindelöf on the two quarter planes $\{0 < \theta < \pi/2\}$ and $\{\pi/2 < \theta < \pi\}$ with $\lambda = \pi/2$, $\rho = 1$, to conclude that

$$|e^{i(\sigma+\delta)z}f(z)| \le \max(\|f\|_{L^\infty(\mathbb{R})}, C_\delta) \text{ for } \operatorname{im} z \ge 0.$$

Next, since $e^{i(\sigma+\delta)z}f(z)$ is bounded, we can apply Phragmen-Lindelöf on $\{0 < \theta < \pi\}$ with $\lambda = \pi$ and $\rho = 0$ to conclude that

$$|e^{i(\sigma+\delta)z}f(z)| \le \|f\|_{L^\infty(\mathbb{R})}.$$

Taking $\delta \to 0$ completes the proof for the upper half-plane. □

Lemma 5. *Suppose that $f \in L^\infty(\mathbb{R})$. We have $f \in \mathcal{E}_{2\pi\sigma}$ if and only if the Fourier transform $\mathcal{F}f$ is supported in $[-\sigma, \sigma]$.*

Proof. Suppose that $\mathcal{F}f$ is supported in $[-\sigma, \sigma]$. From the nature of the Frechet topology on the Schwarz class $\mathcal{S}$ and the duality between continuous functions on $[-\sigma, \sigma]$ with measures, we can deduce that

$$\mathcal{F}f = \sum_{j=0}^{n} (-1)^j \frac{d^j}{dt^j} \mu_j$$

for some finite complex Borel measures μ_j on $[-\sigma, \sigma]$. This implies that

$$f(z) = \sum_{j=0}^{n} \int_{-\sigma}^{\sigma} \frac{d^j}{dt^j} e^{2\pi i z t}\, d\mu_j(t) = \sum_{j=0}^{n} (2\pi i z)^j \int_{-\sigma}^{\sigma} e^{2\pi i z t}\, d\mu_j(t).$$

Note that f extends to an entire function using Morera's theorem and $|f(z)| \leq C_\delta (1+|z|)^n e^{(2\pi\sigma+\delta)|\operatorname{im} z|}$ for any $\delta > 0$, which implies that $f \in \mathcal{E}_\sigma$.

Conversely, suppose that $f \in \mathcal{E}_{2\pi\sigma}$. For $\operatorname{im} z \geq 0$, we have $1 - iz$ in the right half-plane, so we can define $\sqrt{1-iz}$ to be holomorphic there with positive real part. Moreover, $\operatorname{Re}\sqrt{1-iz} \geq c|\sqrt{1-iz}| \geq c\sqrt{|z|}$ for $\operatorname{Re} z \geq 0$. For $\epsilon > 0$, define

$$f_\epsilon(z) = f(z) e^{-c\epsilon\sqrt{1-iz}} \text{ for } \operatorname{Re} z > -1.$$

Note that

$$|f_\epsilon(z)| \leq |f(z)| e^{-c\epsilon\sqrt{|z|}} \leq \|f\|_{L^\infty(\mathbb{R})} e^{-c\epsilon\sqrt{|z|}} e^{\sigma|\operatorname{im} z|}.$$

In particular, $f_\epsilon \in L^1(\mathbb{R})$. If $t < -\sigma$, then

$$\mathcal{F}f_\epsilon(t) = \int_{\mathbb{R}} f_\epsilon(z) e^{-2\pi i t z}\, dz.$$

Because the integrand is holomorphic on $\operatorname{im} z > -1$, we have for $R > 0$

$$\int_{-R}^{R} f_\epsilon(z) e^{-2\pi i t z}\, dz = -\int_{\gamma_R} f_\epsilon(z) e^{-2\pi i t z}\, dz,$$

where γ_R is the semi-circular arc from R to $-R$ in the upper half-plane. But

$$\left| \int_{\gamma_R} f_\epsilon(z) e^{-2\pi i t z}\, dz \right| \leq \int_{\gamma_R} \|f\|_{L^\infty} e^{-c\epsilon\sqrt{|z|}} e^{2\pi(-\sigma - t)|\operatorname{im} z|}\, |dz| \leq \|f\|_{L^\infty} \pi R e^{-c\epsilon\sqrt{R}}$$

Taking $R \to +\infty$ shows that

$$\mathcal{F}f_\epsilon(t) = \int_{\mathbb{R}} f_\epsilon(z) e^{-2\pi i t z}\, dz = 0 \text{ for } t < -\sigma.$$

Therefore, $\mathcal{F}f_\epsilon$ is supported in $[-\sigma, +\infty)$. Because $f_\epsilon \to f$ in $\mathcal{S}'$, we have $\mathcal{F}f_\epsilon \to \mathcal{F}f$ in $\mathcal{S}'$ and hence $\mathcal{F}f$ is supported in $[-\sigma, +\infty)$. A symmetrical argument shows that $\mathcal{F}f$ vanishes for $t > \sigma$, so that $\mathcal{F}f$ is supported in $[-\sigma, \sigma]$. ☐

Lemma 6. *Let $z_k = \frac{1}{2}(k+\frac{1}{2})$ for $k \in \mathbb{Z}$. Then $g_k(z) = \sqrt{2}\cos(2\pi z)/2\pi(z-z_k)$ is an orthonormal basis for $\mathcal{E}_{2\pi} \cap L^2(\mathbb{R})$ in the L^2 inner product.*

Proof. By the last lemma and Plancherel's theorem, the Fourier transform maps $\mathcal{E}_{2\pi} \cap L^2(\mathbb{R})$ isometrically onto $L^2([-1,1]) \subset L^2(\mathbb{R})$. An orthonormal basis for $L^2([-1,1])$ is given by

$$h_k(t) = \frac{1}{\sqrt{2}}\chi_{[-1,1]}(t)e^{-\pi i(k+1/2)t} = \frac{1}{\sqrt{2}}\chi_{[-1,1]}(t)e^{-2\pi i z_k t}, \quad k \in \mathbb{Z}.$$

Then

$$\mathcal{F}^{-1}h_k(z) = \frac{1}{\sqrt{2}}[\mathcal{F}^{-1}\chi_{[-1,1]}](z-z_k),$$

and

$$\mathcal{F}^{-1}\chi_{[-1,1]}(z) = \int_{-1}^{1} e^{2\pi i z t}\,dt = \frac{1}{\pi z}\sin 2\pi z.$$

Hence, since $z_k = \frac{1}{2}(k+\frac{1}{2})$, we have

$$\mathcal{F}^{-1}h_k(z) = \frac{\sqrt{2}\sin 2\pi(z-z_k)}{2\pi(z-z_k)} = \frac{(-1)^{k+1}\sqrt{2}\cos 2\pi z}{2\pi(z-z_k)} = (-1)^{k+1}g_k(z),$$

which implies g_k is an orthonormal basis for $\mathcal{E}_{2\pi} \cap L^2(\mathbb{R})$. □

Lemma 7. *If $f \in \mathcal{E}_{2\pi} \cap L^2(\mathbb{R})$ and g_k is as above, then*

$$\langle f, g_k\rangle_{L^2(\mathbb{R})} = (-1)^{k+1}f(k\pi + \pi/2).$$

In particular,

$$f(z) = \sum_{k\in\mathbb{Z}}(-1)^{k+1}f(z_k)\frac{\cos 2\pi z}{2\pi(z-z_k)},$$

where the sum converges in $L^2(\mathbb{R})$.

Proof. Let f_ϵ be as in the proof of Lemma 5. For $z \in \mathbb{R}$, let $\gamma_r^+(z)$ denote the upper semicircle centered at z with radius r, oriented counterclockwise. By the Cauchy integral formula, for $r > 0$ and $R > r + |z_k|$,

$$\int_{[-R,R]\setminus(z_k-r,z_k+r)} \frac{f_\epsilon(z)\sqrt{2}e^{2\pi i z}}{2\pi(z-z_k)}\,dz = \int_{\gamma_r^+(z_k)} \frac{f_\epsilon(z)\sqrt{2}e^{2\pi i z}}{2\pi(z-z_k)}\,dz - \int_{\gamma_R^+(0)} \frac{f_\epsilon(z)\sqrt{2}e^{2\pi i z}}{2\pi(z-z_k)}\,dz.$$

Note that $\mathcal{F}f \in L^2([-1,1])$, hence in $L^1([-1,1])$, which implies

$$|f(z)| \leq \|\mathcal{F}f\|_{L^1([-1,1])}e^{2\pi|\operatorname{im} z|},$$

and hence $|f_\epsilon(z)| \leq Ce^{2\pi|\operatorname{im} z|}e^{-\epsilon\sqrt{|z|}}$. Thus, the integral over $\gamma_R^+(0)$ approaches zero as $R \to +\infty$, which implies

$$\int_{\mathbb{R}\setminus(z_k-r,z_k+r)} \frac{f_\epsilon(z)\sqrt{2}e^{2\pi i z}}{z-z_k}\,dz = \int_{\gamma_r^+(z_k)} \frac{f_\epsilon(z)\sqrt{2}e^{2\pi i z}}{2\pi(z-z_k)}\,dz.$$

Because $f_\epsilon \to f$ in $L^2(\mathbb{R})$ and locally uniformly on $\{\operatorname{im} z \geq 0\}$ as $\epsilon \to 0$, we have

$$\int_{\mathbb{R}\setminus(z_k-r,z_k+r)} \frac{f(z)\sqrt{2}e^{2\pi i z}}{2\pi(z-z_k)}\,dz = \int_{\gamma_r^+(z_k)} \frac{f(z)\sqrt{2}e^{2\pi i z}}{2\pi(z-z_k)}\,dz.$$

Observe that on $\{|z-z_k| = r\}$, we have

$$\frac{f(z)\sqrt{2}e^{2\pi i z}}{2\pi(z-z_k)} = \frac{f(z_k)\sqrt{2}e^{2\pi i z_k}}{2\pi(z-z_k)} + O(1) = \frac{i(-1)^k\sqrt{2}f(z_k)}{2\pi(z-z_k)} + O(1).$$

Therefore,

$$\int_{\mathbb{R}\setminus(z_k-r,z_k+r)} \frac{f(z)\sqrt{2}e^{2\pi i z}}{2\pi(z-z_k)}\,dz = \frac{(-1)^{k+1}}{\sqrt{2}} f(z_k) + O(r).$$

A symmetrical computation for $e^{-2\pi i z}$ instead of $e^{2\pi i z}$ will yield the same answer; averaging the two results,

$$\int_{\mathbb{R}\setminus(z_k-r,z_k+r)} \frac{f(z)\sqrt{2}\cos 2\pi z}{2\pi(z-z_k)}\,dz = \frac{(-1)^{k+1}}{\sqrt{2}} f(z_k) + O(r).$$

Taking $r \to 0$ with dominated convergence yields

$$\langle f, g_k\rangle = \frac{(-1)^{k+1}}{\sqrt{2}} f(z_k).$$

Therefore,

$$f(z) = \sum_{z\in\mathbb{Z}} \langle f, g_k\rangle g_k(z) = \sum_{k\in\mathbb{Z}} (-1)^{k+1} f(z_k) \frac{\cos 2\pi z}{2\pi(z-z_k)}.$$

□

Theorem 8 (Bernstein's inequality)**.** *Let $f \in \mathcal{E}_\sigma$. Then*

$$|f(s) - f(t)| \leq \sigma|s-t|\|f\|_{L^\infty(\mathbb{R})}.$$

Proof. By rescaling the domain of f, we can assume that $\sigma = 2\pi$. Next, by translating f, it suffices to show that

$$|f(z) - f(-z)| \leq 2\pi \cdot 2|z|\|f\|_{L^\infty(\mathbb{R})} \text{ for } z \geq 0.$$

Define

$$F(z) = \frac{f(z) - f(-z)}{2z}.$$

Since $f \in L^\infty(\mathbb{R})$, we have $F \in L^2(\mathbb{R})$, and clearly $F \in \mathcal{E}_{2\pi}$. Thus, by the previous lemma,

$$F(z) = \sum_{k\in\mathbb{Z}} (-1)^{k+1} F(z_k) \frac{\cos 2\pi z}{2\pi(z-z_k)} = \sum_{k\in\mathbb{Z}} (-1)^{k+1} \frac{f(z_k) - f(-z_k)}{2z_k} \frac{\cos 2\pi z}{2\pi(z-z_k)}.$$

Hence, for $z \in [0, 1/4]$, noting that $z_k(z - z_k) < 0$, we have

$$\begin{aligned}|F(z)| &\le \|f\|_{L^\infty(\mathbb{R})} \sum_{k\in\mathbb{Z}} \frac{-1}{z_k} \frac{\cos 2\pi z}{2\pi(z - z_k)} \\ &\le \|f\|_{L^\infty(\mathbb{R})} \sum_{k\in\mathbb{Z}} (-1)^{k+1} \frac{\sin(2\pi z_k) - \sin(-2\pi z_k)}{2z_k} \frac{\cos 2\pi z}{2\pi(z - z_k)} \\ &\le \|f\|_{L^\infty(\mathbb{R})} \frac{\sin(2\pi z) - \sin(-2\pi z)}{2z} \le \|f\|_{L^\infty(\mathbb{R})} \cdot 2\pi,\end{aligned}$$

where the last line follows from applying the preceding identity with $f(z) = \sin 2\pi z$. This complete the proof for $z \in [0, 1/4]$. On the other hand, for $z > 1/4$,

$$|F(z)| \le \frac{\|f\|_{L^\infty(\mathbb{R})}}{|z|} \le 4\|f\|_{L^\infty(\mathbb{R})} \le 2\pi\|f\|_{L^\infty(\mathbb{R})},$$

which is what we wanted to prove. □

Corollary 9. *If X is a Banach space and $f \in \mathcal{E}_\sigma(X)$, then*

$$\|f(s) - f(t)\|_X \le \sigma|s - t|\|f\|_{L^\infty(\mathbb{R},X)} \text{ for } s, t \in \mathbb{R}.$$

Proof. Let (x, ϕ) denote the bilinear pairing between $x \in X$ and $\phi \in X^*$. For any $\phi \in X^*$, we know that $(f(t), \phi) \in \mathcal{E}_\sigma$. Therefore, by the previous theorem,

$$|(f(s) - f(t), \phi)| \le \sigma|s - t|\|(f(\cdot), \phi)\|_{L^\infty(\mathbb{R})} \le \sigma|s - t|\|f\|_{L^\infty(\mathbb{R},X)}.$$

Taking the supremum over ϕ on the left-hand side completes the proof. □

Theorem 10. *Let $f \in \mathcal{E}_\sigma$, and let A and B be bounded self-adjoint operators on a Hilbert space H. Then*

$$\|f(A) - f(B)\| \le \sigma\|A - B\|\|f\|_{L^\infty(\mathbb{R})}.$$

Proof. Define $g : \mathbb{R} \to \mathcal{L}(H)$ by $g(z) = f(A + z(B - A))$. We claim that $g \in \mathcal{E}_\sigma(L(H))$. By the spectral mapping theorem $\|f(A+z(B-A))\| \le \|f\|_{L^\infty(\mathbb{R})}$ for $z \in \mathbb{R}$, so g is bounded on the real line. To check the exponential growth we use power series. Note that by iterative application of Bernstein's inequality,

$$\|f^{(k)}\|_{L^\infty(\mathbb{R})} \le \sigma^k\|f\|_{L^\infty}.$$

In particular,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \text{ with } |f^{(k)}(0)| \le \sigma^k\|f\|_{L^\infty(\mathbb{R})}.$$

Substituting $A + z(B - A)$ for z yields,

$$\begin{aligned}\|f(A + z(B - A))\| &\le \|f\|_{L^\infty} \sum_{k=0}^{\infty} \frac{\sigma^k}{k!} \|A + z(B - A)\|^k \\ &\le \|f\|_{L^\infty} e^{\sigma\|A+z(B-A)\|} \\ &\le \|f\|_{L^\infty} e^{\sigma\|A\|} e^{\sigma z\|B-A\|}.\end{aligned}$$

This implies $g \in \mathcal{E}_{\sigma\|A-B\|}(\mathcal{L}(H))$. Therefore, by the previous result,

$$\|f(A)-f(B)\| = \|g(0)-g(1)\| \leq \sigma\|A-B\|\|g\|_{L^\infty(\mathbb{R},\mathcal{L}(H))} = \sigma\|A-B\|\|f\|_{L^\infty(\mathbb{R})}.$$

□

Remark. It is in fact true even if A and B are unbounded, so long as $A - B$ is bounded. See [1, Theorem 5.4].

3 Operator Moduli of Continuity

A **modulus of a continuity** is a function $\omega : [0, +\infty) \to [0, +\infty)$ such that

- $\omega(0) = 0$,
- $\omega(x) > 0$ for $x > 0$,
- ω is increasing,
- ω is continuous,
- $\omega(x + y) \leq \omega(x) + \omega(y)$.

For $f : \mathbb{R} \to \mathbb{C}$, we define

$$\|f\|_{\Lambda_\omega} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\omega(|x - y|)},$$

and denote by $\Lambda_\omega = \Lambda_\omega(\mathbb{R})$ the space of f for which this seminorm is finite.

For any modulus of continuity ω, there is another modulus of continuity ω_* given by $\omega_*(0) = 0$ and

$$\omega_*(x) = x \int_x^\infty \frac{\omega(t)}{t^2}\, dt \text{ for } x > 0,$$

assuming the integral is finite for some x. Our goal is to prove

Theorem 11. *If $f \in \Lambda_\omega(\mathbb{R})$, and if A and B are bounded self-adjoint operators on a Hilbert space, then*

$$\|f(A) - f(B)\| \leq C\omega_*(\|A - B\|)\|f\|_{\Lambda_\omega},$$

where C is a universal constant.

Example. If $\omega(x) = x^\alpha$ for some $\alpha \in (0, 1)$, then $\omega_*(x) = (1 - \alpha)^{-1}x^\alpha$. Thus, any Hölder continuous function on $\mathbb{R}$ is also operator Hölder continuous. If $\omega(x) = \min(x, c)$, then $\omega_*(x) = x(1/c + \log c - \log x)$ for $x < c$.

Lemma 12. *If ω is a modulus of continuity and ω_* is finite, then ω_* is a modulus of continuity.*

Proof. Note that

$$\omega_*(x) = \int_1^\infty \frac{\omega(sx)}{s^2}\,ds,$$

and it follows that ω_* is increasing and subadditive. It is clear that if $x > 0$, then $\omega_*(x) > 0$. Moreover, continuity is clear away from 0, so it remains to show that $\omega_*(x) \to 0$ as $x \to 0$. If $\int_0^\infty \omega(x)/x^2\,dx < \infty$, then this would be trivial. If $\int_0^\infty \omega(x)/x^2\,dx = +\infty$, then using L'Hopital's rule,

$$\lim_{x\to 0^+} \frac{\int_x^\infty \omega(y)/y^2\,dy}{1/x} = \lim_{x\to 0^+} \frac{-\omega(x)/x^2}{-1/x^2} = \lim_{x\to 0^+} \omega(x) = 0. \qquad \square$$

Our strategy for proving Theorem 11 will be to perform dyadic decomposition on $\mathcal{F}f$. Using standard bump function constructions, we can create a $w \in C_c^\infty(\mathbb{R}, [0,1])$ supported in $[1/2, 2]$ such that

$$w(x) = 1 - w(x/2) \text{ for } x \in [1,2].$$

We observe that

$$\sum_{n\in\mathbb{Z}} w(x/2^n) + w(-x/2^n) = 1 \text{ for } x \neq 0,$$

and so we aim to write

$$f = \sum_{n\in\mathbb{Z}} \mathcal{F}^{-1}(\mathcal{F}f \cdot w(x/2^n)) + \sum_{n\in\mathbb{Z}} \mathcal{F}^{-1}(\mathcal{F}f \cdot w(-x/2^n)).$$

(Technically, this is only true up to a term with Fourier transform supported at 0, i.e. a polynomial.) Then because $\mathcal{F}f \cdot w(x/2^n)$ is supported in in $[-2^n, 2^n]$, we can apply the operator Bernstein's inequality to $\mathcal{F}^{-1}(\mathcal{F}f \cdot w(x/2^n))$, and the same with $w(x/2^n)$ replaced by $w(-x/2^n)$.

We introduce the following notation: We define $v \in C_c^\infty(\mathbb{R})$ by

$$v(x) = \begin{cases} 1, & |x| \le 1, \\ w(|x|), & |x| \ge 1. \end{cases},$$

and define

$$\begin{aligned} W_n^+ &= \mathcal{F}^{-1}[w(x/2^n)] \\ W_n^- &= \mathcal{F}^{-1}[w(-x/2^n)] \\ V_n &= \mathcal{F}^{-1}[v(x/2^n)]. \end{aligned}$$

Then in $\mathcal{S}'$, we have

$$\mathcal{F}V_N + \sum_{n<N} (\mathcal{F}W_n^+ + \mathcal{F}W_n^-) = 1,$$

so heuristically at least

$$f = V_N * f + \sum_{n<N} (W_n^+ * f + W_n^- * f).$$

Now let us give the details of the argument.

Lemma 13. *There is a universal constant $C > 0$ such that*

$$\begin{aligned}\|f - V_n * f\|_{L^\infty(\mathbb{R})} &\le C\omega(2^{-n})\|f\|_{\Lambda_\omega(\mathbb{R})}\\ \|W_n^+ * f\|_{L^\infty(\mathbb{R})} &\le C\omega(2^{-n})\|f\|_{\Lambda_\omega(\mathbb{R})}\\ \|W_n^- * f\|_{L^\infty(\mathbb{R})} &\le C\omega(2^{-n})\|f\|_{\Lambda_\omega(\mathbb{R})}.\end{aligned}$$

Proof. Note that if $f \in \Lambda_\omega(\mathbb{R})$, then subaddivity of ω implies that $|\omega(x)| \le B|x|$ for some constant B and hence $|f(x)| \le A + B|x|$ for some constants A and B. Since V_n is a Schwarz function, we can express $f * V_n$ using Lebesgue integration. Moreover, since $\int V_n = v(0) = 1$, we have

$$\begin{aligned}|f(x) - V_n * f(x)| &= \left|\int_{\mathbb{R}} [f(x) - f(x-y)]V_n(y)\,dy\right|\\ &= \left|2^n \int_{\mathbb{R}} [f(x) - f(x-y)]V_0(2^n y)\,dy\right|\\ &\le \|f\|_{\Lambda_\omega} \int_{\mathbb{R}} 2^n \omega(|y|)|V_0(2^n y)|\,dy.\end{aligned}$$

Break the integral into three regions $(-\infty, -2^{-n})$, $[2^{-n}, 2^{-n}]$, and $(2^{-n}, +\infty)$, and then combine the two outer terms:

$$|f(x) - V_n * f(x)| \le \|f\|_{\Lambda_\omega} \left(2^n \int_{-2^{-n}}^{2^{-n}} \omega(|y|)|V_0(2^n y)|\,dy + 2^{n+1} \int_{2^{-n}}^{+\infty} \omega(|y|)|V_0(2^n y)|\,dy\right).$$

The first integral can clearly be estimated by $\omega(2^{-n})\|V_0\|_{L^1(\mathbb{R})}$. For the second term, we observe that since $y \ge 2^{-n}$ and choose $k \ge -n$ such that $2^k \le y < 2^{k+1}$, so that

$$\omega(y) \le \omega(2^{k+1}) = \omega(2^{n+k+1} \cdot 2^{-n}) \le 2^{n+k+1}\omega(2^{-n}) \le 2^{n+1} y\omega(2^{-n}).$$

Therefore,

$$\begin{aligned}2^{n+1} \int_{2^{-n}}^{+\infty} \omega(y)|V_0(2^n y)|\,dy &\le 4 \int_{2^{-n}}^{\infty} \omega(2^{-n}) 2^n y |V_0(2^n y)|\, 2^n dy\\ &\le 4\omega(2^{-n}) \int_1^\infty y|V_0(y)|\,dy \le C\omega(2^{-n}).\end{aligned}$$

This implies $|f(x) - V_n * f(x)| \le C\|f\|_{\Lambda_\omega}\omega(2^{-n})$. To prove the estimates for $W_n^\pm$, note that $\int W_n^\pm = 0$ since the Fourier transform vanishes at the origin, and hence

$$f * W_n^\pm(x) = \int_{\mathbb{R}} [f(x) - f(x-y)]W_n^\pm(y)\,dy,$$

and therefore we can use the same argument as for V_n. ∎

Proof of Theorem 11. Let A and B be bounded self-adjoint operators, $f \in \Lambda_\omega$. Since A and B are bounded, we can modify f for large x to make f bounded, without increasing $\|f\|_{\Lambda_\omega}$. Note that for $M < N$, we have

$$f(A) = (f - f * V_N)(A) + \sum_{n=M+1}^{N} f_n(A) + f * V_M(A),$$

where $f_n = f * W_n^+ + f * W_n^-$. Of course, the same holds for B, hence,

$$\|f(A)-f(B)\| \leq 2\|f-f*V_N\|_{L^\infty(\mathbb{R})} + \sum_{n=M+1}^{N} \|f_n(A)-f_n(B)\| + \|f*V_M(A)-f*V_M(B)\|.$$

Now $f * V_M$ has Fourier transform supported in $[-2^{M+1}, 2^{M+1}]$, so by the operator Bernstein's inequality,

$$\begin{aligned}
\|f * V_M(A) - f * V_M(B)\| &\leq 2^{M+1}\|f * V_M\|_{L^\infty(\mathbb{R})}\|A - B\| \\
&\leq 2^{M+1}\|f\|_{L^\infty(\mathbb{R})}\|V_0\|_{L^1(\mathbb{R})}\|A - B\| \\
&\to 0 \text{ as } M \to -\infty.
\end{aligned}$$

Thus, taking $M \to -\infty$ in the above inequality, we have

$$\|f(A) - f(B)\| \leq 2\|f - f * V_N\|_{L^\infty(\mathbb{R})} + \sum_{n=-\infty}^{N} \|f_n(A) - f_n(B)\|.$$

Choose N so that $2^{-N} \leq \|A - B\| < 2^{-N+1}$, and observe that

$$2\|f - f * V_N\|_{L^\infty(\mathbb{R})} \leq C\omega(2^{-N})\|f\|_{\Lambda_\omega} \leq C\omega_*(\|A - B\|)\|f\|_{\Lambda_\omega}.$$

For the other terms, apply the operator Bernstein inequality to f_n to conclude that

$$\begin{aligned}
\sum_{n=-\infty}^{N} \|f_n(A) - f_n(B)\| &\leq \sum_{n=-\infty}^{N} 2^{n+1}\|A - B\|\|f_n\|_{L^\infty(\mathbb{R})} \\
&\leq C \sum_{n=-\infty}^{N} 2^{n+1}\|A - B\|\omega(2^{-n})\|f\|_{\Lambda_\omega} \\
&\leq C \left(\sum_{k=0}^{\infty} \frac{\omega(2^{-N+k})}{2^{-N+k}}\right) \|A - B\|\|f\|_{\Lambda_\omega} \\
&\leq C \left(\sum_{k=0}^{\infty} \int_{2^{-N+k}}^{2^{-N+k+1}} \frac{2\omega(t)}{t^2}\, dt\right) 2^{-N+1}\|f\|_{\Lambda_\omega} \\
&\leq C \cdot 4 \cdot 2^{-N} \int_{2^{-N}}^{\infty} \frac{\omega(t)}{t^2}\, dt \cdot \|f\|_{\Lambda_\omega} \\
&\leq 4C\omega_*(2^{-N})\|f\|_{\Lambda_\omega} \leq 4C\omega_*(\|A - B\|)\|f\|_{\Lambda_\omega}.
\end{aligned}$$

Hence, $\|f(A) - f(B)\| \leq 5C\omega_*(\|A - B\|)\|f\|_{\Lambda_\omega}$ as desired. □

References

[1] A.B. Aleksandrov and V.V. Peller. "Functions of Perturbed Unbounded Self-Adjoint Operators. Operator Bernstein Type Inequalities."

[2] A. B. Aleksandrov and V.V. Peller. "Operator Hölder-Zygmund Functions."

[3] B. Ya Levin. *Lectures on Entire Functions.*